1) In class we showed that the general solution to

\[ au_x + bu_y + u = 0 \]

was

\[ u(x, t) = f(bx - ay)e^{\frac{ax+by}{a^2+b^2}}, \]

for any function (of one variable) \( f \). Assuming \( a \neq 0 \), show that this is equivalent to saying

\[ u(x, t) = f(bx - ay)e^{-\frac{x}{a}}, \]

for any function (of one variable) \( f \).

2) Find the solution of

\[ \sqrt{1-x^2} u_x + u_y = 0, \quad u(0, y) = y. \]

3) Find the solution of

\[ u_x + u_y + u = e^{x+2y}, \quad u(x, 0) = 0 \]

4) Consider the PDE

\[ yu_x + xu_y = 0. \]

What are the characteristics for this PDE? Describe the general solution in terms of these characteristics (first in words, then more precisely in a formula).

5) Consider the electromagnetic fields \( \mathbf{B} \) (magnetic field vector) and \( \mathbf{E} \) (electric field vector). As you may know, their evolution is governed by Maxwell’s equations which for a vacuum can be stated as:

\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \times \mathbf{B} = \mu \varepsilon \frac{\partial \mathbf{E}}{\partial t}, \quad \nabla \cdot \mathbf{E} = 0, \quad \nabla \cdot \mathbf{B} = 0, \]

where \( \mu \) and \( \varepsilon \) are constants. But aren’t \( \mathbf{B} \) and \( \mathbf{E} \) suppose to propagate as waves, hence the name electromagnetic waves?? Yes indeed: show, for example, that if \( \mathbf{B} = (0, B(x, t), 0) \) and \( \mathbf{E} = (0, 0, E(x, t)) \) then both \( E \) and \( B \) satisfy the wave equation

\[ u_{tt} - c^2 u_{xx} = 0, \]

where \( c = (\mu \varepsilon)^{-\frac{1}{2}} \) (in fact \( c \) turns out to be the speed of light!). One can also show this in the general case with components in all three directions depending on all of (three) space and time. Of course then they solve the equation \( u_{tt} - c^2 \Delta u = 0 \) where \( \Delta \) is the Laplacian operator in \( x, y, z \).

6) Use d’Alemberts formula to solve

\[ u_{tt} - c^2 u_{xx} = 0, \quad u(x, 0) = \sin x, \quad u_t(x, 0) = x. \]
7) **Stability of the solution to the wave equation**: Fix a time $t$, say $t^*$. Use the d’Alembert formula to show that for every $\epsilon > 0$ we can find $\delta > 0$ such that if $||\phi_1(x) - \phi_2(x)|| < \delta$ and $||\psi_1(x) - \psi_2(x)|| < \delta$ then $||u_1(x, t^*) - u_2(x, t^*)|| < \epsilon$. Here $u_i$ ($i = 1, 2$) is the solution to

$$\frac{\partial^2 u_i}{\partial t^2} - c^2 \frac{\partial^2 u_i}{\partial x^2} = 0, \quad u_i(x, 0) = \phi_i(x), \quad \frac{\partial u_i}{\partial t} (x, 0) = \psi_i(x),$$

and for any function $f(x)$, $||f(x)||$ denotes the maximum value $f$ on whole line (we really should use supremum instead of maximum - if you know about such things - but assume $\phi_i$ and $\psi_i$ are continuous).

8) Consider the model (described in class) for a vibrating string with only small vibrations in the transverse direction. Suppose now that the string is also subjected to gravity. Derive the equation which governs the motion of the string. You may assume that the acceleration due to gravity is constant with magnitude $g$.

9) Find the general solution of the equation derived in the previous question.

10) Solve

$$u_{xx} - 3u_{xt} - 4u_{tt} = 0, \quad u(x, 0) = x^2, \quad u_t(x, 0) = e^x.$$

11) Find the general solution for $u(x, y, z)$ to

$$u_x + u_y + u_z = 0.$$