Calculus 141, section 9.6 Ratio Test and Root Tests
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- The geometric series \( \sum_{n=m}^{\infty} cr^n = \frac{cr^m}{1-r} \) if and only if \( |r| < 1 \).

- The \( p \)-series \( \sum_{n=1}^{\infty} \frac{1}{n^p} \) converges whenever \( p > 1 \) and diverges whenever \( 0 < p \leq 1 \).

- The Integral Test states a series \( \sum_{n=1}^{\infty} a_n \) converges if and only if \( \int_1^{\infty} f(x) \, dx \) converges.

- In the Direct Comparison Test, \( \sum_{n=1}^{\infty} a_n \) converges if its terms are less than those of a known convergent series, and diverges if its terms are greater than those of a known convergent series.

- The Limit Comparison Test states: If \( \lim_{n \to \infty} \frac{a_n}{b_n} \) exists and is a positive number, then positive series \( \sum_{n=1}^{\infty} a_n \) and \( \sum_{n=1}^{\infty} b_n \) either both converge or both diverge.

A downside to the Comparison Tests is that they require a suitable series to use for the comparison. In contrast, the Ratio Test and the Root Test require only the series itself.

**Ratio Test** (Theorem 9.15) Given a positive series \( \sum_{n=1}^{\infty} a_n \) for which \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = r \): a. If \( 0 \leq r < 1 \), then \( \sum_{n=1}^{\infty} a_n \) converges. b. If \( r > 1 \), then \( \sum_{n=1}^{\infty} a_n \) diverges. c. If \( r = 1 \) or \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} \) does not exist, no conclusion.

The proof of part a. relies upon the definition of limits and the "creation" of a geometric series which converges to which we compare our original series. Briefly, by the nature of inequalities there exists a value \( s \) for which \( 0 \leq r \leq s < 1 \). From the definition of limits, there exists a value \( N \) such that for \( n \geq N \), \( \frac{a_{n+1}}{a_n} \leq s \Rightarrow a_{n+1} \leq a_n s \).

Then \( 0 < a_{N+2} \leq a_{N+1} s \leq (a_N s)^2 \Rightarrow 0 < a_{N+n} \leq a_N s^n \). The latter value is the basis for a geometric series which converges. The application of the Comparison Test finishes the proof of part a. The proof of part b. is analogous, with \( r > s > 1 \).

**Hint:** The Ratio Test works best for series such as \( \sum \frac{1}{n!} \), \( \sum r^n \), and \( \sum \frac{1}{2^n + c} \) for which \( n \) is a factorial or an exponent.

**Example A:** Does \( \sum_{n=0}^{\infty} \frac{100^n}{n!} \) converge? **Answer:** yes
Example B: Does \( \sum_{n=1}^{\infty} \frac{n^n}{n!} \) converge?  \textit{Answer: no}

Example B upside down: Does the "reciprocal" series \( \sum_{n=1}^{\infty} \frac{n!}{n^n} \) converge?  \textit{Answer: yes}

In Lecture 9.4 it was noted that since \( \left\{ \frac{n!}{n^n} \right\} \) converges to 0, although we could say that \( \sum_{n=1}^{\infty} \frac{n!}{n^n} \) might converge, we could not be certain that it \textit{does} converge.

The nature of the Ratio Test is such that if it shows a series converges, then the series involving the reciprocals of the terms must diverge, and vice-versa.

Example C: Use the Ratio Test to test convergence of \( \sum_{n=1}^{\infty} \frac{1}{n} \) and \( \sum_{n=1}^{\infty} \frac{1}{n^2} \).  \textit{Answer: inconclusive}
Root Test (Theorem 9.16) Given a positive series $\sum_{n=1}^{\infty} a_n$ for which $\lim_{n \to \infty} \sqrt[n]{a_n} = r$:  

- a. If $0 \leq r < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.  
- b. If $r > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.  
- c. If $r = 1$ or $\lim_{n \to \infty} \sqrt[n]{a_n} = r$ DNE, no conclusion.

The proof is a little simpler than that for the Ratio Test. For values $s$ and $N$ as described above,  

$$\sqrt[n]{a_n} \leq s \Rightarrow a_n \leq s^n,$$  

the basis for a geometric series which converges. The Comparison Test finishes the proof of part a. The proof of part b. is analogous, with $r > s > 1$. The Root Test is especially useful in series that involve a $k$th power and which have no complications such as factorials.

Example D: Does $\sum_{n=1}^{\infty} a_n$ converge?  

**Answer:** yes

Example E: Does $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^n}$ converge?  

**Answer:** no