Geometric structures on manifolds

William M. Goldman

July 21, 2018

Mathematics Department, University of Maryland, College Park, MD 20742 USA
E-mail address: wmg@math.umd.edu
URL: http://www.math.umd.edu/~wmg
2010 Mathematics Subject Classification. Primary 54C40, 14E20;
Secondary 46E25, 20C20

Key words and phrases. Euclidean geometry, affine geometry,
projective geometry, manifold, coordinate atlas, convexity,
connection, parallel transport, homogeneous coordinates, Lie groups,
homogeneous spaces, metric space, Riemannian metric, geodesic,
completeness, developing map, holonomy homomorphism, proper
transformation group, Lie algebra, vector field

The author gratefully acknowledges research support from NSF
Grants DMS1065965, DMS1406281, DMS1065965 as well as the
Research Network in the Mathematical Sciences DMS1107367 (GEAR).

Abstract. The study of locally homogeneous geometric structures on manifolds was initiated by Charles Ehresmann in 1936,
who first proposed the classification of putting a “classical geometry” on a topological manifold. In the late 1970’s, locally homo-
gegeneous Riemannian structures on 3-manifolds formed the context for Bill Thurston Geometrization Conjecture, later proved by
Perelman. This book develops the theory of geometric structures modeled on a homogeneous space of a Lie group, which are not
necessarily Riemannian. Drawing on a diverse collection of techniques, we hope to invite researchers at all levels to this fascinating
and currently extremely active area of mathematics.
Contents

Introduction 1
Organization of the text 2
Part One: Affine and Projective Geometry 3
Part Two: Geometric Manifolds 4
Part Three: Affine and projective structures 7
Prerequisites 8
Acknowledgements 9

Part 1. Affine and projective geometry 11

Chapter 1. Affine space 13
  1.1. Affine spaces 13
  1.1.1. Euclidean geometry and its isometries 13
  1.1.2. Affine spaces 15
  1.1.3. Affine transformations 16
  1.1.4. Tangent spaces 17
  1.1.5. Acceleration and geodesics 19
  1.2. Parallel structures 20
  1.2.1. Parallel Riemannian structures 20
  1.2.2. Parallel tensor fields 21
  1.2.3. Complex affine geometry 21
  1.3. Affine vector fields 21
  1.3.1. Parallel vector fields 23
  1.3.2. Homotheties and radiant vector fields 23
  1.3.3. Affine subspaces 24
  1.4. Volume in affine geometry 24
  1.4.1. Centers of gravity 25
  1.5. Linearizing affine geometry 26

Chapter 2. Projective space 27
  2.1. Ideal points 27
  2.1.1. Affine patches 28
  2.2. Homogeneous coordinates 28
  2.2.1. The basic dictionary 32
2.3. Affine patches 32
2.4. Classical projective Geometry 34
  2.4.1. Projective reflections 34
  2.4.2. Fundamental theorem of projective geometry 36
  2.4.3. Products of reflections 37
2.5. Asymptotics of projective transformations 38
  2.5.1. Some examples 40
  2.5.2. Limits of similarity transformations 41

Chapter 3. Duality and non-Euclidean geometry 43
  3.1. Duality 43
  3.2. Correlations and polarities 44
  3.3. Intrinsic metrics 46
    3.3.1. Examples of convex cones 46
    3.3.2. The Hilbert metric 48
    3.3.3. The Hilbert metric on a triangle 48
    3.3.4. The Kobayashi metric 50
  3.4. Asymptotics of hyperbolic geometry 54

Chapter 4. Convex domains 55
  4.1. Convex cones 55
  4.2. Convex bodies in projective space 63
  4.3. Spaces of convex bodies in projective space 65
    4.3.1. Collineations and convex bodies 67

Part 2. Geometric manifolds 73

Chapter 5. Geometric structures on manifolds 75
  5.1. Geometric atlases 76
    5.1.1. The pseudogroup of local mappings 77
    5.1.2. \((G,X)\)-automorphisms 78
  5.2. Development, holonomy 79
    5.2.1. Construction of the developing map 80
    5.2.2. Role of the holonomy group 82
    5.2.3. Extending geometries 83
    5.2.4. Simple applications of the developing map 84
  5.3. The graph of a geometric structure 84
    5.3.1. The tangent \((G,X)\)-bundle 84
    5.3.2. Developing sections 86
    5.3.3. The associated principal bundle 86
  5.4. The classification of geometric 1-manifolds 87
    5.4.1. Compact Euclidean 1-manifolds and flat tori 87
    5.4.2. Compact affine 1-manifolds and Hopf manifolds 88
5.4.3. Classification of projective 1-manifolds 89
5.4.4. Grafting 91

Chapter 6. Examples of Geometric Structures 93
6.1. The hierarchy of geometries 94
6.1.1. Projective structures from non-Euclidean geometry 94
6.1.2. Flat tori and Euclidean structures 94
6.1.3. Hopf manifolds and radiant similarity manifolds 95
6.1.4. Enlarging and refining 95
6.2. Fibrations 97
6.2.1. The sphere of directions 97
6.2.2. Hopf manifolds 98
6.2.3. Lifting holonomy representations 100
6.3. Suspensions 101
6.3.1. Parallel suspensions 102
6.3.2. Radiant suspensions 103

Chapter 7. Classification 107
7.1. Marking geometric structures 107
7.1.1. Marked Riemann surfaces 107
7.1.2. Moduli of flat tori 108
7.1.3. Marked geometric manifolds 108
7.1.4. The infinitesimal approach 109
7.2. The Ehresmann-Weil-Thurston holonomy principle 109
7.3. Representation varieties 110
7.4. Deformation spaces of geometric structures 110
7.5. Affine structures and connections 111

Chapter 8. Completeness 113
8.1. Locally homogeneous Riemannian manifolds 113
8.2. Completeness and convexity of affine connections 115
8.2.1. Review of affine connections 115
8.2.2. Geodesic completeness and the developing map 117
8.3. Complete affine structures on the 2-torus 119
8.4. Examples of incomplete structures 123
8.4.1. A manifold with only complete structures 123
8.4.2. Geodesics on Hopf manifolds 124
8.4.3. Radiant affine manifolds 124
8.5. Cartesian products 125
8.6. Complete affine manifolds 125
8.6.1. The Bieberbach theorems 126
8.6.2. Complete affine solvmanifolds 127
Part 3. Affine and projective structures

Chapter 9. Affine structures on surfaces and the Euler characteristic

9.1. Benzécri’s theorem on affine 2-manifolds
9.1.1. The surface as an identification space.
9.1.2. The turning number
9.1.3. The Milnor-Wood inequality
9.2. Higher dimensions
9.2.1. The Chern-Gauss-Bonnet Theorem
9.2.2. Smillie’s examples of flat tangent bundles
9.2.3. The Kostant-Sullivan Theorem

Chapter 10. Affine structures on Lie groups and algebras

10.1. Koszul-Vinberg algebras
10.1.1. Bi-invariant affine structures and associativity
10.1.2. Locally simply transitive affine actions
10.1.3. Two-dimensional commutative associative algebras
10.1.4. Two-dimensional noncommutative associative algebras
10.2. Some locally simply transitive affine actions
10.2.1. Simply transitive action of non-unimodular group
10.2.2. An incomplete homogeneous flat Lorentzian structure
10.2.3. Right parabolic halfplane
10.2.4. Parabolic cylinders
10.2.5. Nonradiant deformations of radiant halfspace quotients

Chapter 11. Parallel volume and completeness

11.1. The volume obstruction
11.2. Smillie’s nonexistence theorem
11.3. Radiant affine structures
11.4. Fried’s classification of closed similarity manifolds
11.4.1. Completeness versus radiance
11.4.2. Canonical metrics and incompleteness
11.4.3. Incomplete geodesics are recurrent
11.4.4. Degenerate similarities

Chapter 12. Hyperbolicity

12.1. Benoist’s theory of divisible convex sets
12.2. Vey’s semisimplicity theorem
12.3. Kobayashi hyperbolicity
12.3.1. Hessian manifolds
12.3.2. Completely incomplete manifolds
12.3.2.1. Hyperbolic torus bundles
CONTENTS

Chapter 13. Projective structures on surfaces 185
  13.1. Pathological developing maps 185
  13.2. Generalized Fenchel-Nielsen twist flows 185
  13.3. Bulging deformations 185
  13.4. Fock-Goncharov coordinates 191
  13.5. Affine spheres and Labourie-Loftin parametrization 193
  13.6. Choi’s convex decomposition theorem 193

Chapter 14. Complex-projective structures 195
  14.1. Schwarzian derivative 196
  14.2. Affine structures and the complex exponential map 196
  14.3. Thurston parametrization 196
  14.4. Fuchsian holonomy 196
    14.4.1. Normality domains 196
  14.5. Higher dimensions: flat conformal and spherical
    CR-structures 199

Chapter 15. Geometric structures on 3-manifolds 201
  15.1. Complete affine structures on 3-manifolds 201
    15.1.1. Complete affine 3-manifolds 201
    15.1.2. Complete affine structures on noncompact 3-manifolds 202
  15.2. Margulis spacetimes 203
    15.2.1. Affine boosts and crooked planes 204
    15.2.2. Marked length spectra 206
    15.2.3. Deformations of hyperbolic surfaces 207
    15.2.4. Classification 208
  15.3. Nilpotent holonomy 210
  15.4. Dupont’s classification of hyperbolic torus bundles 210
  15.5. Examples of projective three-manifolds 210

Appendix A. Transformation groups 211

Appendix B. Tensor analysis 213

Bibliography 215

List of Figures 225

List of Tables 227

Index 229
Introduction

Symmetry powerfully unifies the various notions of geometry. Based on ideas of Sophus Lie, Felix Klein’s 1972 Erlanger program proposed that geometry is the study of properties of a space $X$ invariant under a group $G$ of transformations of $X$. For example Euclidean geometry is the geometry of $n$-dimensional Euclidean space $\mathbb{R}^n$ invariant under its group of rigid motions. This is the group of transformations which transforms an object $\xi$ into an object congruent to $\xi$. In Euclidean geometry one can speak of points, lines, parallelism of lines, angles between lines, distance between points, area, volume, and many other geometric concepts. All these concepts can be derived from the notion of distance, that is, from the metric structure of Euclidean geometry. Thus any distance-preserving transformation or isometry preserves all of these geometric entities.

Notions more primitive than that of distance are the length and speed of a smooth curve. Namely, the distance between points $a, b$ is the infimum of the length of curves $\gamma$ joining $a$ and $b$. The length of $\gamma$ is the integral of its speed $\|\gamma'(t)\|$. Thus Euclidean geometry admits an infinitesimal description in terms of the Riemannian metric tensor, which allows a measurement of the size of the velocity vector $\gamma'(t)$. In this way standard Riemannian geometry generalizes Euclidean geometry by imparting Euclidean geometry to each tangent space.

Other geometries “more general” than Euclidean geometry are obtained by removing the metric concepts, but retaining other geometric notions. Similarity geometry is the geometry of Euclidean space where the equivalence relation of congruence is replaced by the broader equivalence relation of similarity. It is the geometry invariant under similarity transformations. In similarity geometry does not involve distance, but rather involves angles, lines and parallelism. Affine geometry arises when one speaks only of points, lines and the relation of parallelism. And when one removes the notion of parallelism and only studies lines, points and the relation of incidence between them (for example, three points being collinear or three lines being concurrent) one arrives at projective geometry. However in projective geometry, one must enlarge the space to projective space, which is the space upon while all the projective transformations are defined.

Here is a basic example illustrating the differences among the various geometries. Consider a particle moving along a smooth path; it has a well-defined velocity vector field (this uses only the differentiable structure of $\mathbb{R}^n$). In Euclidean geometry, it makes sense to discuss its “speed,” so “motion at unit speed” (that is, “arc-length-parametrized
geodesic”) is a meaningful concept there. But in affine geometry, the concept of “speed” or “arc-length” must be abandoned: yet “motion at constant speed” remains meaningful since the property of moving at constant speed can be characterized as parallelism of the velocity vector field (zero acceleration). In projective geometry this notion of “constant speed” (or “parallel velocity”) must be further weakened to the concept of “projective parameter” introduced by J. H. C. Whitehead [150].

Synthetic projective geometry was developed by the architect Desargues in 1636–1639 out of attempts to understand the geometry of perspective. Two hundred years later non-Euclidean (hyperbolic) geometry was developed independently and practically simultaneously by Bolyai in 1833 and Lobachevsky in 1826–1829. These geometries were unified in 1871 by Klein who noticed that Euclidean, affine, hyperbolic and elliptic geometry were all “present” in projective geometry.

The plethora of different geometries suggests that, at the least superficial level, there is no inclusive theory of locally homogeneous structures. Each geometry has its own features and idiosyncrasies, and special techniques particular to each geometry are used in each case. For example, a surface modeled on $\mathbb{C}P^1$ has the underlying structure of a Riemann surface, and viewing a $\mathbb{C}P^1$-structure as a projective structure on a Riemann surface provides a satisfying classification of $\mathbb{C}P^1$-structures. Namely, as was presumably understood by Poincaré, the deformation space of $\mathbb{C}P^1$-structures on a closed surface $\Sigma$ with $\chi(\Sigma) < 0$ identifies with a holomorphic affine bundle over the Teichmüller space of $\Sigma$. When $X$ is a complex manifold upon which $G$ acts biholomorphically, holomorphic mappings provide a powerful tool in the study, a class of local mappings more flexible than “constant” maps (maps which are “locally in $G$”) but more rigid than general smooth maps. Another example occurs when $X$ admits a $G$-invariant connection (such as an invariant (pseudo-)Riemannian structure). Then the geodesic flow provides a powerful tool for the study of $(G, X)$-manifolds.

We emphasize the interplay between different mathematical techniques as an attractive aspect of this general subject.

**Organization of the text**

The book divides into three parts. Part One describes affine and projective geometry and provides some of the main background on these extensions of Euclidean geometry. As noted by Lie and Klein, most classical geometries can be modeled in projective geometry. We
introduce projective geometry as an extension of affine geometry, so we begin with a detailed discussion of affine geometry.

**Part One: Affine and Projective Geometry**

The first chapter introduces affine geometry as the geometry of parallelism. Two objects are parallel if they are related by a translation. Translations form a vector space $V$, and act simply transitively on affine space. That is, for two points $p, q \in A$ there is a unique translation taking $p$ to $q$. In this way, points in $A$ identify with the vector space $V$, but this identification depends on the (arbitrary) choice of a basepoint, or origin which identifies with the zero vector in $V$. One might say that an affine space is a vector space, where the origin is forgotten. More accurately, the special role of the zero vector is suppressed, so that all points are regarded equally.

The action by translations now allows the definition of acceleration of a smooth curve. A curve is a geodesic if its acceleration is zero, that is, if its velocity is parallel. In affine space itself, unparametrized geodesics are straight lines; a parametrized geodesic is a curve following a straight line at “constant speed”. Of course, the “speed” itself is undefined, but the notion of “constant speed” just means that the acceleration is zero.

These notions of parallelism is is a special case of the notion of an affine connection, except the existence of globally defined translations effecting the notion of parallelism is a special feature to our setting — the setting of flat connections. Just as Euclidean geometry is affine geometry with a parallel Riemannian metric, other linear-algebraic notions enhance affine geometry with parallel tensor fields. The most notable (and best understood) are flat Lorentzian (and pseudo-Riemannian) structures.

Chapter Two develops the geometry of projective space, viewed as the compactification of affine space. Ideal points arise as “where parallel lines meet.” A more formal definition of an ideal point is an equivalence class of lines, where the equivalence relation is parallelism of lines. Linear families (or pencils) of lines form planes, and indeed the set of ideal points in a projective space form a projective hyperplane, that is, a projective space of one lower dimension. Projective geometry appears when the ideal points lose their special significance, just as affine geometry appears when the zero vector $0$ in a vector space loses its special significance.

However, we prefer a more efficient (if less synthetic) approach to projective geometry in terms of linear algebra. Namely, the projective
space associated to a vector space V is the space \( P(V) \) of 1-dimensional linear subspaces of V (that is, lines in V passing through 0). Homogeneous coordinates are introduced on projective space as follows. Since a 1-dimensional linear subspace is determined by any nonzero element, its coordinates determine a point in projective space. Furthermore the homogeneous coordinates are uniquely defined up to projective equivalence, that is, the equivalence relation defined by multiplication by nonzero scalars. Projectivizing linear subspaces of V produces projective subspaces of \( P(V) \), and projectivizing linear automorphisms of V yield projective automorphisms, or collineations of \( P(V) \).

The equivalence of the geometry of incidence in \( P(V) \) with the algebra of V is remarkable. Homogeneous coordinates provide the “dictionary” between projective geometry and and linear algebra. The collineation group is compactified as a projective space of “projective endomorphisms;” this will be useful for studying limits of sequences of projective transformations. These “singular projective transformations” are important in controlling developing maps of geometric structures, as developed in the second part.

The third chapter discusses, first from the classical viewpoint of polarities, the Cayley-Beltrami-Klein model for hyperbolic geometry. Polarities are the geometric version of nondegenerate symmetric bilinear forms on vector spaces. They provide a natural context for hyperbolic geometry, which is one of the principal examples of geometry in this study.

The Hilbert metric on a properly convex domain in projective space is introduced and is shown to be equivalent to the categorically defined Kobayashi metric \([83, 85]\). Later this notion is extended to manifolds with projective structure.

The fourth chapter develops notions of convexity, and proves Benzécri’s theorem that the collineation group acts properly on the space of convex bodies in projective space, and the quotient is a compact (Hausdorff) manifold. Recently Benzécri’s theorem has been used by Cooper, Long and Tillman \([36]\) in their study of cusps of \( \mathbb{RP}^n \)-manifolds.

Part Two: Geometric Manifolds

The second part globalizes these geometric notions to manifolds, introducing locally homogeneous geometric structures in the sense of Ehresmann \([44]\) in the fifth chapter. We associate to every transformation group \((G, X)\) a category of geometric structures on manifolds locally modeled on the geometry of \( X \) invariant under the group \( G \). Because of the “rigidity” of the local coordinate changes of open sets
in $X$ which arise from transformations in $G$, these structures on $M$ intimately relate to the fundamental group $\pi_1(M)$.

Chapter 5 discusses three viewpoints for these structures. First we describe the coordinate atlases for the pseudogroup arising from $(G, X)$. Using the aforementioned rigidity, these are globalized in terms of a developing map

$$\tilde{M} \overset{\text{dev}}{\longrightarrow} X,$$

defined on the universal covering space $\tilde{M}$ of the geometric manifold $M$. The developing map is equivariant with respect to the holonomy homomorphism

$$\pi_1(M) \overset{h}{\longrightarrow} G$$

which represents the group $\pi_1(M)$ of deck transformations of $\tilde{M} \rightarrow M$ in $G$. Each of these two viewpoints represent $M$ as a quotient: in the coordinate atlas description, $M$ is the quotient of the disjoint union

$$\mathcal{U} := \coprod_{\alpha \in \Lambda} U_\alpha$$

of the coordinate patches $U_\alpha$; in the second description, $M$ is represented as the quotient of $\tilde{M}$ by the action of the group $\pi_1(M)$. While a map defined on a connected space $\tilde{M}$ may seem more tractable than a map defined on the disjoint union $\mathcal{U}$, the space $\tilde{M}$ can still be quite large. The third viewpoint replaces $\tilde{M}$ with $M$ and replaces the developing map by a section of a bundle defined over $M$. The bundle is a flat bundle, (that is, has discrete structure group in the sense of Steenrod). The corresponding developing section is characterized by transversality with respect to the foliation arising from the flat structure. This replaces the coordinate charts (respectively the developing map) being local diffeomorphisms into $X$.

Chapter 6 discusses examples of geometric manifolds from these three points of view. Although the main interest in these notes are structures modeled on affine and projective geometry, we describe other interesting structures.

All these structures are inter-related, because some geometries "contain" or "refine others." For example, affine geometry contains Euclidean geometry, when the metric notions are abandoned, but notions of parallelism are retained. This corresponds to the inclusion of the Euclidean isometry group as a subgroup of the affine automorphism group. Other examples include the projective and conformal models for non-Euclidean geometry. In these examples, the model space of the refined geometry is an open subset of the larger model space, and the
transformations in the refined geometry are restrictions of transformations in the larger geometry.

This hierarchy of geometries plays a crucial role in the theory. This is simply the geometric interpretation of the inclusion relations between closed subgroups of Lie groups. This algebraization of classical geometries in the nineteenth century by Lie and Klein organized the proliferation of classical geometries. We adopt that point of view here. Indeed, we use this as a cornerstone in the construction and classification of geometric structures. The classification of geometric manifolds often shows that a manifold modeled on one geometry may actually have a stronger geometry. For example, Fried’s theorem (discussed in §11.4) shows that a closed manifold with a similarity structure is either a Euclidean manifold or a manifold manifold on $\mathbb{E}^n\setminus\{0\} \cong S^{n-1} \times \mathbb{R}$ with its invariant (product) Riemannian metric.

Chapter 7 deals with the general classification of $(G,X)$-structures from the point of view of developing sections. The main result is an important observation due to Thurston [140] that the deformation space of marked $(G,X)$-structures on a fixed topology $\Sigma$ is itself “locally modeled” on the quotient of the space $\text{Hom}(\pi_1(\Sigma),G)$ by the group $\text{Inn}(G)$ of inner automorphisms of $G$. The description of $\mathbb{RP}^1$-manifolds is described in this framework.

Chapter 8 deals with the important notion of completeness, for taming the developing map. In general, the developing map may be quite pathological — even for closed $(G,X)$-manifolds — but under various hypotheses, can be proved to be a covering space onto its image. However, the main techniques borrow from Riemannian geometry, and involves geodesic completeness of the Levi-Civita connection (the Hopf-Rinow theorem). As an example, we classify complete affine structures on the 2-torus (due to Kuiper). The Hopf manifolds introduced in §??? are fundamental examples of incomplete structures. That affine structures on compact manifolds are generally incomplete is one dramatic difference between affine geometry and traditional Riemannian geometry.

This requires, of course, relating geometric structures to connections, since all of the locally homogeneous geometric structures discussed in this book can be approached through this general concept. However, we do not discuss the general notion of Cartan connections, but rather refer to the excellent introduction to this subject by R. Sharpe [127].
Part Three: Affine and projective structures

Chapter 9 begins the classification of affine structures on surfaces. We prove Benzécri's theorem [18] that a closed surface $\Sigma$ admits an affine structure if and only if its Euler characteristic vanishes. We discuss the famous conjecture of Chern that the Euler characteristic of a closed affine manifold vanishes. Following Kostant-Sullivan [87] we prove this in the complete case. Chern's conjecture has recently been proved in the volume-preserving case by Klingler [81].

Chapter 10 offers a detailed study of left-invariant affine structures on Lie groups. These provide many examples; in particular all the non-radiant affine structures on $T^2$ are invariant affine structures on the Lie group $T^2$. For these structures the holonomy homomorphism and the developing map blend together in an intriguing way, and this perhaps provides a conceptual basis for the unexpected relation between the one-dimensional property of geodesic completeness and the top-dimensional property of volume-preserving holonomy. Covariant differentiation of left-invariant vector fields lead to well-studied non-associative algebras $\text{algèbres symétriques à gauche}$ (left-symmetric algebras) (so defined as their associators are symmetric in the left two arguments). Commutator defines the structure of an underlying Lie algebra. Associative algebras correspond to bi-invariant affine structures, so the “group objects” in the category of affine manifolds correspond naturally to associative algebras. As these structures were introduced by Ernest Vinberg [148] in his study of homogeneous convex cones in affine space, and further developed by Jean-Louis Koszul and his school, we can these algebras Koszul-Vinberg algebras. We take a decidedly geometric approach to these ubiquitous mathematical structures.

Most closed affine surfaces are invariant affine structures on the torus group.

Chapter 11 describes the question (apparently first raised by L. Markus) of whether, for an closed orientable affine manifold, completeness is equivalent to parallel volume. The existence of a parallel volume form is equivalent to unimodularity of the linear holonomy group, that is, whether the holonomy is volume-preserving. This tantalizing question has led to much research, and subsumes various questions which we discuss. Carrière's proof that compact flat Lorentzian manifolds are complete [27] is a special case of this conjecture. In particular we give the sharp classification of closed similarity manifolds by D. Fried [50] (a much different proof is independently due to Vaisman-Reischer [144]). The analog of this question for left-invariant affine structures on Lie
groups is the conceptual and suggestive result that completeness is equivalent to parallelism of right-invariant vector fields, proved in §\textsection ??.

Chapter 12 expounds the notions of “hyperbolicity” of Vey [146] and Kobayashi [85]. Hyperbolic affine manifolds are quotients of properly convex cones. Compact such manifolds are radiant suspensions of \( \mathbb{R}P^n \)-manifolds which are quotients of divisible domains. In particular we describe how a completely incomplete closed affine manifold must be affine hyperbolic in this sense. (That is, we tame the developing map of an affine structure with no two-ended complete geodesics.) This striking result is similar to the tameness where all geodesics are complete — complete manifolds are also quotients.

Chapter 12 summarizes the now blossoming subject of \( \mathbb{R}P^2 \)-structures on surfaces, in terms of the explicit coordinates and deformations which extend some of the classic geometric constructions on the deformation space of hyperbolic structures on closed surfaces. The thirteenth chapter surveys the classical (yet still extremely active) subject of \( \mathbb{C}P^1 \)-structures on surfaces, and extensions to higher dimensions. In one direction, \( \mathbb{C}P^1 \)-structures relate to Kleinian groups and hyperbolic 3-manifolds, and in another direction to Riemann surfaces. Playing off these relationships against each other provides powerful tools for their study. We briefly describe this theory and how it generalizes to flat conformal structures in higher dimensions.

Chapter 13 describes the classic subject of \( \mathbb{C}P^1 \)-manifolds, which traditionally identify with projective structures on Riemann surfaces. Using the Schwarzian derivative, these structures are classified by the points of a holomorphic affine bundle over the Teichmüller space of \( \Sigma \). This parametrization (presumably known to Poincaré), is remarkable in that is completely formal, using standard facts from the theory of Riemann surfaces. One knows precisely the deformation space without any knowledge of the developing map (besides it being a local biholomorphism). This is notable because the developing maps can be pathological; indeed the first examples of pathological developing maps were \( \mathbb{C}P^1 \)-manifolds on hyperbolic surfaces.

Chapter 14 surveys known results, and the many open questions, in dimension three. This complements Thurston’s book [141] and expository articles of Scott [126] and Bonahon [21], which deal with geometrization and the relations to 3-manifold topology.

**Prerequisites**

This book is aimed roughly at first-year graduate students, although some knowledge of advanced material will be useful. In particular, we
assume basic familiarity with elementary topology, smooth manifolds, and the rudiments of Lie groups and Lie algebras. Much of this can be found in Lee’s book “Introduction to Smooth Manifolds” [103], including its appendices. For topology, we require basic familiarity with the notion of metric spaces, covering spaces and fundamental groups.

Fiber bundles, as discussed in the still excellent treatise of Steenrod [135], or the more modern treatment of principal bundles is Sontz [134], will be used.

Some familiarity with the properties of proper maps and proper group actions will also be useful.

Some familiarity with the theory of connections in fiber bundles and vector bundles is useful, for example, Kobayashi-Nomizu [86], or Milnor [111], do Carmo [40] Lee [102], O’Neill [119].

Acknowledgements

This book grew out of lecture notes Projective Geometry on Mani-
folds, from a course at the University of Maryland in 1988. Since then I have given minicourses at international conferences, graduate courses at the University of Maryland and Brown University in Fall 2017, which have been extremely useful in developing the material here. ICTP, Lisbon lectures in geometry, minicourse at Austin.

I am grateful for notes written by Son Lam Ho and Greg Laun in later iterations of this course at Maryland.

Comments and suggestions by Suhyoung Choi, Jeff Danciger, Ludovic Marquis, Fanny Kassel, Stephan Tillmann, and various anonymous referees have been extremely helpful.

I am extremely grateful to Ed Dunne, Ina Mettes, and Eriko Hironaka for many informative suggestions and comments.
Part 1

Affine and projective geometry
CHAPTER 1

Affine space

This section introduces the geometry of affine spaces. After a rigorous definition of affine spaces and affine maps, we discuss how linear algebraic constructions define geometric structures on affine spaces. Affine geometry is then transplanted to manifolds. The section concludes with a discussion of affine subspaces, affine volume and the notion of center of gravity.

1.1. Affine spaces

We begin with a short summary of Euclidean geometry in terms of its underlying space and its group of isometries. From that we develop the less familiar subject of affine geometry as Euclidean geometry where certain notions are removed. The particular context is crucial here, and context is emphasized using notation. Thus \( E^n \) denotes Euclidean \( n \)-space, \( \mathbb{R}^n \) denotes the standard \( n \)-dimensional vector space (consisting of ordered \( n \)-tuples of real numbers), and \( A^n \) denotes affine \( n \)-space. All three of these objects agree as sets but are used in different ways. The vector space \( \mathbb{R}^n \) appears as the group of translations, whereas \( E^n \) and \( A^n \) are sets of points with a geometric structure. The geometric structure of \( E^n \) involves metric quantities and relations such as distance, angle, area and volume. The geometric structure of \( A^n \) is based around the notion of parallelism.

1.1.1. Euclidean geometry and its isometries. Euclidean geometry is the familiar geometry of \( \mathbb{R}^n \) involving points, lines, distance, line segments, angles, and flat subspaces. However, \( \mathbb{R}^n \) is a vector space, and is thus a group under vector addition. Therefore, like any group, it has a has a distinguished basepoint, the identity element, which is the zero vector, denoted \( 0_n \in \mathbb{R}^n \). Euclidean geometry has no distinguished point: all points in Euclidean geometry are to be treated equally. To distinguish the space of points in Euclidean space, we shall denote by \( E^n \) the set of all vectors \( \mathbf{v} \in \mathbb{R}^n \), but with the structure of Euclidean geometry, reserving the notation \( \mathbb{R}^n \) for the vector space of all \( n \)-tuples \( \mathbf{v} = (v^1, \ldots, v^n) \) of real numbers \( v^1, \ldots, v^n \in \mathbb{R} \). The point
corresponding to the zero vector in \( \mathbb{R}^n \) is the *origin* but it plays no special role in the context of Euclidean geometry.

*Translations* remove the special role of the origin. Translations are transformations which preserve Euclidean geometry, (in other words, they are *Euclidean isometries*). They are simply defined by vector addition: if \( \mathbf{v} \in \mathbb{R}^n \) is a vector, and \( p \in \mathbb{E}^n \) is a point in Euclidean space, then *translation by* \( \mathbf{v} \) is the transformation

\[
\mathbb{E}^n \xrightarrow{\tau_{\mathbf{v}}} \mathbb{E}^n \\
x \mapsto x + \mathbf{v}
\]

where \( x \in \mathbb{E}^n \) corresponds to the vector \( \mathbf{x} \), that is, has coordinates \((x^1, \ldots, x^n)\). More accurately, the map

\[
\mathbb{R}^n \longrightarrow \mathbb{E}^n \\
\mathbf{v} \mapsto \tau_{\mathbf{v}}(0)
\]

identifies Euclidean space \( \mathbb{E}^n \) with the vector space \( \mathbb{R}^n \).

Translations enjoy the key property that given two points \( p, q \in \mathbb{E}^n \), there is a unique translation \( \tau_{\mathbf{v}} \), represented by the vector difference

\[
\mathbf{v} := q - p \in \mathbb{R}^n
\]

taking \( p \) to \( q \).

Euclidean geometry possesses many other geometric structures not present in a vector space: distance, orthogonality, angle, area and volume are geometric quantities which are not preserved under Euclidean isometries. These can all be defined by the Euclidean inner product

\[
\mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R} \\
(\mathbf{x}, \mathbf{y}) \mapsto \mathbf{x} \cdot \mathbf{y} := x^1 y^1 + \cdots + x^n y^n.
\]

A linear map \( \mathbb{R}^n \rightarrow \mathbb{R}^n \) preserving this inner product we call a *linear isometry* and is represented in the standard basis by an *orthogonal matrix*. Denote the group of orthogonal matrices by \( \text{O}(n) \). Then any Euclidean isometry \( \mathbb{E}^n \overset{g}{\rightarrow} \mathbb{E}^n \) decomposes as the composition of a translation and a linear isometry as follows. Let \( \tau_{\mathbf{v}} \) be the translation taking the origin \( O \) to \( g(O) \); the \( \mathbf{v} \) is just the vector having the same coordinates as \( g(O) \). Its inverse is the translation corresponding to \(-\mathbf{v}\):

\[
(\tau_{\mathbf{v}})^{-1} = \tau_{-\mathbf{v}}.
\]

Then the composition \( \tau_{-\mathbf{v}} \circ g \) fixes the origin \( O \), and preserves both the Euclidean geometry and \( O \); therefore it is a linear isometry, and represented by an orthogonal matrix \( A \in \text{O}(n) \). Thus a Euclidean
isometry is represented by

\[(1) \quad x \xrightarrow{g} Ax + v\]

where \(A \in O(n)\) is the linear part (denoted \(L(g)\)) and \(v\) is the translational part, the vector corresponding to the unique translation taking \(O\) to \(g(O)\).

In this way, a transformation of \(E^n\) which preserves Euclidean geometry is given by (1). A more general transformation defined by (1) where \(L(g) = A\) is only required to be a linear transformation of \(\mathbb{R}^n\) is called an affine transformation. An affine transformation, therefore, is the composition of a linear map with a translation.

### 1.1.2. Affine spaces.

What geometric properties of \(E^n\) do not involve the metric notions of distance, angle, area and volume? For example, the notion of straight line is invariant under translations, linear maps, and more general affine transformations. What more fundamental geometric property is preserved by affine transformations?

**Definition 1.1.1.** Two subsets \(A, B \subset E^n\) are parallel if and only if there exists a translation \(\tau_v\), where \(v \in \mathbb{R}^n\) such that

\[\tau_v(A) = B.\]

We write \(A \parallel B\). A transformation \(g\) is affine if and only if whenever \(A \parallel B\), then \(g(A) \parallel g(B)\).

**Exercise 1.1.2.** Show that \(g\) is affine if and only if it is given by the formula (1), where the linear part \(A = L(g)\) is only required to lie in \(GL(n, \mathbb{R})\).

Here is a more formal definition of an affine space. Although it is less intuitive, it embodies the idea that affine geometry is the geometry of parallelism. First we begin with important definitions about group actions.

**Definition 1.1.3.** Suppose that \(G\) is a topological group which acts on a topological space \(X\). The action is free if and only if whenever \(g \in G\) fixes \(x \in X\) (that is, \(g(x) = x\)), then \(g = 1_G\), the identity element of \(G\). The action is transitive if and only if for some \(x \in X\), the orbit \(G(x) = X\). The action is simply transitive if and only if it is transitive and free.

Equivalently, \(G\) acts simply transitively on \(X\) if for some (and then necessarily every) \(x \in X\), the evaluation map

\[G \longrightarrow X\]

\[g \longmapsto g \cdot x\]
1. AFFINE SPACE

is bijective: that is, for all \( x, y \in X \), a unique \( g \in G \) takes \( x \) to \( y \).

**Definition 1.1.4.** Let \( G \) be a group. A \( G \)-torsor is a space \( X \) with a simply transitive \( G \)-action.

Now we give the formal definition of a an affine space:

**Definition 1.1.5.** An affine space is a \( V \)-torsor \( A \), where \( V \) is a vector space. We call \( V \) the vector space underlying \( A \), and denote it by \( \tau_A \), the elements of which are the translations of \( A \).

1.1.3. Affine transformations. Affine maps are maps between affine spaces which are compatible with these simply transitive actions of vector spaces. Suppose \( A, A' \) are affine spaces. Then a map

\[
A \xrightarrow{f} A'
\]

is affine if for each \( v \in \tau_A \), there exists a translation \( v' \in \tau_{A'} \) such that

\[
\begin{array}{ccc}
A & \xrightarrow{f} & A' \\
v & \downarrow & v' \\
A & \xrightarrow{f} & A'
\end{array}
\]

commutes. Necessarily \( v' \) is unique and it is easy to see that the correspondence

\[
L(f) : v \mapsto v'
\]

defines a homomorphism

\[
\tau_A \longrightarrow \tau_{A'},
\]

that is, a linear map between the vector spaces \( \tau_A \) and \( \tau_{A'} \), called the linear part \( L(f) \) of \( f \). Denoting the space of all affine maps \( A \longrightarrow A' \) by \( \text{aff}(A, A') \) and the space of all linear maps \( \tau_A \longrightarrow \tau_{A'} \) by \( \text{Hom}(\tau_A, \tau_{A'}) \), linear part defines a map

\[
\text{aff}(A, A') \xrightarrow{L} \text{Hom}(\tau_A, \tau_{A'})
\]

The set of affine endomorphisms of an affine space \( A \) will be denoted by \( \text{aff}(A) \) and the group of affine automorphisms of \( A \) will be denoted \( \text{Aff}(A) \).

The space \( \text{aff}(A, A') \) has the natural structure of an affine space. Namely the vector space

\[
\text{Hom}(\tau_A, \tau_{A'}) \oplus \tau_{A'}
\]

acts simply transitively on \( \text{aff}(A, A') \). Furthermore the group \( \text{Aff}(A) \times \text{Aff}(A') \) acts by composition on \( \text{aff}(A, A') \), preserving the affine structure.
1.1. AFFINE SPACES

\textbf{Aff}(A) is a Lie group and its Lie algebra identifies with \textbf{aff}(A). Furthermore \textbf{Aff}(A) is isomorphic to the semidirect product}

\[ \textbf{Aut}(\tau_A) \cdot \tau_A \]

\textit{where} $\tau_A$ the normal subgroup consisting of translations and 

\[ \textbf{Aut}(\tau_A) = \text{GL}(A) \]

is the group of linear automorphisms of the vector space $\tau_A$. 

The kernel of \[ \textbf{aff}(A, A') \xrightarrow{L} \text{Hom}(\tau_A, \tau_{A'}) \] (that is, the inverse image of 0) is the vector space $\tau_{A'}$ of translations of $A'$. Choosing an origin $x \in A$, we write, for \[ f \in \textbf{aff}(A, A'), \]

\[ f(y) = f(x + (y - x)) = (Lf)(y - x) + t \]

Since every affine map \[ f \in \textbf{aff}(A, A') \] may be written as 

\[ f(x) = L(f)(x) + f(0), \]

where $f(0) \in A'$ is the \textit{translational part} of $f$. (Strictly speaking one should say the translational part of $f$ with respect to 0, that is, the translation taking 0 to $f(0)$.)

\textit{Affine geometry} is the study of affine spaces and affine maps between them. If $U \subset A$ is an open subset, then a map $U \xrightarrow{f} A'$ is \textit{locally affine} if for each connected component $U_i$ of $U$, there exists an affine map $f_i \in \textbf{aff}(A, A')$ such that the restrictions of $f$ and $f_i$ to $U_i$ are identical. Note that two affine maps which agree on a nonempty open set are identical.

\textbf{1.1.4. Tangent spaces.} We wish to study smooth paths in an affine space $A$. Let $\gamma(t)$ denote a \textit{smooth curve} in $A$; that is, in coordinates

\[ \gamma(t) = (x^1(t), \ldots, x^n(t)) \]

where $x^j(t)$ are smooth functions of the time parameter, which ranges in an interval $[t_0, t_1] \subset \mathbb{R}$. The vector $\gamma(t) - \gamma(t_0)$ corresponds to the unique translation taking $\gamma(t_0)$ to $\gamma(t)$, and lies in the vector space $V$ underlying $A$. It represents the displacement of the curve $\gamma$ as it goes from $t = t_0$ to $t$. Its \textit{velocity vector} $\gamma'(t) \in V$ is defined as the derivative of this path in the vector space $V$ of translations. It represents the \textit{infinitesimal displacement} of $\gamma(t)$ as $t$ varies.

We consider more general \textit{smooth manifolds}, built from open subsets (\textit{coordinate patches}) of $A$ under coordinate changes which are general smooth locally invertible maps. Thus we want to distinguish the infinitesimal displacements at \textit{different} points $p \in A$. Therefore we attach to each point $p \in A$, a “copy” of $V$, represents the vector space of \textit{infinitesimal displacements} of $p$. 

Such an object is a tangent vector at $p$, which can be defined in many equivalent ways:

- Equivalence classes of smooth curves $\gamma(t)$ with $\gamma(0) = p$, where $\gamma_1 \sim \gamma_2$ if and only if
  \[
  \left. \frac{d}{dt} \right|_{t=0} f \circ \gamma_1(t) = \left. \frac{d}{dt} \right|_{t=0} f \circ \gamma_2(t)
  \]
  for all smooth functions $U \xrightarrow{f} \mathbb{R}$, where $U \subset A$ is an open neighborhood of $p$.
- Linear operators $C^\infty(A) \xrightarrow{D} \mathbb{R}$ satisfying $D(fg) = D(f)g(p) + f(p)D(g)$.

The set of tangent vectors is a vector space, denoted $T_pA$, and naturally identifies with $V$ as follows. If $v \in V$ is a vector, then the path $\gamma_{(p,v)}(t)$ defined by:

\[
(2) \quad t \mapsto p + tv = \tau_{tv}(p)
\]

is a smooth path with $\gamma(0) = p$ and velocity vector $\gamma'(0) = v$. Conversely, definition above of an infinitesimal displacement, shows that every smooth path through $p = \gamma(0)$ with velocity $\gamma'(0) = v$ is tangent to the curve (2) as above.

The tangent spaces $T_pA$ linearize $A$ as follows. A mapping

\[
A \xrightarrow{f} A'
\]

is differentiable at $p$ if every infinitesimal displacement $v \in T_pA$ maps to an infinitesimal displacement $D_p f(v) \in T_qA'$, where $q = f(p)$. That is, if $\gamma$ is a smooth curve with $\gamma(0) = q$ and $\gamma'(0) = v$, then we require that $f \circ \gamma$ is a smooth curve through $q$ at $t = 0$; then we call the new velocity $(f \circ \gamma)'(0)$ the value of the derivative

\[
T_pA \xrightarrow{D_p f} T_qA'
\]

\[
v \mapsto (f \circ \gamma)'(0)
\]

Exercise 1.1.6. Prove the Chain Rule: If $\Omega \subset A$ and $\Omega' \subset A'$ are open subsets of affine spaces $A, A'$ respectively and and $\Omega \xrightarrow{f} A' \xrightarrow{g} A''$ are smooth maps, then

\[
(D(g \circ f))_p = (Dg)_{f(p)} \circ (Df)_p
\]

for all $p \in \Omega$. 
1.1.5. **Acceleration and geodesics.** The velocity vector field \( \gamma'(t) \) of a smooth curve \( \gamma(t) \) is an example of a *vector field along the curve* \( \gamma(t) \): For each \( t \), the tangent vector \( \gamma'(t) \in T_{\gamma(t)}A \). Differentiating the velocity vector field, then raises a significant difficulty: since the values of the vector field live in different vector spaces, we need a way to compare, or to *connect* them. The natural way is use the simply transitive action of the group \( V \) of translations of \( A \). That is, suppose that \( \gamma(t) \) is a smooth path, and \( \xi(t) \) is a vector field along \( \gamma(t) \). Let \( \tau^t_s \) denote the translation taking \( \gamma(t+s) \) to \( \gamma(t) \), that is, in coordinates:

\[
\begin{align*}
A & \xrightarrow{\tau^t_s} A \\
p & \mapsto p + (\gamma(t+s) - \gamma(t))
\end{align*}
\]

Its differential

\[
T_{\gamma(t+s)}A \xrightarrow{D\tau^t_s} T_{\gamma(t)}A
\]

then maps \( \xi(t+s) \) into \( T_{\gamma(t)}A \) and the *covariant derivative* \( \frac{D}{dt} \xi(t) \) is the derivative of this smooth path in the *fixed* vector space \( T_{\gamma(t)}A \):

\[
\frac{D}{dt} \xi(t) := \frac{d}{ds} \bigg|_{s=0} (D\tau^t_s)(\xi(t+s))
\]

\[
= \lim_{s \to 0} \frac{(D\tau^t_s)(\xi(t+s)) - \xi(t)}{s}
\]

In this way, define the *acceleration* as the covariant derivative of the velocity:

\[
\gamma''(t) := \frac{D}{dt} \gamma'(t)
\]

A curve with zero acceleration is called a *geodesic*.

**Exercise 1.1.7.** Given a point \( p \) and a tangent vector \( v \in T_pA \), show that the unique geodesic \( \gamma(t) \) with

\[
(\gamma(0), \gamma'(0)) = (p, v)
\]

is given by (2).

In other words, geodesics in \( A \) are parametrized curves which are Euclidean straight lines traveling at *constant* speed. However, in affine geometry the *speed* itself is not defined, but “motion along a straight line at constant speed” is affinely invariant (zero acceleration).

This leads to the following important definition:
1. AFFINE SPACE

**Definition 1.1.8.** Let $p \in A$ and $v \in T_p(A) \cong V$. Then the exponential mapping is defined by:

$$T_pA \xrightarrow{\exp_p} A$$

$$v \mapsto p + v.$$ 

Thus the unique geodesic with initial position and velocity $(p, v)$ equals

$$t \mapsto \exp_p(tv) = p + tv.$$ 

1.2. Parallel structures

The most familiar geometry is Euclidean geometry, extremely rich with metric notions such as distance, angle, area and volume. We have seen that affine geometry underlies it with the more primitive notion of parallelism. Now we shall recover Euclidean geometry from affine geometry by introducing a Riemannian structure on $A$, namely a parallel Riemannian metric.

**1.2.1. Parallel Riemannian structures.** Let $B$ be an inner product on $V$ and $O(V; B) \subset \text{GL}(A)$ the corresponding orthogonal group. Then $B$ defines a flat Riemannian metric on $A$ and the inverse image

$$L^{-1}(O(V; B)) \cong O(V; B) \cdot \tau_A$$

is the full group of isometries, that is, the Euclidean group. If $B$ is a non-degenerate indefinite form, then there is a corresponding flat pseudo-Riemannian metric on $A$ and the inverse image $L^{-1}(O(V; B))$ is the full group of isometries of this pseudo-Riemannian metric.

**Exercise 1.2.1.** Show that an affine automorphism $g$ of Euclidean $n$-space $\mathbb{R}^n$ is conformal (that is, preserves angles) if and only if its linear part is the composition of an orthogonal transformation and scalar multiplication.

Such a transformation will be called a similarity transformation and the group of similarity transformations will be denoted $\text{Sim}(E^n)$. The scalar multiple is called the scale factor $\lambda(g) \in \mathbb{R}^+$ and defines a homomorphism $\text{Sim}(E^n) \xrightarrow{\lambda} \mathbb{R}^+$. In general, if $g \in \text{Sim}(E^n)$, then $\exists A \in O(n), b \in \mathbb{R}^n$ such that

$$g(x) = \lambda(g)Ax + b.$$
1.2.2. Parallel tensor fields. The parallel vector fields introduced in §1.3.1 are similar. Namely, any tangent vector $v \in T_p A$ extends uniquely to a vector field on $A$ invariant under the group of translations. As we saw in §1.2.1, Euclidean structures are defined by extending an inner product from a single tangent space to all of $E$.

Explain parallel volume forms, parallel 1-forms.

1.2.3. Complex affine geometry. We have been working entirely over $\mathbb{R}$, but it is clear one may study affine geometry over any field. If $k \supset \mathbb{R}$ is a field extension, then every $k$-vector space is a vector space over $\mathbb{R}$ and thus every $k$-affine space is an $\mathbb{R}$-affine space. In this way we obtain more refined geometric structures on affine spaces by considering affine maps whose linear parts are linear over $k$.

Exercise 1.2.2. Show that 1-dimensional complex affine geometry is the same as (orientation-preserving) 2-dimensional similarity geometry.

This structure is another case of a paralle structure on an affine space, as follows. Recall a complex vector space has an underlying structure as a real vector space $V$. The difference is a notion of scalar multiplication by $\sqrt{-1}$, which is given by a linear map

$$V \xrightarrow{J} V$$

such that $J \circ J = -\mathbb{I}$. Such an automorphism is called a complex structure on $V$, and “turns $V$ into” a complex vector space.

If $M$ is a manifold, an endomorphism field (that is, a $(1,1)$-tensor field) $J$ where, for each $p \in M$, the value $J_p$ is a complex structure on the tangent space $T_p M$ is called an almost complex structure. Necessarily $\dim(M)$ is even.

Recall that a complex manifold is a manifold with an atlas of coordinate charts where coordinate changes are biholomorphic. (Such an atlas is called a holomorphic atlas. Every complex manifold admits an almost complex structure, but not every almost complex structure arises from a holomorphic atlas, except in dimension two.

Exercise 1.2.3. Prove that a complex affine space is the same as a affine space with a parallel almost complex structure.

1.3. Affine vector fields

A vector field $X$ on $A$ is said to be affine if it generates a one-parameter group of affine transformations. $X$ is parallel if generates a one-parameter group of translations, and radiant if it generates a one-parameter group of homotheties.
We obtain equivalent criteria for these conditions in terms of the covariant differential operation
\[ \mathcal{T}^p(M; TM) \xrightarrow{\nabla} \mathcal{T}^{p+1}(M; TM) \]
where \( \mathcal{T}^p(M; TM) \) denotes the space of \( TM \)-valued covariant \( p \)-tensor fields on \( M \), that is, the tensor fields of type \((1, p)\). Thus \( \mathcal{T}^0(M; TM) = \text{Vec}(M) \), the space of vector fields on \( M \), and
\[ \mathcal{T}^1(M; TM) = \text{Vec}(M), \]
the space of \( T^*M \)-valued vector fields on \( M \). Alternatively elements of \( \mathcal{T}^1(M; TM) = \text{Vec}(M) \) identify with \( TM \)-valued 1-forms, or equivalently endomorphism fields on \( M \).

**Exercise 1.3.1.** \( X \) is affine if and only if it satisfies any of the following equivalent conditions:
- For all \( Y, Z \in \text{Vec}(A) \),
  \[ \nabla_Y \nabla_Z X = \nabla_{(\nabla_Y Z)} X. \]
- \( \nabla \nabla X = 0 \).
- The coefficients of \( X \) are affine functions, that is,
  \[ X = \sum_{i,j=1}^{n} (a^i_j x^j + b^i) \frac{\partial}{\partial x_i} \]
  for constants \( a^i_j, b^i \in \mathbb{R} \).

Write
\[ L(X) = \sum_{i,j=1}^{n} a^i_j x^j \frac{\partial}{\partial x_i} \]
for the linear part (which corresponds to the matrix \( (a^i_j) \in \mathfrak{gl}(\mathbb{R}^n) \)) and
\[ X(0) = \sum_{i=1}^{n} b^i \frac{\partial}{\partial x_i} \]
for the translational part (the translational part of an affine vector field is a parallel vector field). The Lie bracket of two affine vector fields is given by:
- \( L([X, Y]) = [L(X), L(Y)] = L(X)L(Y) - L(X)L(Y) \) (matrix multiplication)
- \([X, Y](0) = L(X)Y(0) - L(Y)X(0)\).
In this way the space \( \text{aff}(A) = \text{aff}(A, A) \) of affine endomorphisms \( A \to A \) is a Lie algebra.

### 1.3.1. Parallel vector fields.

**Definition 1.3.2.** A vector field \( X \) on \( A \) is parallel if, for every \( p, q \in A \), the values \( X_p \in T_pA \) and \( X_q \in T_qA \) are parallel.

Since translation \( \tau \) by \( v = q - p \) is the unique translation taking \( p \) to \( q \), this simply means that the differential \( D\tau \) maps \( X_p \) to \( X_q \).

**Exercise 1.3.3.** A vector field is parallel if and only if its coefficients (in affine coordinates) are constant.

- \( X \) is parallel if and only if \( \nabla_Y X = 0 \) for all \( Y \in \text{Vec}(A) \) if and only if \( \nabla X = 0 \).
- An affine vector field is parallel if and only if its linear part is zero.

The vector space \( \tau_A \) identifies with the parallel vector fields on \( A \). The parallel vector fields form an abelian Lie algebra of vector fields on \( A \).

### 1.3.2. Homotheties and radiant vector fields.

Another important class of affine vector fields are the radiant vector fields, or infinitesimal homotheties:

**Definition 1.3.4.** An affine transformation \( \phi \in \text{aff}(A) \) is a homothety if it is conjugate by a translation to scalar multiplication \( v \mapsto \lambda v \), for some scalar \( \lambda \in \mathbb{R}^* \). An affine vector field is radiant if it generates a one-parameter group of homotheties.

**Exercise 1.3.5.** \( X \) is radiant if and only if \( \nabla X = \mathbb{I}_A \) (where \( \mathbb{I}_A \in T^1(A; TA) \) is the identity map \( TA \to TA \), regarded as an endomorphism field on \( A \)) if and only if there exists \( b^i \in \mathbb{R} \) for \( i = 1, \ldots, n \) such that

\[
X = \sum_{i=1}^{n} (x^i - b^i) \frac{\partial}{\partial x_i}.
\]

Note that \( b = (b^1, \ldots, b^n) \) is the unique zero of \( X \) and that \( X \) generates the one-parameter group of homotheties fixing \( b \). (Thus a radiant vector field is a special kind of affine vector field.) Furthermore \( X \) generates the center of the isotropy group of \( \text{Aff}(A) \) at \( b \), which is conjugate (by translation by \( b \)) to \( \text{GL}(A) \). Show that the radiant vector fields on \( A \) form an affine space isomorphic to \( A \).
1.3.3. Affine subspaces. Suppose that \( A_1 \hookrightarrow A \) is an injective affine map; then we say that \( \iota(A_1) \) (or with slight abuse, \( \iota \) itself) is an affine subspace. If \( A_1 \) is an affine subspace then for each \( x \in A_1 \) there exists a linear subspace \( V_1 \subset \tau_A \) such that \( A_1 \) is the orbit of \( x \) under \( V_1 \) (that is, “an affine subspace in a vector space is just a coset (or translate) of a linear subspace \( A_1 = x + V_1 \).”) An affine subspace of dimension 0 is thus a point and an affine subspace of dimension 1 is a line.

Exercise 1.3.6. Show that if \( l, l' \) are (affine) lines and 
\[
(x, y) \in l \times l, \ x \neq y \\
(x', y') \in l' \times l', \ x' \neq y'
\]
are pairs of distinct points. Then there is a unique affine map \( l \xrightarrow{f} l' \) such that 
\[
f(x) = x', \\
f(y) = y'.
\]
If \( x, y, z \in l \) (with \( x \neq y \)), then define \([x, y, z] \) to be the image of \( z \) under the unique affine map \( l \xrightarrow{f} \mathbb{R} \) with \( f(x) = 0 \) and \( f(y) = 1 \). Show that if \( l = \mathbb{R} \), then \([x, y, z] \) is given by the formula
\[
[x, y, z] = \frac{z - x}{y - x}.
\]
This is called an affine parameter along the line.

1.4. Volume in affine geometry

Although an affine automorphism of an affine space \( A \) need not preserve a natural measure on \( A \), Euclidean volume nonetheless does behave rather well with respect to affine maps. The Euclidean volume form \( \omega \) can almost be characterized affinely by its parallelism: it is invariant under all translations. Moreover two \( \tau_A \)-invariant volume forms differ by a scalar multiple but there is no natural way to normalize. Such a volume form will be called a parallel volume form. If \( g \in \text{Aff}(A) \), then the distortion of volume is given by
\[
g^*\omega = \det \mathsf{L}(g) \cdot \omega.
\]
Thus although there is no canonically normalized volume or measure there is a natural affinely invariant line of measures on an affine space. The subgroup \( \text{SAff}(A) \) of \( \text{Aff}(A) \) consisting of volume-preserving affine transformations is the inverse image \( \mathsf{L}^{-1}(\text{SL}(V)) \), sometimes called the
special affine group of $\mathbb{A}$. Here $\text{SL}(\mathbb{V})$ denotes, as usual, the special linear group

$$\text{Ker}(\text{GL}(\mathbb{V}) \xrightarrow{\text{det}} \mathbb{R}^*) = \{ g \in \text{GL}(\mathbb{V}) \mid \text{det}(g) = 1 \}.$$  

**1.4.1. Centers of gravity.** Given a finite subset $F \subset \mathbb{A}$ of an affine space, its center of gravity or centroid $\bar{F} \in \mathbb{A}$ is point associated with $F$ in an affinely invariant way: that is, given an affine map $\mathbb{A} \xrightarrow{\phi} \mathbb{A}'$ we have

$$\phi(\bar{F}) = \bar{\phi(F)}.$$  

This operation can be generalized as follows.

**Theorem 1.4.1.** Let $\mu$ be a probability measure on an affine space $\mathbb{A}$. Then there exists a unique point $\bar{x} \in \mathbb{A}$ (the centroid of $\mu$) such that for all affine maps $\mathbb{A} \xrightarrow{f} \mathbb{R}$,

$$(3) \quad f(x) = \int_{\mathbb{A}} f \, d\mu$$  

**Proof.** Let $(x^1, \ldots, x^n)$ be an affine coordinate system on $\mathbb{A}$. Let $\bar{x} \in \mathbb{A}$ be the points with coordinates $(\bar{x}^1, \ldots, \bar{x}^n)$ given by

$$\bar{x}^i = \int_{\mathbb{A}} x^i \, d\mu.$$  

This uniquely determines $\bar{x} \in \mathbb{A}$; we must show that (3) is satisfied for all affine functions. Suppose $\mathbb{A} \xrightarrow{f} \mathbb{R}$ is an affine function. Then there exist $a_1, \ldots, a_n, b$ such that

$$f = a_1 x^1 + \cdots + a_n x^n + b$$  

and thus

$$f(\bar{x}) = a_1 \int_{\mathbb{A}} x^1 \, d\mu + \cdots + a_n \int_{\mathbb{A}} x^n \, d\mu + b \int_{\mathbb{A}} d\mu = \int_{\mathbb{A}} f \, d\mu$$  

as claimed. $\square$

Now let $C \subset \mathbb{A}$ be a convex body, that is, a convex open subset having compact closure. Then $C$ determines a probability measure $\mu_C$ on $\mathbb{A}$ by

$$\mu_C(X) = \frac{\int_{X \cap C} \omega}{\int_{C} \omega}$$  

where $\omega$ is any parallel volume form on $\mathbb{A}$. 


Proposition 1.4.2. Let $C \subset A$ be a convex body. Then the centroid $\bar{C}$ of $C$ lies in $C$.

**Proof.** $C$ is the intersection of halfspaces, that is, $C$ consists of all $x \in A$ such that $f(x) > 0$ for all affine maps $A \xrightarrow{f} \mathbb{R}$ such that $f|_C > 0$. If $f$ is such an affine map, then clearly $f(C) > 0$. Therefore $\bar{C} \in C$. □

1.5. Linearizing affine geometry

Associated to every affine space $A$ is an embedding of $A$ as an *affine hyperplane* in a vector space $V'(A)$. Recall that every vector space $U$ may be considered as an affine space by “forgetting the origin.” Just as a linear hyperplane in $U$ is the kernel $\text{Ker}(\psi) = \psi^{-1}(0)$ of a linear functional $U \xrightarrow{\psi} \mathbb{R}$, an affine hyperplane in $U$ is the preimage $A_\psi := \psi^{-1}(1) \subset U$.

**Exercise 1.5.1.** Show that the $\text{Ker}(\psi)$ identifies with the vector space $\tau_{A_\psi}$ underlying $A_\psi$.

Now let $A$ be an affine space and consider its underlying vector space $\tau_A$. Let $V'(A)$ denote the direct sum $\tau_A \oplus \mathbb{R}$. Choose an arbitrary point $p_0 \in A$ to serve as an origin. Then the map $A \rightarrow V'(A)$

$$p \mapsto \tau_{p_0,p} \oplus 1$$

embeds $A$ an affine hyperplane in $V'(A)$. Furthermore the affine group $\text{Aff}(A)$ identifies with the subgroup of $\text{GL}(V'(A))$ which preserves this hyperplane. Equivalently this subgroup is the stabilizer in $\text{GL}(V'(A))$ of the linear functional

$$V'(A) \xrightarrow{\psi} \mathbb{R}$$

$$p \oplus r \mapsto r$$

If $\left[ A \mid a \right]$ represents the affine transformation as in (42), the corresponding linear transformation of $V'(A)$ is represented by the block matrix

$$
\begin{bmatrix}
A & a \\
0 & 1
\end{bmatrix}
$$

where 0 is the row vector representing the zero map $V \rightarrow \mathbb{R}$. 
CHAPTER 2

Projective space

Projective geometry may be constructed as a way of “closing off” (that is, compactifying) affine geometry. To develop an intuitive feel for projective geometry, consider how points in \( \mathbb{A}^n \) may “go to infinity.”

Naturally it takes the least work to move to infinity along straight lines moving at constant speed (zero acceleration) and two such geodesic paths go to the “same point at infinity” if they are parallel. Imagine two railroad tracks running parallel to each other; they meet at “infinity.” We will thus force parallel lines to intersect by attaching to affine space a space of “points at infinity,” where parallel lines intersect.

2.1. Ideal points

Let \( A \) be an affine space; then the relation of two lines in \( A \) being parallel is an equivalence relation. We define an ideal point of \( A \) to be a parallelism class of lines in \( A \). The ideal set of an affine space \( A \) is the space \( \mathbb{P}_\infty(A) \) of ideal points, with the quotient topology. If \( l, l' \subset A \) are parallel lines, then the point in \( \mathbb{P}_\infty \) corresponding to their parallelism class is defined to be their intersection. So two lines are parallel if and only if they intersect at infinity.

Projective space is defined to be the union \( \mathbb{P}(A) = A \cup \mathbb{P}_\infty(A) \). The natural structure on \( \mathbb{P}(A) \) is perhaps most easily seen in terms of an alternate, maybe more familiar description. We may embed \( A \) as an affine hyperplane in a vector space \( V \) as follows. Let \( V = \tau_A \oplus \mathbb{R} \) and choose an origin \( x_0 \in A \); then the map \( A \to V \) which assigns to \( x \in A \) the pair \( (x - x_0, 1) \) embeds \( A \) as an affine hyperplane in \( V \) which misses 0. Let \( \mathbb{P}(V) \) denote the space of all lines through 0 \( \in V \) with the quotient topology. The composition

\[
A \xrightarrow{i} V - \{0\} \to \mathbb{P}(V)
\]

embeds \( A \) as an open dense subset of \( \mathbb{P}(V) \). Now the complement \( \mathbb{P}(V) - i(A) \) consists of all lines through the origin in \( \tau_A \oplus \{0\} \) and naturally bijectively corresponds with \( \mathbb{P}_\infty(A) \): given a line \( l \) in \( A \), the 2-plane \( \text{span}(l) \) it spans meets \( \tau_A \oplus \{0\} \) in a line corresponding to a
point in \( P_{\infty}(A) \). Conversely lines \( l_1, l_2 \) in \( A \) are parallel if
\[
\text{span}(l_1) \cap (\tau_A \oplus \{0\}) = \text{span}(l_2) \cap (\tau_A \oplus \{0\}).
\]
In this way we topologize projective space \( P(A) = A \cup P_{\infty}(A) \) in a natural way.

Projective geometry arose historically out of the efforts of Renaissance artists to understand perspective. Imagine a one-eyed painter looking at a 2-dimensional canvas (the affine plane \( A \)), his eye being the origin in the 3-dimensional vector space \( A \). As the painter moves around or tilts the canvas, the metric geometry of the canvas as he sees it changes. As the canvas is tilted, parallel lines no longer appear parallel (like railroad tracks viewed from above ground) and distance and angle are distorted. But lines stay lines and the basic relations of collinearity and concurrence are unchanged. The change in perspective given by “tilting” the canvas or the painter changing position is determined by a linear transformation of \( V \), since a point on \( A \) is determined completely by the 1-dimensional linear subspace of \( V \) containing it. (One must solve systems of linear equations to write down the effect of such transformation.) Projective geometry is the study of points, lines and the incidence relations between them.

2.1.1. Affine patches. The inclusion of affine geometry in projective geometry arises by considering the affine subspace \( A \) embedded as a hyperplane \( A \subset V \) as in §1.5. If \( \tau_A \) denotes the vector space of translations of \( A \), then \( V \) identifies with the direct sum \( \tau_V \oplus \mathbb{R} \) and \( A \) with the hyperplane
\[
\tau_V \oplus \{1\} = \{(v \oplus 1) \mid v \in \tau_A\}.
\]
Such an inclusion is called an affine patch. Every affine automorphism of \( A \) extends to a projective automorphism of \( P \).

2.2. Homogeneous coordinates

A point of \( P^n \) then corresponds to a nonzero vector in \( \mathbb{R}^{n+1} \), uniquely defined up to a nonzero scalar multiple. If \( a^1, \ldots, a^{n+1} \in \mathbb{R} \) and not all of the \( a^i \) are zero, then we denote the point in \( P^n \) corresponding to the nonzero vector
\[
a = (a^1, \ldots, a^{n+1}) \in \mathbb{R}^{n+1}
\]
by \([a] = [a^1 : \cdots : a^{n+1}]\); the \( a^i \) are called the homogeneous coordinates of the corresponding point in \( P^n \). The original affine space \( A^n \) is the subset comprising points with homogeneous coordinates \([a^1, \ldots, a^n, 1]\) where \((a^1, \ldots, a^n)\) are the corresponding (affine) coordinates.
2.2. HOMOGENEOUS COORDINATES

Corresponding to the \( n+1 \) homogeneous coordinates on \( \mathbb{P}^n \) are \( n+1 \) affine patches which cover \( \mathbb{P}^n \).

**Exercise 2.2.1.** Compute the coordinate change between two affine patches in \( \mathbb{P}^n \) in affine coordinates.

In general, the topology of projective space is complicated. Since it arises from a *quotient* and not a *subset* construction, it is more sophisticated than a subset. Indeed, projective space generally does not arise as a *hypersurface* in Euclidean space.

**Exercise 2.2.2.** Let \( A = \mathbb{R}^n \) and let \( P = \mathbb{P}^n \) be the projective space obtained from \( A \) as above. Exhibit \( \mathbb{P}^n \) as a quotient of the unit sphere \( S^n \subset \mathbb{R}^{n+1} \) by the antipodal map. Thus \( \mathbb{P}^n \) is compact and for \( n > 1 \) has fundamental group of order two. Show that \( \mathbb{P}^n \) is orientable if and only if \( n \) is odd.

Thus to every projective space \( P \) there exists a vector space \( V = V(P) \) such that the points of \( P \) correspond to the lines through 0 in \( V \). Denote the quotient map by

\[
V \setminus \{0\} \xrightarrow{\Pi} P.
\]

If \( P, P' \) are projective spaces and \( U \subset P \) is an open set then a map \( U \xrightarrow{f} P' \) is *locally projective* if for each component \( U_i \subset U \) there exists a linear map

\[
V(P) \xrightarrow{\tilde{f}_i} V(P')
\]

such that the restrictions of \( f \circ \Pi \) and \( \Pi \circ \tilde{f}_i \) to \( \Pi^{-1}U_i \) agree. A *projective automorphism* or *collineation* of \( P \) is an invertible locally projective map \( P \to P \). We denote the space of locally projective maps \( U \to P' \) by \( \text{Proj}(U, P') \).

Locally projective maps (and hence also locally affine maps) satisfy the Unique Extension Property: if \( U \subset U' \subset P \) are open subsets of a projective space with \( U \) nonempty and \( U' \) connected, then any two locally projective maps \( f_1, f_2 : U' \to P' \) which agree on \( U \) must be identical. (Compare §5.1.1.)

**Exercise 2.2.3.** Show that the projective automorphisms of \( P \) form a group and that this group (which we denote \( \text{Aut}(P) \)) has the following description. If \( P \xrightarrow{f} P \) is a projective automorphism, then there exists a linear isomorphism \( V \xrightarrow{\tilde{f}} V \) inducing \( f \). Indeed there is a short exact sequence

\[
1 \xrightarrow{} \mathbb{R}^n \xrightarrow{} \text{GL}(V) \xrightarrow{} \text{Aut}(P) \xrightarrow{} 1
\]
where \( \mathbb{R}^* \to \text{GL}(V) \) is the inclusion of the group of multiplications by nonzero scalars. (Sometimes this quotient
\[
\text{GL}(V)/\mathbb{R}^* \cong \text{Aut}(\mathbb{P}^n)
\]
(the projective general linear group) is denoted by \( \text{PGL}(V) \) or \( \text{PGL}(n+1, \mathbb{R}) \).) Show that if \( n \) is even, then
\[
\text{Aut}(\mathbb{P}^n) \cong \text{SL}(n+1; \mathbb{R})
\]
and if \( n \) is odd, then \( \text{Aut}(\mathbb{P}^n) \) has two connected components, and its identity component is doubly covered by \( \text{SL}(n+1; \mathbb{R}) \).

If \( V, V' \) are vector spaces with associated projective spaces \( \mathbb{P}, \mathbb{P}' \) then a linear map \( \tilde{f} : V \to V' \) maps lines through 0 to lines through 0. But \( \tilde{f} \) only induces a map \( \mathbb{P} \to \mathbb{P}' \) if it is injective, since \( f(x) \) can only be defined if \( \tilde{f}(\tilde{x}) \neq 0 \) (where \( \tilde{x} \) is a point of \( \Pi^{-1}(x) \subset V - \{0\} \)). Suppose that \( \tilde{f} \) is a (not necessarily injective) linear map and let
\[
\cup(f) = \Pi(\text{Ker}(\tilde{f})).
\]
The resulting projective endomorphism of \( \mathbb{P} \) is defined on the complement \( \mathbb{P} - \cup(f) \). If \( \cup(f) \neq \emptyset \), the corresponding projective endomorphism is by definition a singular projective transformation of \( \mathbb{P} \). If \( f \) is singular, its image is a proper projective subspace, called the range of \( f \) and denoted \( \mathcal{R}(f) \).

A projective map \( \mathbb{P}_1 \to \mathbb{P} \) corresponds to a linear map \( V_1 \to V \) between the corresponding vector spaces (well-defined up to scalar multiplication). Since \( \iota \) is defined on all of \( \mathbb{P}_1, \tilde{\iota} \) is an injective linear map and hence corresponds to an embedding. Such an embedding (or its image) will be called a projective subspace. Projective subspaces of dimension \( k \) correspond to linear subspaces of dimension \( k+1 \). (By convention the empty set is a projective space of dimension -1.) Note that the “bad set” \( \cup(f) \) of a singular projective transformation is a projective subspace. Two projective subspaces of dimensions \( k, l \) where \( k+l \geq n \) intersect in a projective subspace of dimension at least \( k+l-n \). The rank of a projective endomorphism is defined to be the dimension of its image.

**Exercise 2.2.4.** Let \( \mathbb{P} \) be a projective space of dimension \( n \). Show that the (possibly singular) projective transformations of \( \mathbb{P} \) form themselves a projective space of dimension \( (n+1)^2 - 1 \). We denote this projective space by \( \text{End}(\mathbb{P}) \). Show that if \( f \in \text{End}(\mathbb{P}) \), then
\[
\dim N(f) + \text{rank}(f) = n - 1.
\]
Show that $f \in \text{End}(\mathbb{P})$ is nonsingular (in other words, a collineation) if and only if $\text{rank}(f) = n$, that is, $\mathbb{U}(f) = \emptyset$. Equivalently, $\mathbb{R}(f) = \mathbb{P}$.

An important kind of projective endomorphism is a projection, also called a perspectivity. Let $A^k, B^l \subset \mathbb{P}^n$ be disjoint projective subspaces whose dimensions satisfy $k + l = n - 1$. We define the projection $\Pi = \Pi_{A^k, B^l}$ onto $A^k$ from $B^l$

$$\mathbb{P}^n - B^l \xrightarrow{\Pi} A^k$$

as follows. For every $x \in \mathbb{P}^n - A^k$ there is a unique projective subspace $\text{span}(\{x\} \cup B^l)$ of dimension $l + 1$ containing $\{x\} \cup B^l$ which intersects $A^k$ in a unique point. Let $\Pi_{A^k, B^l}(x)$ be this point. (Clearly such a perspectivity is the projectivization of a linear projection $V \rightarrow V$.) It can be shown that every projective map defined on a projective subspace can be obtained as the composition of projections.

Exercise 2.2.5. Suppose that $n$ is even. Show that a collineation of $\mathbb{P}^n$ which has order two fixes a unique pair of disjoint projective subspaces $A^k, B^l \subset \mathbb{P}^n$ where $k + l = n - 1$. Conversely suppose that $A^k, B^l \subset \mathbb{P}^n$ where $k + l = n - 1$ are disjoint projective subspaces; then there is a unique collineation of order two whose set of fixed points is $A^k \cup B^l$. If $n$ is odd find a collineation of order two which has no fixed points.

Such a collineation will be called a projective reflection. Consider the case $\mathbb{P} = \mathbb{P}^2$. Let $R$ be a projective reflection with fixed line $l$ and isolated fixed point $p$. Choosing homogeneous coordinates $[u^0, u^1, u^2]$ so that $p = [1, 0, 0]$ and

$$l = \{[0, u^1, u^2]|(u^1, u^2) \neq (0, 0)\},$$

$R$ is represented by the diagonal matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \in \text{SL}(3; \mathbb{R}).$$

Note that near $l$ the reflection looks like a Euclidean reflection in $l$ and reverses orientation. (Indeed $R$ is given by

$$(y^1, y^2) \xrightarrow{R} (-y^1, y^2)$$

in affine coordinates

$$y^1 = u^0/u^2, \ y^2 = u^1/u^2.$$
On the other hand, near $p$, the reflection looks like reflection in $p$ (that is, a rotation of order two about $p$) and preserves orientation:

$$(x^1, x^2) \mapsto (-x^1, -x^2)$$

in the affine coordinates

$$x^1 = u^1/u^0, x^2 = u^2/u^0.$$  

Of course there is no global orientation on $\mathbb{P}^2$. That a single reflection can appear simultaneously as a symmetry in a point and reflection in a line indicates the topological complexity of $\mathbb{P}^2$. A reflection in a line reverses a local orientation about a point on the line, and a symmetry in a point preserves a local orientation about the point.

**2.2.1. The basic dictionary.** We can consider the passage between the geometry of $\mathbb{P}$ and the algebra of $V$ as a kind of “dictionary” between linear algebra and projective geometry. Linear maps and linear subspaces correspond geometrically to projective maps and projective subspaces; inclusions, intersections and linear spans correspond to incidence relations in projective geometry. In this way we can either use projective geometry to geometrically picture linear algebra or linear algebra to prove theorems in geometry.

**Exercise 2.2.6.** Let $U \subset \mathbb{P}$ be a connected open subset of a projective space of dimension greater than 1. Let $U \overset{f}{\to} \mathbb{P}$ be a local diffeomorphism. Then $f$ is locally projective if and only if for each line $l \subset \mathbb{P}$, the image $f(l \cap U)$ is a line.

**2.3. Affine patches**

Let $H \subset \mathbb{P}$ be a projective hyperplane (projective subspace of codimension one). Then the complement $\mathbb{P} - H$ has a natural affine geometry, that is, is an affine space in a natural way. Indeed the group of projective automorphisms $\mathbb{P} \to \mathbb{P}$ leaving fixed each $x \in H$ and whose differential $T_x\mathbb{P} \to T_x\mathbb{P}$ is a volume-preserving linear automorphism is a vector group acting simply transitively on $\mathbb{P} - H$. Moreover the group of projective transformations of $\mathbb{P}$ leaving $H$ invariant is the full group of automorphisms of this affine space. In this way affine geometry embeds in projective geometry.

Here is how it looks in terms of matrices. Let $A = \mathbb{R}^n$; then the affine subspace of

$$V = \tau_A \oplus \mathbb{R} = \mathbb{R}^{n+1}$$

corresponding to $A$ is $\mathbb{R}^n \times \{1\} \subset \mathbb{R}^{n+1}$, the point of $A$ with affine or inhomogeneous coordinates $(x^1, \ldots, x^n)$ has homogeneous coordinates
[x^1, \ldots, x^n, 1]. Let \( f \in \text{Aff}(E) \) be the affine transformation with linear part \( A \in \text{GL}(n; \mathbb{R}) \) and translational part \( b \in \mathbb{R}^n \), that is, \( f(x) = Ax + b \), is then represented by the \((n + 1)\)-square matrix

\[
\begin{bmatrix}
A & b \\
0 & 1
\end{bmatrix}
\]

where \( b \) is a column vector and 0 denotes the \( 1 \times n \) zero row vector.

**Exercise 2.3.1.** Let \( O \in \mathbb{P}^n \) be a point, say \([0, \ldots, 0, 1]\). Show that the group \( G_{-1} = G_{-1}(O) \) of projective transformations fixing \( O \) and acting trivially on the tangent space \( T_O \mathbb{P}^n \) is given by matrices of the form

\[
\begin{bmatrix}
I_n & 0 \\
\xi & 1
\end{bmatrix}
\]

where \( I_n \) is the \( n \times n \) identity matrix and \( \xi = (\xi_1, \ldots, \xi_n) \in (\mathbb{R}^n)^* \) is a row vector; in affine coordinates such a transformation is given by

\[
(x^1, \ldots, x^n) \mapsto \left( \frac{x^1}{1 + \sum_{i=1}^n \xi_i x^i}, \ldots, \frac{x^n}{1 + \sum_{i=1}^n \xi_i x^i} \right).
\]

Show that this group is isomorphic to a \( n \)-dimensional vector group and that its Lie algebra consists of vector fields of the form

\[
\left( \sum_{i=1}^n \xi_i x^i \right) \rho
\]

where

\[
\rho = \sum_{i=1}^n x^i \frac{\partial}{\partial x_i}
\]

is the radiant vector field radiating from the origin and \( \xi \in (\mathbb{R}^n)^* \). Note that such vector fields comprise an \( n \)-dimensional abelian Lie algebra of polynomial vector fields of degree 2 in affine coordinates.

Let \( H \) be a hyperplane containing \( O \), for example,

\[
H = \left\{ [x^1, \ldots, x^n, 0] \mid (x^1, \ldots, x^n) \in \mathbb{R}^n \right\}.
\]

Let \( G_1 = G_1(H) \) denote the group of translations of the affine space \( \mathbb{P} - H \) and let

\[
G_0 = G_0(O, H) \cong \text{GL}(n, \mathbb{R})
\]

denote the group of collineations of \( \mathbb{P} \) fixing \( O \) and leaving invariant \( H \). (Alternatively \( G_0(O, H) \) is the group of collineations centralizing the radiant vector field \( \rho = \rho(O, H) \) above.) Let \( \mathfrak{g} \) denote the Lie
algebra of $\text{Aut}(\mathbb{P})$ and let $\mathfrak{g}_{-1}, \mathfrak{g}_0, \mathfrak{g}_1$ be the Lie algebras of $G_{-1}, G_0, G_1$ respectively. Show that there is a vector space decomposition

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

where

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$$

for $i, j = \pm 1, 0$ (where $\mathfrak{g}_i = 0$ for $|i| > 1$). Furthermore show that the stabilizer of $O$ has Lie algebra $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0$ and the stabilizer of $H$ has Lie algebra $\mathfrak{g}_0 \oplus \mathfrak{g}_1$.

2.4. Classical projective Geometry

2.4.1. Projective reflections. Let $l$ be a projective line $x, z \in l$ be distinct points. Then there exists a unique reflection (a harmonic homology in classical terminology)

$$l \xrightarrow{\rho_{x,z}} l$$

whose fixed-point set equals $\{x, z\}$. We say that a pair of points $y, w$ are harmonic with respect to $x, z$ if $\rho_{x,z}$ interchanges them. In that case one can show that $x, z$ are harmonic with respect to $y, w$. Furthermore this relation is equivalent to the existence of lines $p, q$ through $x$ and lines $r, s$ through $z$ such that

$$(4) \quad y = \overrightarrow{(p \cap r)(q \cap s)} \cap l$$

$$(5) \quad z = \overrightarrow{(p \cap s)(q \cap r)} \cap l$$

This leads to a projective-geometry construction of reflection, as follows. Let $x, y, z \in l$ be fixed; we seek the harmonic conjugate of $y$ with respect to $x, z$, that is, the image $R_{x,z}(y)$. Erect arbitrary lines (in general position) $p, q$ through $x$ and a line $r$ through $z$. Through $y$ draw the line through $r \cap q$; join its intersection with $p$ with $z$ to form line $s$,

$$s = \overrightarrow{z \cap (p \cap y r \cap q)}.$$

Then $R_{x,z}(y)$ will be the intersection of $s$ with $l$. 
Figure 1. Non-Euclidean tessellations by equilateral triangles
Exercise 2.4.1. Consider the projective line $\mathbb{P}^1 = \mathbb{R} \cup \{\infty\}$. Show that for every rational number $x \in \mathbb{Q}$ there exists a sequence $x_0, x_1, x_2, x_3, \ldots, x_n \in \mathbb{P}^1$ such that:

- $x = \lim_{i \to \infty} x_i$;
- $\{x_0, x_1, x_2\} = \{0, 1, \infty\}$;
- For each $i \geq 3$, there is a harmonic quadruple $(x_i, y_i, z_i, w_i)$ with $y_i, z_i, w_i \in \{x_0, x_1, \ldots, x_{i-1}\}$.

If $x$ is written in reduced form $\frac{p}{q}$ then what is the smallest $n$ for which $x$ can be reached in this way?

Exercise 2.4.2 (Synthetic arithmetic). Using the above synthetic geometry construction of harmonic quadruples, show how to add, subtract, multiply, and divide real numbers by a straightedge-and-pencil construction. In other words, draw a line $l$ on a piece of paper and choose three points to have coordinates $0, 1, \infty$ on it. Choose arbitrary points corresponding to real numbers $x, y$. Using only a straightedge (not a ruler!) construct the points corresponding to $x + y, x - y, xy$, and $x/y$ if $y \neq 0$.

2.4.2. Fundamental theorem of projective geometry. If $l \subset \mathbb{P}$ and $l' \subset \mathbb{P}'$ are projective lines, the Fundamental Theorem of Projective Geometry asserts that for given triples $x, y, z \in l$ and $x', y', z' \in l'$ of distinct points there exists a unique projective map

$$
l \longrightarrow l' \quad \begin{array}{ccc} x & \mapsto & x' \\ y & \mapsto & y' \\ z & \mapsto & z' \end{array}
$$

If $w \in l$ then the cross-ratio $[x, y, w, z]$ is defined to be the image of $w$ under the unique collineation $f : l \longrightarrow \mathbb{P}^1$ with

$$
x \overset{f}{\mapsto} 0 \quad y \overset{f}{\mapsto} 1 \quad z \overset{f}{\mapsto} \infty
$$

If $l = \mathbb{P}^1$, then the cross-ratio is given by the formula

$$[x, y, w, z] = \frac{w - x}{w - z} / \frac{y - x}{y - z}.$$
The cross-ratio can be extended to quadruples of four points, of which at least three are distinct.

**Exercise 2.4.3.** Let \( \sigma \) be a permutation on four symbols. Show that there exists a linear fractional transformation \( \Phi_\sigma \) such that

\[
[x_\sigma(1), x_\sigma(2), x_\sigma(3), x_\sigma(4)] = \Phi_\sigma([x_1, x_2, x_3, x_4]).
\]

In particular determine which permutations leave the cross-ratio invariant.

A pair \( y, w \) is harmonic with respect to \( x, z \) (in which case we say that \( (x, y, w, z) \) is a harmonic quadruple) if and only if the cross-ratio \( [x, y, w, z] = -1 \).

**Exercise 2.4.4.** Show that a homeomorphism \( \mathbb{P}^1 \xrightarrow{f} \mathbb{P}^1 \) is projective if and only if \( f \) preserves harmonic quadruples if and only if \( f \) preserves cross-ratios, that is, for all quadruples \( (x, y, w, z) \), the cross-ratios satisfy

\[
[f(x), f(y), f(w), f(z)] = [x, y, w, z].
\]

**2.4.3. Products of reflections.** If \( \phi, \psi \) are collineations, each of which fix a point \( O \in \mathbb{P} \), their composition fixes \( \phi \circ \psi \) fixes \( O \). In particular its derivative

\[
D(\phi \circ \psi)_O = D\phi_O \circ D\psi_O
\]

acts linearly on the tangent space \( T_O \mathbb{P} \). We consider the case when \( \phi, \psi \) are reflections in \( \mathbb{P}^2 \). As in §2.4.1, a reflection \( \phi \) is completely determined by its set \( \text{Fix}(\phi) \) which consists of a point \( p_\phi \) and a line \( l_\phi \) such that \( p_\phi \notin l_\phi \). Define

\[
O := l_\phi \cap l_\psi
\]

and \( \mathbb{P}_O \) the projective line whose points are the lines incident to \( O \).

**Exercise 2.4.5.** Let \( \rho \) denote the cross-ratio of the four lines

\[
l, \overrightarrow{Op}, \overrightarrow{Op}', l'
\]

as elements of \( \mathbb{P}_O \).

- The linear automorphism

\[
T_O \mathbb{P} \xrightarrow{D_O() \rho} T_O \mathbb{P}
\]

of the tangent space \( T_O \mathbb{P} \) leaves invariant a postive definite inner product \( g_O \) on \( T_O \mathbb{P} \).
Furthermore \( D_\theta(\phi \psi) \) represents a rotation of angle \( \theta \) in the tangent space \( T_\theta(P) \) with respect to \( g_\theta \) if and only if

\[
\rho = \frac{1}{2} (1 + \cos \theta)
\]

for \( 0 < \theta < \pi \) and is a rotation of angle \( \pi \) (that is, an involution) if and only if \( p \in l' \) and \( p' \in l \).

### 2.5. Asymptotics of projective transformations

We shall be interested in the singular projective transformations since they occur as limits of nonsingular projective transformations. The collineation group \( \text{Aut}(P) \) of \( P = P^n \) is a large noncompact group which is naturally embedded in the projective space \( \text{End}(P) \) as an open dense subset as in Exercise 2.2.4. Thus understanding precisely what it means for a sequence of collineations to converge to a (possibly singular) projective transformation is crucial.

A **singular projective transformation** of \( P \) is a projective map \( f \) defined on the complement of a projective subspace \( U(f) \subset P \), called the **undefined subspace** of \( f \) and taking values in a projective subspace \( R(f) \subset P \), called the **range** of \( f \). Furthermore

\[
\dim P = \dim U(f) + \dim R(f) + 1.
\]

**Proposition 2.5.1.** Let \( g_m \in \text{Aut}(P) \) be a sequence of collineations of \( P \) and let \( g_\infty \in \text{End}(P) \). Then the sequence \( g_m \) converges to \( g_\infty \) in \( \text{End}(P) \) if and only if the restrictions \( g_m|_K \) converge uniformly to \( g_\infty|_K \) for all compact sets \( K \subset P - U(g_\infty) \).

**Proof.** Convergence in \( \text{End}(P) \) may be described as follows. Let \( P = P(V) \) where \( V \cong \mathbb{R}^{n+1} \) is a vector space. Then \( \text{End}(P) \) is the projective space associated to the vector space \( \text{End}(V) \) of \((n+1)\)-square matrices. If \( a = (a_{ij}) \in \text{End}(V) \) is such a matrix, let

\[
\|a\| = \sqrt{\sum_{i,j=1}^{n+1} |a_{ij}|^2}
\]

denote its Euclidean norm; projective endomorphisms then correspond to matrices \( a \) with \( \|a\| = 1 \), uniquely determined up to the antipodal map \( a \mapsto -a \). The following lemma will be useful in the proof of Proposition 2.5.1.
Lemma 2.5.2. Let $V, V'$ be vector spaces and let $V \xrightarrow{f_n} V'$ be a sequence of linear maps converging to $V \xrightarrow{f_\infty} V'$. Let $\tilde{K} \subset V$ be a compact subset of $V - \ker(f_\infty)$ and let $f_i$ be the map defined by

$$f_i(x) = \frac{\tilde{f}_i(x)}{\|\tilde{f}_i(x)\|}.$$

Then $f_n$ converges uniformly to $f_\infty$ on $\tilde{K}$ as $n \to \infty$.

Proof. Choose $C > 0$ such that $C \leq \|\tilde{f}_\infty(x)\| \leq C^{-1}$ for $x \in \tilde{K}$. Let $\epsilon > 0$. There exists $N$ such that if $n > N$, then

$$\|\tilde{f}_\infty(x) - \tilde{f}_n(x)\| < \frac{C\epsilon}{2}, \tag{6}$$

$$\left| 1 - \frac{\tilde{f}_n(x)}{\|\tilde{f}_\infty(x)\|} \right| < \frac{\epsilon}{2} \tag{7}$$

for $x \in \tilde{K}$. It follows that

$$\left\| f_n(x) - f_\infty(x) \right\| = \left\| \frac{\tilde{f}_n(x)}{\|\tilde{f}_n(x)\|} - \frac{\tilde{f}_\infty(x)}{\|\tilde{f}_\infty(x)\|} \right\|$$

$$= \frac{1}{\|\tilde{f}_\infty(x)\|} \left( \left\| \frac{\tilde{f}_\infty(x)}{\|\tilde{f}_n(x)\|} \tilde{f}_n(x) - \tilde{f}_\infty(x) \right\| \right.$$  

$$\leq \frac{1}{\|\tilde{f}_\infty(x)\|} \left( \left\| \frac{\tilde{f}_\infty(x)}{\|\tilde{f}_n(x)\|} \tilde{f}_n(x) - \tilde{f}_\infty(x) \right\| \right.$$  

$$+ \|\tilde{f}_n(x) - \tilde{f}_\infty(x)\| $$

$$= \left( 1 - \frac{\|\tilde{f}_n(x)\|}{\|\tilde{f}_\infty(x)\|} \right)$$

$$\left] + \frac{1}{\|\tilde{f}_\infty(x)\|} \|\tilde{f}_n(x) - \tilde{f}_\infty(x)\| \right]$$

$$< \frac{\epsilon}{2} + C^{-1}(\frac{C\epsilon}{2}) = \epsilon$$

for all $x \in \tilde{K}$ as desired. \qed

The proof of Proposition 2.5.1 proceeds as follows. Suppose $g_m$ is a sequence of locally projective maps defined on a connected domain $\Omega \subset P$ converging uniformly on all compact subsets of $\Omega$ to a map

$$\Omega \xrightarrow{g_\infty} P'.$$
Lift $g_{\infty}$ to a linear transformation $\tilde{g}_{\infty}$ of norm 1, and lift $g_{m}$ to linear transformations $\tilde{g}_{m}$, also linear transformations of norm 1, converging to $\tilde{g}_{\infty}$. Then

$$g_{m} \longrightarrow g_{\infty}$$

in $\text{End}(P)$. Conversely if $g_{m} \longrightarrow g_{\infty}$ in $\text{End}(P)$ and

$$K \subset P - \mathbb{U}(g_{\infty}),$$

choose lifts as above and a compact set $\tilde{K} \subset V$ such that $\Pi(\tilde{K}) = K$. By Lemma 2.5.2, the normalized linear maps $\tilde{g}_{m}/|\tilde{g}_{m}|$ converge uniformly to $g_{\infty}/|g_{\infty}|$ on $\tilde{K}$ and hence $g_{m}$ converges uniformly to $g_{\infty}$ on $K$. The proof of Proposition 2.5.1 is now complete. \[\square\]

2.5.1. Some examples. Let us consider a few examples of this convergence. Consider the case first when $n = 1$. Let $\lambda_{m} \in \mathbb{R}$ be a sequence converging to $+\infty$ and consider the projective transformations given by the diagonal matrices

$$g_{m} = \begin{bmatrix} \lambda_{m} & 0 \\ 0 & (\lambda_{m})^{-1} \end{bmatrix}$$

Then $g_{m} \longrightarrow g_{\infty}$ where $g_{\infty}$ is the singular projective transformation corresponding to the matrix

$$g_{\infty} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

— this singular projective transformation is undefined at $\mathbb{U}(g_{\infty}) = \{[0, 1]\}$; every point other than $[0, 1]$ is sent to $[1, 0]$. It is easy to see that a singular projective transformation of $\mathbb{P}^1$ is determined by the ordered pair of points $\mathbb{U}(f), \mathbb{R}(f)$ (which may be coincident).

Exercise 2.5.3. Show that the sequence

$$\phi_{n}(x) := \frac{x}{1 - nx}$$

converges to a singular projective transformation whose undefined set and image each equals $\{0\}$ but $\phi_{n}$ does not converge uniformly on $\mathbb{P}^1$ to the constant function 0.

More interesting phenomena arise when $n = 2$. Let $g_{m} \in \text{Aut}(P^2)$ be a sequence of diagonal matrices

$$\begin{bmatrix} \lambda_{m} & 0 & 0 \\ 0 & \mu_{m} & 0 \\ 0 & 0 & \nu_{m} \end{bmatrix}$$
where $0 < \lambda_m < \mu_m < \nu_m$ and $\lambda_m \mu_m \nu_m = 1$. Corresponding to the three eigenvectors (the coordinate axes in $\mathbb{R}^3$) are three fixed points

$$p_1 = [1, 0, 0], \quad p_2 = [0, 1, 0], \quad p_3 = [0, 0, 1].$$

They span three invariant lines

$$l_1 = \overrightarrow{p_2 p_3}, \quad l_2 = \overrightarrow{p_3 p_1}, \quad l_3 = \overrightarrow{p_3 p_1}.$$

Since $0 < \lambda_m < \mu_m < \nu_m$, the collineation has a repelling fixed point at $p_1$, a saddle point at $p_2$ and an attracting fixed point at $p_3$. Points on $l_2$ near $p_1$ are repelled from $p_1$ faster than points on $l_3$ and points on $l_2$ near $p_3$ are attracted to $p_3$ more strongly than points on $l_1$. Suppose that $g_m$ does not converge to a nonsingular matrix; it follows that $\nu_m \to +\infty$ and $\lambda_m \to 0$ as $m \to \infty$. Suppose that $\mu_m / \nu_m \to \rho$; then $g_m$ converges to the singular projective transformation $g_{\infty}$ determined by the matrix

$$\begin{bmatrix}
0 & 0 & 0 \\
0 & \rho & 0 \\
0 & 0 & 1
\end{bmatrix}$$

which, if $\rho > 0$, has undefined set $U(g_{\infty}) = p_1$ and range $l_1$; otherwise $U(g_{\infty}) = l_2$ and $\text{Image}(g_{\infty}) = p_2$.

### 2.5.2. Limits of similarity transformations.

Convergence to singular projective transformations is perhaps easiest for translations of affine space, or more generally Euclidean isometries.

**Exercise 2.5.4.** Suppose $g_m \in \text{Isom}(E^n)$ be a divergent sequence of Euclidean isometries. Show that $\exists p \in P_\infty^{n-1}$ a subsequence $g_{m_k}$, that $g_{m_k}|_K \Rightarrow p$ for every compact $K \subset\subset E^n$.

Indeed the boundary of the translation group $V$ of $A^n$ is the projective space $P_n^{\infty-1}$. More generally the boundary of $\text{Isom}(E^n)$ equals $P_n^{\infty-1}$.

**Exercise 2.5.5.** Suppose $g_m \in \text{Sim}(E^n)$ be a divergent sequence of similarities of Euclidean space. Then $\exists$ a subsequence $g_{m_k}$, and point $p \in E^n \coprod P_\infty^{n-1}$ such that one of the three possibilities occur:

- $p \in P_\infty^{n-1}$ and
  $$g_{m_k}|_K \Rightarrow p, \quad \forall K \subset\subset E^n;$$
- $p \in P_\infty^{n-1}$ and $\exists q \in E^n$ such that
  $$g_{m_k}|_K \Rightarrow p, \quad \forall K \subset\subset E^n \setminus \{q\};$$
- $p \in E_\infty^n$ and
  $$g_{m_k}|_K \Rightarrow p, \quad \forall K \subset\subset E^n.$$
The two latter cases occur when \( \lim_{k \to \infty} \lambda(g_{m_k}) = \infty \) (respectively \( \lim_{k \to \infty} \lambda(g_{m_k}) = 0 \), where \( \text{Sim}(E^n) \xrightarrow{\lambda} \mathbb{R}^+ \) is the scale factor homomorphism defined in §1.2.1 of Chapter 1.

These results will be used in Fried’s classification of closed similarity manifolds (§11.4 of Chapter 11).
This chapter introduces the projective models for elliptic and hyperbolic geometry through the classical notion of polarities. The Beltrami-Klein metric generalizes to the Hilbert metric on convex domains. For an extensive modern discussion of Hilbert geometry, see Papadopoulos-Troyanov [122].

3.1. Duality

In an axiomatic development of projective geometry, there is a basic symmetry: A pair of distinct points lie on a unique line and a pair of distinct lines meet in a unique point (in dimension two). As a consequence any statement about the geometry of $\mathbb{P}^2$ can be “dualized” by replacing “point” by “line,” “line” by “point,” “collinear” with “concurrent,” etc in a completely consistent fashion.

Perhaps the oldest nontrivial theorem of projective geometry is Pappus’ theorem (300 A.D.), which asserts that if $l, l' \subset \mathbb{P}^2$ are distinct lines and $A, B, C \in l$ and $A', B', C' \in l'$ are triples of distinct points, then the three points

$$\overrightarrow{AB'} \cap \overrightarrow{A'B}, \overrightarrow{BC'} \cap \overrightarrow{B'C}, \overrightarrow{CA'} \cap \overrightarrow{C'A}$$

are collinear. The dual of Pappus’ theorem is therefore: if $p, p' \in \mathbb{P}^2$ are distinct points and $a, b, c$ are distinct lines all passing through $p$ and $a', b', c'$ are distinct lines all passing through $p'$, then the three lines

$$\overrightarrow{(a \cap b')} \left( \overrightarrow{a' \cap b} \right), \overrightarrow{(b \cap c')} \left( \overrightarrow{b' \cap c} \right), \overrightarrow{(c \cap a')} \left( \overrightarrow{c' \cap a} \right)$$

are concurrent. (According to [38], Hilbert observed that Pappus’ theorem is equivalent to the commutative law of multiplication.)

In terms of our projective geometry/linear algebra dictionary, projective duality translates into duality between vector spaces as follows. Let $\mathbb{P}$ be a projective space and let $V$ be the associated vector space. A nonzero linear functional $V \xrightarrow{\psi} \mathbb{R}$ defines a projective hyperplane $H_\psi$ in $\mathbb{P}$; two such functionals define the same hyperplane if and only if they differ by a nonzero scalar multiple, that is, they determine the same line in the vector space $V^*$ dual to $V$. (Alternately $\psi$ defines the
constant projective map \( P - H_\psi \rightarrow P^0 \) which is completely specified by its undefined set \( H_\psi \). We thus define the projective space dual to \( P \) as follows. The dual projective space \( P^* \) of lines in the dual vector space \( V^* \) correspond to hyperplanes in \( P \). The line joining two points in \( P^* \) corresponds to the intersection of the corresponding hyperplanes in \( P \), and a hyperplane in \( P^* \) corresponds to a point in \( P \). In general if \( P \) is an \( n \)-dimensional projective space there is a natural correspondence \( \{ k\text{-dimensional subspaces of } P \} \leftrightarrow \{ l\text{-dimensional subspaces of } P^* \} \) where \( k + l = n - 1 \). In particular we have an isomorphism of \( P \) with the dual of \( P^* \).

Let \( P \xrightarrow{f} P' \) be a projective map between projective spaces. Then for each hyperplane \( H' \subset P' \) the inverse image \( f^{-1}(H') \) is a hyperplane in \( P \). There results a map \( (P')^* \xrightarrow{f^!} P^* \), the transpose of the projective map \( f \). (Evidently \( f^! \) is the projectivization of the transpose of the linear map \( \tilde{f} : V \rightarrow V' \).)

### 3.2. Correlations and polarities

Let \( P \) be an \( n \)-dimensional projective space and \( P^* \) its dual. A correlation of \( P \) is a projective isomorphism \( P \xrightarrow{\theta} P^* \). That is, \( \theta \) associates to each point in \( P \) a hyperplane in \( P \) in such a way that if \( x_1, x_2, x_3 \in P \) are collinear, then the hyperplanes

\[
\theta(x_1), \theta(x_2), \theta(x_3) \subset P
\]

are incident, that is, the intersection \( \theta(x_1) \cap \theta(x_2) \cap \theta(x_3) \) is a projective subspace of codimension two (rather than three, as would be the case if they were in general position). The transpose correlation \( \theta^! \) is also a projective isomorphism \( P \rightarrow P^* \) (using the reflexivity \( P^{**} \cong P \)). A correlation is a polarity if it is equal to its transpose.

Using the dictionary between projective geometry and linear algebra, one sees that if \( V \) is the vector space corresponding to \( P = P(V) \), then \( P^* = P(V^*) \) and a correlation \( \theta \) is realized as a linear isomorphism \( V \xrightarrow{\tilde{\theta}} V^* \), which is uniquely determined up to homotheties. Linear maps \( V \xrightarrow{\tilde{\theta}} V^* \), correspond to bilinear forms

\[
V \times V \xrightarrow{B_{\tilde{\theta}}} \mathbb{R}
\]

under the correspondence

\[
\tilde{\theta}(v)(w) = B_{\tilde{\theta}}(v, w)
\]

and \( \tilde{\theta} \) is an isomorphism if and only if \( B_{\tilde{\theta}} \) is nondegenerate. Thus correlations can be interpreted analytically as projective equivalence.
3.2. CORRELATIONS AND POLARITIES

classes of nondegenerate bilinear forms. Furthermore a correlation $\theta$ is self-inverse (that is, a polarity) if and only if a corresponding bilinear form $B_\theta$ is symmetric.

Let $\theta$ be a polarity on $P$. A point $p \in P$ is conjugate if it is incident to its polar hyperplane, that is, if $p \in \theta(p)$. By our dictionary we see that the conjugate points of a polarity correspond to null vectors of the associated quadratic form, that is, to nonzero vectors $v \in V$ such that $B_\theta(v, v) = 0$. A polarity is said to be elliptic if it admits no conjugate points; elliptic polarities correspond to symmetric bilinear forms which are definite. For example here is an elliptic polarity of $P = P^2$: a point $p$ in $P^2$ corresponds to a line $\Pi^{-1}(p)$ in Euclidean 3-space and its orthogonal complement $\Pi^{-1}(p)\perp$ is a 2-plane corresponding to a line $\theta(p) \in P^*$. It is easy to check that $\theta$ defines an elliptic polarity of $P$.

In general the set of conjugate points of a polarity is a quadric, which up to a collineation is given in homogeneous coordinates as

$$Q = Q_{p,q} = \left\{ [x^1 : \cdots : x^{n+1}] \mid (x^1)^2 + \cdots + (x^p)^2 - (x^{p+1})^2 - \cdots - (x^{p+q})^2 = 0 \right\}$$

where $p + q = n + 1$ (since the corresponding symmetric bilinear form is given by the diagonal matrix $I_p \oplus -I_q$). We call $(p, q)$ the signature of the polarity. The quadric $Q$ determines the polarity $\theta$ as follows. For brevity we consider only the case $q = 1$, in which case the complement $P - Q$ has two components, a convex component

$$\Omega = \{ [x^0 : x^1 : \cdots : x^n] \mid (x^0)^2 - (x^1)^2 - \cdots - (x^n)^2 < 0 \}$$

and a nonconvex component

$$\Omega^\perp = \{ [x^0 : x^1 : \cdots : x^n] \mid (x^0)^2 - (x^1)^2 - \cdots - (x^n)^2 > 0 \}$$

diffeomorphic to the total space of the tautological line bundle over $P^{n-1}$ (for $n = 2$ this is a M"obius band). If $x \in Q$, let $\theta(x)$ denote the hyperplane tangent to $Q$ at $x$. If $x \in \Omega^\perp$ the points of $Q$ lying on tangent lines to $Q$ containing $x$ all lie on a hyperplane which is $\theta(x)$. If $H \in P^*$ is a hyperplane which intersects $Q$, then either $H$ is tangent to $Q$ (in which case $\theta(H)$ is the point of tangency) or there exists a cone tangent to $Q$ meeting $Q$ in $Q \cap H$ — the vertex of this cone will be $\theta(H)$. If $x \in \Omega$, then there will be no tangents to $Q$ containing $x$, but by representing $x$ as an intersection $H_1 \cap \cdots \cap H_n$, we obtain $\theta(x)$ as the hyperplane containing $\theta(H_1), \ldots, \theta(H_n)$.

**Exercise 3.2.1.** Show that $P \overset{\theta}{\to} P^*$ is projective.
Observe that a polarity on $\mathcal{P}$ of signature $(p, q)$ determines, for each non-conjugate point $x \in \mathcal{P}$ a unique reflection $R_x$ which preserves the polarity. The group of collineations preserving such a polarity is the projective orthogonal group $\text{PO}(p, q)$, that is, the image of the orthogonal group $\text{O}(p, q) \subset \text{GL}(n + 1, \mathbb{R})$ under the projectivization homomorphism

$$\text{GL}(n + 1, \mathbb{R}) \longrightarrow \text{PGL}(n + 1, \mathbb{R})$$

having kernel the scalar matrices $\mathbb{R}^* \subset \text{GL}(n + 1, \mathbb{R})$. Let

$$\Omega = \{ \Pi(v) \in \mathcal{P} \mid B(v, v) < 0 \};$$

then by projection from the origin $\Omega$ can be identified with the hyperquadric

$$\{ v \in \mathbb{R}^{p,q} \mid B(v, v) = -1 \}$$

whose induced pseudo-Riemannian metric has signature $(q, p - 1)$ and constant nonzero curvature. In particular if $(p, q) = (1, n)$ then $\Omega$ is a model for hyperbolic $n$-space $\mathbb{H}^n$ in the sense that the group of isometries of $\mathbb{H}^n$ are represented precisely as the group of collineations of $\mathcal{P}^n$ preserving $\Omega^n$. In this model, geodesics are the intersections of projective lines in $\mathcal{P}$ with $\Omega$; more generally intersections of projective subspaces with $\Omega$ define totally geodesic subspaces.

Consider the case that $\mathcal{P} = \mathbb{P}^2$. Points “outside” $\Omega$ correspond to geodesics in $\mathbb{H}^2$. If $p_1, p_2 \in \Omega^l$, then $\overrightarrow{p_1 p_2}$ meets $\Omega$ if and only if the geodesics $\theta(p_1), \theta(p_2)$ are ultra-parallel in $\mathbb{H}^2$; in this case $\theta(\overrightarrow{p_1 p_2})$ is the geodesic orthogonal to both $\theta(p_1), \theta(p_2)$. (Geodesics $\theta(p)$ and $l$ are orthogonal if and only if $p \in l$.) Furthermore $\overrightarrow{p_1 p_2}$ is tangent to $Q$ if and only if $\theta(p_1)$ and $\theta(p_2)$ are parallel. For more information on this model for hyperbolic geometry, see [38] or [140], §2. This model for non-Euclidean geometry seems to have first been discovered by Cayley in 1858.

### 3.3. Intrinsic metrics

We shall discuss the metric on hyperbolic space, however, in the more general setting of the Hilbert-Carathéodory-Kobayashi metric on a convex domain $\mathcal{P} = \mathbb{P}^n$. This material can be found in Kobayashi [83, 84, 85] and Vey [146].

#### 3.3.1. Examples of convex cones.

Let $V = \mathbb{R}^{n+1}$ be the corresponding vector space. A subset $\Omega \subset V$ is a cone if and only if $\mathbb{R}^+ (\Omega) = \Omega$, that is, if it is invariant under positive homotheties. A subset $\Omega \subset V$ is convex if whenever $x, y \in \Omega$, then the line segment $\overline{xy} \subset \Omega$. A convex domain $\Omega \subset V$ is *sharp* if and only if there is no
Figure 2. The inside of a properly convex domain admits a projectively invariant distance defined in terms of cross-ratio. This is called the Hilbert distance. When the domain is the interior of a conic, then this distance is a Riemannian metric of constant negative curvature. This is the Klein-Beltrami projective model of the hyperbolic plane.

Entire affine line contained in \( \Omega \). For example, \( V \) itself and the upper half-space
\[
\mathbb{R}^n \times \mathbb{R}^+ = \{(x^0, \ldots , x^n) \in V \mid x^0 > 0\}
\]
are both convex cones, neither of which are sharp. The positive orthant
\[
(\mathbb{R}^+)^{n+1} = \{(x^0, \ldots , x^n) \in V \mid x^i > 0 \text{ for } i = 0, 1, \ldots , n\}
\]
and the positive light-cone
\[
C_{n+1} = \{(x^0, \ldots , x^n) \in V \mid x^0 > 0 \text{ and } -(x^0)^2 + (x^1)^2 + \ldots + (x^n)^2 < 0\}
\]
are both properly convex cones. Note that the planar region
\[
\{(x,y) \in \mathbb{R}^2 \mid y > x^2\}
\]
is a properly convex domain but not affinely equivalent to a cone.

Exercise 3.3.1. Show that the set \( \mathcal{P}_n(\mathbb{R}) \) of all positive definite symmetric \( n \times n \) real matrices is a properly convex cone in the \( n(n+1)/2 \)-dimensional vector space \( V \) of \( n \times n \) symmetric matrices. Are there any affine transformations of \( V \) preserving \( \mathcal{P}_n(\mathbb{R}) \)? What is its group of affine automorphisms?

We shall say that a subset \( \Omega \subset \mathcal{P} \) is convex if there is a convex set \( \Omega' \subset V \) such that \( \Omega = \Pi(\Omega') \). Since \( \Omega' \subset V - \{0\} \) is convex, \( \Omega \) must
be disjoint from at least one hyperplane \( H \) in \( P \). (In particular we do not allow \( P \) to itself be convex.) Equivalently \( \Omega \subset P \) is convex if there is a hyperplane \( H \subset P \) such that \( \Omega \) is a convex set in the affine space complementary to \( H \). A domain \( \Omega \subset P \) is properly convex if and only if there exists a properly convex cone \( \Omega' \subset V \) such that \( \Omega = \Pi(\Omega') \). Equivalently \( \Omega \) is properly convex if and only if there is a hyperplane \( H \subset P \) such that \( \overline{\Omega} \) is a convex subset of the affine space \( P - H \). If \( \Omega \) is properly convex, then the intersection of \( \Omega \) with a projective subspace \( P' \subset P \) is either empty or a properly convex subset \( \Omega' \subset P' \). In particular every line intersecting \( \Omega \) meets \( \partial \Omega \) in exactly two points.

3.3.2. The Hilbert metric. In 1894 Hilbert introduced a projectively invariant metric \( d = d_\Omega \) on any properly convex domain \( \Omega \subset P \) as follows. Let \( x, y \in \Omega \) be a pair of distinct points; then the line \( \overrightarrow{xy} \) meets \( \partial \Omega \) in two points which we denote by \( x_\infty, y_\infty \) (the point closest to \( x \) will be \( x_\infty \), etc.). The Hilbert distance
\[
d = d^{\text{Hilb}}
\]

between \( x \) and \( y \) in \( \Omega \) will be defined as the logarithm of the cross-ratio of this quadruple:
\[
d(x, y) = \log [x_\infty, x, y, y_\infty]
\]
It is clear that \( d(x, y) \geq 0 \), that \( d(x, y) = d(y, x) \) and since \( \Omega \) contains no complete affine line, \( x_\infty \neq y_\infty \) so that \( d(x, y) > 0 \) if \( x \neq y \). The same argument shows that this function \( d : \Omega \times \Omega \rightarrow \mathbb{R} \) is proper, or finitely compact: that is, for each \( x \in \Omega \) and \( r > 0 \), the “\( r \)-ball”
\[
B_r(x) = \{ y \in \Omega \mid d(x, y) \leq r \}
\]
is compact. Once the triangle inequality is established, it will follow that \( (\Omega, d) \) is a complete metric space. The triangle inequality results from the convexity of \( \Omega \), although we shall deduce it by showing that the Hilbert metric agrees with the general intrinsic metric introduced by Kobayashi \[85\], where the triangle inequality is enforced as part of its construction. In this metric the geodesics are represented by straight lines.

3.3.3. The Hilbert metric on a triangle. Let \( \triangle \subset P^2 \) denote a domain bounded by a triangle. Then the balls in the Hilbert metric are hexagonal regions. (In general if \( \Omega \) is a convex \( k \)-gon in \( P^2 \) then the unit balls in the Hilbert metric will be interiors of \( 2k \)-gons.) Note that since \( \text{Aut}(\triangle) \) acts transitively on \( \triangle \) \( (\text{Aut}(\triangle) \) is conjugate to the group of diagonal matrices with positive eigenvalues) all the unit balls are isometric.
3.3. INTRINSIC METRICS

Here is a construction which illustrates the Hilbert geometry of \( \triangle \). (Compare Figure 2.4.1.) Start with a triangle \( \triangle \) and choose line segments \( l_1, l_2, l_3 \) from an arbitrary point \( p_1 \in \triangle \) to the vertices \( v_1, v_2, v_3 \) of \( \triangle \). Choose another point \( p_2 \) on \( l_1 \), say, and form lines \( l_4, l_5 \) joining it to the remaining vertices. Let

\[
\rho = \log \left\lvert \frac{[v_1, p_1, p_2, l_1 \cap \overrightarrow{v_2 v_3}]}{\overrightarrow{v_2 v_3}} \right\rvert
\]

where \([,] \) denotes the cross-ratio of four points on \( l_1 \). The lines \( l_4, l_5 \) intersect \( l_2, l_3 \) in two new points which we call \( p_3, p_4 \). Join these two points to the vertices by new lines \( l_i \) which intersect the old \( l_i \) in new points \( p_i \). In this way one generates infinitely many lines and points inside \( \triangle \), forming a configuration of smaller triangles \( T_j \) inside \( \triangle \). For each \( p_i \), the union of the \( T_j \) with vertex \( p_i \) is a convex hexagon which is a Hilbert ball in \( \triangle \) of radius \( \rho \). Note that this configuration is combinatorially equivalent to the tessellation of the plane by congruent equilateral triangles. Indeed, this tessellation of \( \triangle \) arises from an action of a \((3,3,3)\)-triangle group by collineations and converges (in an appropriate sense) to the Euclidean equilateral-triangle tessellation as \( \rho \to 0 \).

**Exercise 3.3.2.** Let \( \triangle := \{(x, y) \in \mathbb{R}^2 \mid x, y > 0\} \) be the positive quadrant. Then the Hilbert distance is given by

\[
d((x, y), (x', y')) = \log \max \left\{ \frac{x}{x'}, \frac{x'}{x}, \frac{y}{y'}, \frac{y'}{y}, \frac{xy}{xy'}, \frac{x'y}{x'y} \right\}.
\]

- Show that the unit balls are hexagons.
- For any two points \( p, p' \in \triangle \), show that there are infinitely many geodesics joining \( p \) to \( p' \).
- Show that there are even non-smooth polygonal curves from \( p \) to \( p' \) having minimal length.

Cooper has called such a Finsler, where the unit balls are hexagons, a *hex-metric*.

Let \( Q \subset \mathbb{P}^n \) be a quadric corresponding to a polarity of signature \((1, n)\) and let \( \Omega \) be the convex region bounded by \( Q \). Indeed, let us take \( \Omega \) to be the unit ball in \( \mathbb{R}^n \) defined by

\[
\|x\|^2 = \sum_{i=1}^{n} (x^i)^2 < 1.
\]
Then the Hilbert metric is given by the Riemannian metric
\[
  ds^2 = \frac{-4}{\sqrt{1 - \|x\|^2}} d^2 \sqrt{1 - \|x\|^2}
\]
\[
  = \frac{4}{(1 - \|x\|^2)^2} \sum_{i=1}^{n} (x^i dx^i)^2 + (1 - \|x\|^2)^2 (dx^i)^2
\]
which has constant curvature $-1$. This is the only case when the Hilbert metric is Riemannian; in general the Hilbert metric is Finsler, given infinitesimally by a norm on the tangent spaces (not necessarily a norm arising from a quadratic form). By changing $\sqrt{1 - \|x\|^2}$ to $\sqrt{1 + \|x\|^2}$ in the above formula, one obtains a metric on $\mathbb{P}^n$ of constant curvature $+1$. In 1866 Beltrami showed that the only Riemannian metrics on domains in $\mathbb{P}^n$ where the geodesics are straight line segments are (up to a collineation and change of scale factor) Euclidean metrics and these two metrics. Hilbert’s fourth problem was to determine all metric space structures on domains in $\mathbb{P}^n$ whose geodesics are straight line segments. There are many unusual such metrics, see Busemann-Kelly [25], and Pogorelov [123].

**3.3.4. The Kobayashi metric.** To motivate Kobayashi’s construction, consider the basic case of intervals in $\mathbb{P}^1$. There are several natural choices to take, for example, the interval of positive real numbers $\mathbb{R}^+ = (0, \infty)$ or the unit ball $I = [-1, 1]$. They are related by the projective transformation $I \xrightarrow{\tau} \mathbb{R}^+$
\[
  x = \tau(u) = \frac{1 + u}{1 - u}
\]
mapping $-1 < u < 1$ to $0 < x < \infty$ with $\tau(0) = 1$. The corresponding Hilbert metrics are given by
\[
  d_{\mathbb{R}^+}(x_1, x_2) = \log \left| \frac{x_1}{x_2} \right| \tag{8}
\]
\[
  d_I(u_1, u_2) = 2 \left| \tanh^{-1}(u_1) - \tanh^{-1}(u_2) \right| \tag{9}
\]
which follows from the fact that $\tau$ pulls back the parametrization corresponding to Haar measure
\[
  \frac{|dx|}{x} = |d \log x|
\]
on $\mathbb{R}^+$ to the Poincaré metric
\[
  2|du| = 2|d \tanh^{-1} u|
\]
on $I$.

**Exercise 3.3.3.** Show that a projective map mapping
\[
-1 \mapsto x_-
\]
\[
0 \mapsto 0
\]
\[
1 \mapsto x_+
\]
is given by:
\[
t \mapsto \frac{2(x_- x_+) t}{(t + 1)x_- + (t - 1)x_+}
\]
and a projective automorphism of $I$ by
\[
t \mapsto \frac{\cosh(s)t + \sinh(s)}{\sinh(s)t + \cosh(s)} = \frac{t + \tanh(s)}{1 + \tanh(s)t}
\]
In terms of the Poincaré metric on $I$ the Hilbert distance $d(x, y)$ can be characterized as an infimum over all projective maps $I \to \Omega$:
\[
d(x, y) = \inf \left\{ d_I(a, b) \middle| f \in \text{Proj}(I, \Omega), f(a) = x, f(b) = y \right\}
\]
We now define the Kobayashi pseudo-metric for any domain $\Omega$ or more generally any manifold with a projective structure. This proceeds by a general universal construction whereby two properties are forced: the triangle inequality and the fact that projective maps are distance-nonincreasing (the projective *Schwarz lemma*). What we must sacrifice in general is positivity of the resulting pseudo-metric.

Let $\Omega \subset P$ be a domain. If $x, y \in \Omega$, a *chain* from $x$ to $y$ is a sequence $C$ of projective maps $f_1, \ldots, f_m \in \text{Proj}(I, \Omega)$ and pairs $a_i, b_i \in I$ such that
\[
f_1(a_1) = x, f_1(b_1) = f_2(a_2), \ldots, f_{m-1}(b_{m-1}) = f_m(a_m), f_m(b_m) = y
\]
and its length is defined as
\[
\ell(C) = \sum_{i=1}^{m} d_I(a_i, b_i).
\]
Let $\mathcal{C}(x, y)$ denote the set of all chains from $x$ to $y$. The *Kobayashi pseudo-distance* $d^Kob(x, y)$ is then defined as
\[
d^Kob(x, y) = \inf \{ \ell(C) \mid C \in \mathcal{C}(x, y) \}.
\]
The resulting function enjoys the following obvious properties:

- $d^Kob(x, y) \geq 0$;
- $d^Kob(x, x) = 0$;
- $d^Kob(x, y) = d^Kob(y, x)$;
• (The triangle inequality) $d^{\text{Kob}}(x,y) \leq d^{\text{Kob}}(y,z) + d^{\text{Kob}}(z,x)$. (The composition of a chain from $x$ to $z$ with a chain from $z$ to $y$ is a chain from $x$ to $y$.)

• (The Schwarz lemma) If $\Omega, \Omega'$ are two domains in projective spaces with Kobayashi pseudo-metrics $d, d'$ respectively and $f : \Omega \to \Omega'$ is a projective map, then $d'(f(x), f(y)) \leq d(x, y)$. (The composition of projective maps is projective.)

• The Kobayashi pseudo-metric on the interval $I$ equals the Hilbert metric on $I$.

• $d^{\text{Kob}}$ is invariant under the group $\text{Aut}(\Omega)$ consisting of all collineations of $P$ preserving $\Omega$.

**Proposition 3.3.4** (Kobayashi [85]). Let $\Omega \subset P$ be properly convex, and $x, y \in \Omega$. Then

$$d^{\text{Hilb}}(x, y) = d^{\text{Kob}}(x, y)$$

**Corollary 3.3.5.** The function $d^{\text{Hilb}} : \Omega \times \Omega \to \mathbb{R}$ is a complete metric on $\Omega$.

**Proof of Proposition 3.3.4.** Let $x, y \in \Omega$ be distinct points and let $l = \overrightarrow{xy}$ be the line incident to them. Now

$$d^{\text{Hilb}}_{\Omega}(x, y) = d^{\text{Hilb}}_{l \cap \Omega}(x, y) = d^{\text{Kob}}_{l \cap \Omega}(x, y) \leq d^{\text{Kob}}_{\Omega}(x, y)$$

by the Schwarz lemma applied to the projective map $l \cap \Omega \hookrightarrow \Omega$. For the opposite inequality, let $S$ be the intersection of a supporting hyperplane to $\Omega$ at $x_\infty$ and a supporting hyperplane to $\Omega$ at $y_\infty$. Projection from $S$ to $l$ defines a projective map

$$\Pi_{S,l} \Omega \to l \cap \Omega$$

which retracts $\Omega$ onto $l \cap \Omega$. Thus

$$d^{\text{Kob}}_{\Omega}(x, y) \leq d^{\text{Kob}}_{l \cap \Omega}(x, y) = d^{\text{Hilb}}_{l \cap \Omega}(x, y) = d^{\text{Hilb}}_{\Omega}(x, y)$$

(again using the Schwarz lemma) as desired. \qed

**Corollary 3.3.6.** Line segments in $\Omega$ are geodesics. If $\Omega \subset P$ is properly convex, $x, y \in \Omega$, then the chain consisting of a single projective isomorphism

$$I \to \overrightarrow{xy} \cap \Omega$$

minimizes the length among all chains in $\mathfrak{C}(x, y)$.

**Exercise 3.3.7.** Prove that the geodesics in $\Omega$ with respect to this metric are straight lines.
3.3. INTRINSIC METRICS

Figure 3. Projective model of a $(3, 3, 4)$-triangle tessellation of $\mathbb{H}^2$

Figure 4. Projective deformation of hyperbolic $(3, 3, 4)$-triangle tessellation
3.4. Asymptotics of hyperbolic geometry

Again, maybe make this an exercise. The point is to describe the ideal boundary of hyperbolic space as singular projective transformations using the Klein-Beltrami model. This might be a good time to relate these notions to the limit set of a discrete group. This will be used later in the “Fuchsian holonomy” section.

Perhaps do the easier case of the Poincaré upper half plane in Chapter 2, using complex numbers. Then in this section describe the mapping from the upper halfplane to the disc in RP2.
CHAPTER 4

Convex domains

4.1. Convex cones

Let \( V \) be a real vector space of dimension \( n \). A convex cone in \( V \) is a subset \( \Omega \subset V \) such that if \( t_1, t_2 > 0 \) and \( x_1, x_2 \in \Omega \), then \( t_1 x_1 + t_2 x_2 \in \Omega \). A convex cone \( \Omega \) is properly convex if it contains no complete affine line.

**Lemma 4.1.1.** Let \( \Omega \subset V \) be an open convex cone in a vector space. Then there exists a unique linear subspace \( W \subset V \) such that:
- \( \Omega \) is invariant under translation by vectors in \( W \) (that is, \( \Omega \) is \( W \)-invariant;)
- There exists a properly convex cone \( \Omega_0 \subset V/W \) such that \( \Omega = \pi^{-1}_W(\Omega_0) \) where \( \pi_W : V \rightarrow V/W \) denotes linear projection with kernel \( W \).

**Proof.** Let
\[
W = \{ w \in V \mid x + tw \in \Omega, \forall x \in \Omega, t \in \mathbb{R} \}.
\]
Then \( W \) is a linear subspace of \( V \) and \( \Omega \) is \( W \)-invariant. Let
\[
\Omega_0 = \pi_W(\Omega) \subset V/W;
\]
then \( \Omega = \pi^{-1}_W(\Omega_0) \). We must show that \( \Omega_0 \) is properly convex. To this end we can immediately reduce to the case \( W = 0 \). Suppose that \( \Omega \) contains a complete affine line \( \{ y + tw \mid t \in \mathbb{R} \} \) where \( y \in \Omega \) and \( w \in V \). Then for each \( s, t \in \mathbb{R} \)
\[
x_{s,t} = \frac{s}{s+1} x + \frac{1}{s+1} \left( y + stw \right) \in \Omega
\]
whence
\[
\lim_{s \rightarrow \infty} x_{s,t} = x + tw \in \bar{\Omega}.
\]
Thus \( x + tw \in \bar{\Omega} \) for all \( t \in \mathbb{R} \). Since \( x \in \Omega \) and \( \Omega \) is open and convex, \( x + tw \in \Omega \) for all \( t \in \mathbb{R} \) and \( w \in W \) as claimed. \( \square \)

Suppose that \( \Omega \subset V \) is a properly convex cone. Its dual cone is defined to be the set
\[
\Omega^* = \{ \psi \in V^* \mid \psi(x) > 0, \forall x \in \bar{\Omega} \}\]
where $V^*$ is the vector space dual to $V$.

**Lemma 4.1.2.** Let $\Omega \subset V$ be a properly convex cone. Then its dual cone $\Omega^*$ is a properly convex cone.

**Proof.** Clearly $\Omega^*$ is a convex cone. We must show that $\Omega^*$ is properly convex and open. Suppose first that $\Omega^*$ contains a line; then there exists $\psi_0, \lambda \in V^*$ such that $\lambda \neq 0$ and $\psi_0 + t\lambda \in \Omega^*$ for all $t \in \mathbb{R}$, that is, for each $x \in \Omega$,

$$\psi_0(x) + t\lambda(x) > 0$$

for each $t \in \mathbb{R}$. Let $x \in \Omega$; then necessarily $\lambda(x) = 0$. For if $\lambda(x) \neq 0$, there exists $t \in \mathbb{R}$ with

$$\psi_0(x) + t\lambda(x) \leq 0,$$

a contradiction. Thus $\Omega^*$ is properly convex. The openness of $\Omega^*$ follows from the proper convexity of $\Omega$. Since $\Omega$ is properly convex, its projectivization $P(\Omega)$ is a properly convex domain; in particular its closure lies in an open ball in an affine subspace $E$ of $P$ and thus the set of hyperplanes in $P$ disjoint from $P(\Omega)$ is open. It follows that $P(\Omega^*)$, and hence $\Omega^*$, is open. \hfill $\Box$

**Lemma 4.1.3.** The canonical isomorphism $V \rightarrow V^{**}$ maps $\Omega$ onto $\Omega^{**}$.

**Proof.** We shall identify $V^{**}$ with $V$; then clearly $\Omega \subset \Omega^{**}$. Since both $\Omega$ and $\Omega^{**}$ are open convex cones, either $\Omega = \Omega^{**}$ or there exists $y \in \partial \Omega \cap \Omega^{**}$. Let $H \subset V$ be a supporting hyperplane for $\Omega$ at $y$. Then the linear functional $\psi \in V^*$ defining $H$ satisfies $\psi(y) = 0$ and $\psi(x) > 0$ for all $x \in \Omega$. Thus $\psi \in \Omega^*$. But $y \in \Omega^{**}$ implies that $\psi(y) > 0$, a contradiction. \hfill $\Box$

**Theorem 4.1.4.** Let $\Omega \subset V$ be a properly convex cone. Then there exists a real analytic $\text{Aff}(\Omega)$-invariant closed 1-form $\alpha$ on $\Omega$ such that its covariant differential $\nabla \alpha$ is an $\text{Aff}(\Omega)$-invariant Riemannian metric on $\Omega$. Furthermore

$$\alpha(\rho_V) = -n < 0$$

where $\rho_V$ is the radiant vector field on $V$.

Let $d\psi$ denote a parallel volume form on $V^*$. The characteristic function $f : \Omega \rightarrow \mathbb{R}$ of the properly convex cone $\Omega$ is defined by the integral

$$f(x) = \int_{\Omega^*} e^{-\psi(x)} d\psi$$
for $x \in \Omega$. This function will define a canonical Riemannian geometry on $\Omega$ which is invariant under the automorphism group $\text{Aff}(\Omega)$ as well as a canonical diffeomorphism $\Omega \rightarrow \Omega^*$. (Note that replacing the parallel volume form $d\psi$ by another one $c \, d\psi$ replaces the characteristic function $f$ by its constant multiple $c \, f$. Thus $\Omega \rightarrow \mathbb{R}$ is well-defined only up to scaling.) For example in the one-dimensional case, where

$$\Omega = \mathbb{R}_+ \subset V = \mathbb{R}$$

the dual cone equals $\Omega^* = \mathbb{R}_+$ and the characteristic function equals

$$f(x) = \int_0^\infty e^{-\psi x} \, d\psi = \frac{1}{x}.$$

We begin by showing the integral (10) converges for $x \in \Omega$. For $x \in V$ and $t \in \mathbb{R}$ consider the hyperplane cross-section

$$V^*_x(t) = \{ \psi \in V^* \mid \psi(x) = t \}$$

and let

$$\Omega^*_x(t) = \Omega^* \cap V^*_x(t).$$

For each $x \in \Omega$ we obtain a decomposition

$$\Omega^* = \bigcup_{t>0} \Omega^*_x(t)$$

and for each $s > 0$ there is a diffeomorphism

$$\Omega^*_x(t) \xrightarrow{h_s} \Omega^*_x(st)$$

$$\psi \mapsto s\psi$$

and obviously $h_s \circ h_t = h_{st}$. We decompose the volume form $d\psi$ on $\Omega^*$ as

$$d\psi = d\psi_t \wedge dt$$

where $d\psi_t$ is an $(n-1)$-form on $V^*_x(t)$. Now the volume form $(h_s)^*d\psi_{st}$ on $\Omega^*_x(t)$ is a parallel translate of $t^{n-1}d\psi_t$. Thus:

$$f(x) = \int_0^\infty \left( e^{-t} \int_{\Omega^*_x(t)} d\psi_t \right) \, dt$$

$$= \int_0^\infty e^{-t} t^{n-1} \left( \int_{\Omega^*_x(1)} d\psi_1 \right) \, dt$$

(11)$$= (n-1)! \, \text{area}(\Omega^*_x(1)) < \infty$$

since $\Omega^*_x(1)$ is a bounded subset of $V^*_x(1)$. Since

$$\text{area}(\Omega^*_x(n)) = n^{n-1} \text{area}(\Omega^*_x(1)),$$
the formula in (11) implies:

\[ f(x) = \frac{n!}{n^n} \text{area}(\Omega^*_x(n)) \]

Let \( \Omega_c \) denote the tube domain \( \Omega + \sqrt{-1} V \subset V \otimes \mathbb{C} \). Then the integral defining \( f(x) \) converges absolutely for \( x \in \Omega_c \). It follows that \( \Omega \xrightarrow{f} \mathbb{R} \) extends to a holomorphic function \( \Omega_c \rightarrow \mathbb{C} \) from which it follows that \( f \) is real analytic on \( \Omega \).

**Lemma 4.1.5.** The function \( f(x) \rightarrow +\infty \) as \( x \rightarrow \partial \Omega \).

**Proof.** Consider a sequence \( \{x_n\}_{n>0} \) in \( \Omega \) converging to \( x_{\infty} \in \partial \Omega \). Then the functions

\[
F_k : \Omega^* \rightarrow \mathbb{R} \\
\psi \mapsto e^{-\psi(x_k)}
\]

are nonnegative functions converging uniformly to \( F_\infty \) on every compact subset of \( \Omega^* \) so that

\[
\lim \inf f(x_k) = \lim \inf \int_{\Omega^*} F_k(\psi)d\psi \geq \int_{\Omega^*} F_\infty(\psi)d\psi.
\]

Suppose that \( \psi_0 \in V^* \) defines a supporting hyperplane to \( \Omega \) at \( x_{\infty} \); then \( \psi_0(x_{\infty}) = 0 \). Let \( K \subset \Omega^* \) be a closed ball; then \( K + \mathbb{R}_+ \psi_0 \) is a cylinder in \( \Omega^* \) with cross-section \( K_1 = K \cap \psi_0^{-1}(c) \) for some \( c > 0 \).

\[
\int_{\Omega^*} F_\infty(\psi)d\psi \geq \int_{K + \mathbb{R}_+ \psi_0} e^{-\psi(x_{\infty})}d\psi \\
\geq \int_{K_1} \left( \int_0^\infty dt \right) e^{-\psi(x_{\infty})}d\psi_1 = \infty
\]

where \( d\psi_1 \) is a volume form on \( K_1 \).

**Lemma 4.1.6.** If \( \gamma \in \text{Aff}(\Omega) \subset \text{GL}(V) \) is an automorphism of \( \Omega \), then

\[ f \circ \gamma = \det(\gamma)^{-1} \cdot f \]

In other words, if \( dx \) is a parallel volume form on \( E \), then \( f(x)dx \) defines an \( \text{Aff}(\Omega) \)-invariant volume form on \( \Omega \).
Proof.

\[ f(\gamma x) = \int_{\Omega^*} e^{-\psi(\gamma x)} \, d\psi \]
\[ = \int_{\gamma^{-1} \Omega^*} e^{-\psi(x)} \gamma^* d\psi \]
\[ = \int_{\Omega^*} e^{-\psi(x)} (\det \gamma)^{-1} \, d\psi \]
\[ = (\det \gamma)^{-1} f(x) \]
\[ \square \]

Since \( \det(\gamma) \) is a constant, it follows from (13) that \( \log f \) transforms under \( \gamma \) by the additive constant \( \log \det(\gamma)^{-1} \) and thus

\[ \alpha = d \log f = f^{-1} df \]

is an \( \text{Aff}(\Omega) \)-invariant closed 1-form on \( \Omega \). Furthermore, taking \( \gamma \) to be the homothety

\[ x \mapsto s x, \]

implies:

\[ f \circ h_s = s^{-n} f, \]

which by differentiation with respect to \( s \) yields:

\[ \alpha(\rho_V) = -n. \]

Let \( X \in T_x \Omega \cong V \) be a tangent vector; then \( df(x) \in T^*_x \Omega \) maps

\[ X \mapsto -\int_{\Omega^*} \psi(X) e^{-\psi(x)} d\psi. \]

Using the identification \( T^*_x \Omega \cong V^* \) we obtain a map

\[ \Phi : \Omega \to V^* \]
\[ x \mapsto -d \log f(x). \]

As a linear functional, \( \Phi(x) \) maps \( X \in V \) to

\[ \frac{\int_{\Omega^*} \psi(X) e^{-\psi(x)} \, d\psi}{\int_{\Omega^*} e^{-\psi(x)} \, d\psi} \]

so if \( X \in \Omega \), the numerator is positive and \( \Phi(x) > 0 \) on \( \Omega \). Thus \( \Phi : \Omega \to \Omega^* \). Furthermore by decomposing the volume form on \( \Omega^* \) we
obtain
\[
\Phi(x) = \frac{\int_0^\infty e^{-tn} \left( \int_{\Omega^*_x(1)} \psi_1 d\psi \right) dt}{\int_0^\infty e^{-tn-1} \left( \int_{\Omega^*_x(1)} d\psi \right) dt}
\]
\[
= n \frac{\int_{\Omega^*_x(1)} \psi_1 d\psi dt}{\int_{\Omega^*_x(1)} d\psi dt}
\]
\[
= n \text{ centroid}(\Omega^*_x(1)).
\]
Since
\[
\Phi(x) \in n \cdot \Omega^*_x(1) = \Omega^*_x(n),
\]
that is, \( \Phi(x) : x \mapsto n \).

(14) \( \Phi(x) = \text{ centroid}(\Omega^*_x(n)) \).

The logarithmic Hessian \( d^2 \log f = \nabla d \log f = \nabla \alpha \) is an Aff(\Omega)-invariant symmetric 2-form on \( \Omega \). Now for any function \( f : \Omega \to \mathbb{R} \) we have
\[
d^2(\log f) = \nabla (f^{-1} df) = f^{-1} d^2 f - (f^{-1} df)^2
\]
and \( d^2 f(x) \in S^2 T^*_x \Omega \) assigns to a pair \( (X,Y) \in T_x \Omega \times T_x \Omega = V \times V \)
\[
\int_{\Omega^*} \psi(X) \psi(Y) e^{-\psi(x)} d\psi
\]
We claim that \( d^2 \log f \) is positive definite:
\[
f(x)^2 \left( d^2 \log f(x) \right)(X,X) = \int_{\Omega^*} e^{-\psi(x)} d\psi \int_{\Omega^*} \psi(X)^2 e^{-\psi(x)} d\psi
\]
\[
- \left( \int_{\Omega^*} \psi(X) e^{-\psi(x)} d\psi \right)^2
\]
\[
= \|e^{-\psi(x)/2}\|_2^2 \|\psi(X) e^{-\psi(x)/2}\|_2^2
\]
\[
- \langle e^{-\psi(x)/2}, \psi(X) e^{-\psi(x)/2} \rangle_2 > 0
\]
by the Schwartz inequality, since the functions
\[
\psi \mapsto e^{-\psi(x)/2},
\]
\[
\psi \mapsto \psi(X) e^{-\psi(x)/2}
\]
on \( \Omega^* \) are not proportional. (Here \( \langle , \rangle_2 \) and \( \| \|_2 \) respectively denote the usual \( L^2 \) inner product and norm on \( (\Omega^*, d\psi) \).) Therefore \( d^2 \log f \) is positive definite and hence defines an Aff(\Omega)-invariant Riemannian metric on \( \Omega \).

We can characterize the linear functional \( \Phi(x) \in \Omega^* \) quite simply. Since \( \Phi(x) \) is parallel to \( df(x) \), each of its level hyperplanes is parallel
to the tangent plane of the level set $S_x$ of $\Omega \xrightarrow{f} \mathbb{R}$ containing $x$. Since $\Phi(x)(x) = n$, we obtain:

**Proposition 4.1.7.** The tangent space to the level set $S_x$ of $\Omega \xrightarrow{f} \mathbb{R}$ at $x$ equals $\Phi(x)^{-1}(n)$.

This characterization yields the following result:

**Theorem 4.1.8.** $\Omega \xrightarrow{\Phi} \Omega^*$ is bijective.

**Proof.** Let $\psi_0 \in \Omega^*$ and let $Q_0 := \{z \in V \mid \psi_0(z) = n\}$.

Then the restriction of $\log f$ to the affine hyperplane $Q_0$ is a convex function which approaches $+\infty$ on $\partial(Q_0 \cap \Omega)$. Therefore the restriction $f|_{Q_0 \cap \Omega}$ has a unique critical point $x_0$, which is necessarily a minimum. Then $T_{x_0}S_{x_0} = Q_0$ from which Proposition 4.1.7 implies that $\Phi(x_0) = \psi_0$. Furthermore, if $\Phi(x) = \psi_0$, then $f|_{Q_0 \cap \Omega}$ has a critical point at $x$ so $x = x_0$. Therefore $\Omega \xrightarrow{\Phi} \Omega^*$ is bijective as claimed. \hfill \Box

If $\Omega \subset V$ is a properly convex cone and $\Omega^*$ is its dual, then let $\Phi_{\Omega^*} : \Omega^* \longrightarrow \Omega$ be the diffeomorphism $\Omega^* \longrightarrow \Omega^{**} = \Omega$ defined above. If $x \in \Omega$, then $\psi = (\Phi^*)^{-1}(x)$ is the unique $\psi \in \Omega^*$ such that:

- $\psi(x) = n$;
- The centroid of $\Omega \cap \psi^{-1}(n)$ equals $x$.

The duality isomorphism $\text{GL}(V) \rightarrow \text{GL}(V^*)$ (given by inverse transpose of matrices) defines an isomorphism $\text{Aff}(\Omega) \rightarrow \text{Aff}(\Omega^*)$. Let

$$\Omega \xrightarrow{\Phi_{\Omega^*}} \Omega^*, \quad \Omega^* \xrightarrow{\Phi_{\Omega^*}} \Omega^{**} = \Omega$$

be the duality maps for $\Omega$ and $\Omega^*$ respectively. Vinberg points out in [148] that in general the composition

$$\Phi_{\Omega^*} \circ \Phi_{\Omega^*} : \Omega \rightarrow \Omega$$

is not the identity, although if $\Omega$ is homogeneous, that is, $\text{Aff}(\Omega) \subset \text{GL}(V)$ acts transitively on $\Omega$, then $\Omega_{\Omega^*} \circ \Phi_{\Omega^*} = \text{id}_{\Omega}$:

**Proposition 4.1.9 (Vinberg [148]).** Let $\Omega \subset V$ be a homogeneous properly convex cone. Then $\Phi_{\Omega^*}$ and $\Phi_{\Omega}$ are inverse maps $\Omega^* \leftrightarrow \Omega$.

**Proof.** Let $x \in \Omega$ and $Y \in V \cong T_x \Omega$ be a tangent vector. Denote by

$$g_x : T_x \Omega \times T_x \Omega \xrightarrow{g_x} \mathbb{R}$$
the canonical Riemannian metric $\nabla \alpha = d^2 \log f$ at $x$. Then the differential of $\Phi_\Omega : \Omega \rightarrow \Omega^*$ at $x$ is the composition

$$T_x \Omega \xrightarrow{\tilde{g}_x} T^*_x \Omega \cong V^* \cong T_{\Phi(x)} \Omega^*$$

where $T_x \Omega \xrightarrow{\tilde{g}_x} T^*_x \Omega$ is the linear isomorphism corresponding to $g_x$ and the isomorphisms

$$T^*_x \Omega \cong V^* \cong T_{\Phi(x)} \Omega^*$$

are induced by parallel translation. Taking the directional derivative of the equation

$$\alpha_x(\rho) = -n$$

with respect to $Y \in V \cong T_x \Omega$, we obtain:

$$0 = (\nabla_Y \alpha)(\rho) + \alpha(\nabla_Y \rho)$$

$$= g_x(\rho_x, Y) + \alpha_x(Y)$$

$$= g_x(x, Y) - \Phi(x)(Y).$$

(15)

Let $\Omega \xrightarrow{f_\Omega} \mathbb{R}$ and $\Omega^* \xrightarrow{f_{\Omega^*}} \mathbb{R}$ be the characteristic functions for $\Omega$ and $\Omega^*$ respectively. Then $f_\Omega(x) \, dx$ is a volume form on $\Omega$ invariant under $\text{Aff}(\Omega)$ and $f_{\Omega^*}(\psi) \, d\psi$ is a volume form on $\Omega^*$ invariant under the induced action of $\text{Aff}(\Omega)$ on $\Omega^*$. Moreover $\Omega \xrightarrow{\Phi} \Omega^*$ is equivariant with respect to the isomorphism $\text{Aff}(\Omega) \rightarrow \text{Aff}(\Omega^*)$. Therefore the tensor field on $\Omega$ defined by

$$f_\Omega(x) \, dx \otimes (f_{\Omega^*} \circ \Phi)(x) \, d\psi$$

$$\in \wedge^n T_x \Omega \omega \otimes \wedge^n T_{\Phi(x)} \Omega^*$$

$$\cong \wedge^n V \otimes \wedge^n V^*$$

is $\text{Aff}(\Omega)$-invariant. Since the tensor field $dx \otimes d\psi \in \wedge^n V \otimes \wedge^n V^*$ is invariant under all of $\text{Aff}(V)$, the coefficient

$$h(x) = f_\Omega(x) \, dx \otimes (f_{\Omega^*} \circ \Phi(x) \, d\psi)$$

is an $\text{Aff}(\Omega)$-invariant function on $\Omega$. Since $\Omega$ is homogeneous, $h$ is constant.

Differentiate $\log h$:

$$0 = d \log f_\Omega(x) + d \log(f_{\Omega^*} \circ \Phi)(x).$$

Since $d \log f_{\Omega^*}(\psi) = \Phi_{\Omega^*}(\psi)$,

$$0 = -\Phi(x)(Y) + \Phi_{\Omega^*}(d\Phi(Y))$$

$$= -\Phi(x)(Y) + g_x(Y, \Phi_{\Omega^*} \circ \Phi_{\Omega}(x))$$

Combining this equation with (15) yields:

$$\Phi_{\Omega^*} \circ \Phi_{\Omega}(x) = x$$
as desired. □

Thus, if $\Omega$ is a homogeneous cone, then $\Phi(x) \in \Omega^*$ is the centroid of the cross-section $\Omega_x^*(n) \subset \Omega^*$ in $V^*$.

4.2. Convex bodies in projective space

Let $P = P(V)$ and $P^* = P(V^*)$ be the associated projective spaces. Then the projectivization $P(\Omega) \subset P$ of $\Omega$ is by definition a properly convex domain and its closure $K = \overline{P(\Omega)}$ a convex body. Then the dual convex body $K^*$ equals the closure of the projectivization $P(\Omega^*)$ consisting of all hyperplanes $H \subset P$ such that $\Omega \cap H = \emptyset$. A pointed convex body consists of a pair $(K, x)$ where $K$ is a convex body and $x \in \text{int}(K)$ is an interior point of $K$. Let $H \subset P$ be a hyperplane and $E = P - H$ its complementary affine space. We say that the pointed convex body $(K, u)$ is centered relative to $E$ (or $H$) if $u$ is the centroid of $K$ in the affine geometry of $E$.

**Proposition 4.2.1.** Let $(K, u)$ be a pointed convex body in a projective space $P$. Then there exists a hyperplane $H \subset P$ disjoint from $K$ such that in the affine space $E = P - H$, the centroid of $K \subset E$ equals $u$.

**Proof.** Let $V = V(P)$ be the vector space corresponding to the projective space $P$ and let $\Omega \subset V$ be a properly convex cone whose projectivization is the interior of $K$. Let $x \in \Omega$ be a point corresponding to $u \in \text{int}(K)$. Let

$$\Omega^* \xrightarrow{\Phi_{\Omega^*}} \Omega$$

be the duality map for $\Omega^*$ and let $\psi = (\Phi_{\Omega^*})^{-1}(y)$. Then the centroid of the cross-section

$$\Omega_\psi(n) = \{x \in \Omega \mid \psi(x) = n\}$$

in the affine hyperplane $\psi^{-1}(n) \subset V$ equals $y$. Let $H = P(\text{Ker}(\psi))$ be the projective hyperplane in $P$ corresponding to $\psi$; then projectivization defines an affine isomorphism

$$\psi^{-1}(n) \longrightarrow P - H$$

mapping $\Omega_\psi(n) \longrightarrow K$. Since affine maps preserve centroids, it follows that $(K, u)$ is centered relative to $H$. □

Thus every pointed convex body $(K, u)$ is centered relative to a unique affine space containing $K$. In dimension one, this means the following: let $K \subset \mathbb{RP}^1$ be a closed interval $[a, b] \subset \mathbb{R}$ and let $a < x < b$ be an
interior point. Then \( x \) is the midpoint of \([a, b]\) relative to the “hyper-plane” \( H \) obtained by projectively reflecting \( x \) with respect to the pair \( \{a, b\} \):

\[
H = R_{[a,b]}(x) = \frac{(a + b)x - 2ab}{2x - (a + b)}
\]

An equivalent version of Proposition 4.2.1 involves using collineations to “move a pointed convex body” into affine space to center it:

**Proposition 4.2.2.** Let \( K \subset E \) be a convex body in an affine space and let \( x \in \text{int}(A) \) be an interior point. Let \( P \supset E \) be the projective space containing \( E \). Then there exists a collineation \( P \xrightarrow{g} P \) such that:

- \( g(K) \subset E \);
- \( (g(K), g(x)) \) is centered relative to \( E \).

The one-dimensional version of this is just the fundamental theorem of projective geometry: if \([a, b]\) is a closed interval with interior point \( x \), then there is a unique collineation taking \( a \mapsto -1, \quad x \mapsto 0, \quad b \mapsto 1 \) thereby centering \(([a, b], x) \in \mathcal{C}_*(P) \).

**Proposition 4.2.3.** Let \( K_i \subset E \) be convex bodies \((i = 1, 2) \) in an affine space \( E \) with centroids \( u_i \), and suppose that \( P \xrightarrow{g} P \) is a collineation such that \( g(K_1) = K_2 \) and \( g(u_1) = u_2 \). Then \( g \) is an affine automorphism of \( E \), that is, \( g(A) = E \).

**Proof.** Let \( V \) be a vector space containing \( E \) as an affine hyperplane and let \( \Omega_i \) be the properly convex cones in \( V \) whose projective images are \( K_i \). By assumption there exists a linear map \( V \xrightarrow{\tilde{g}} V \) and points \( x_i \in \Omega_i \) mapping to \( u_i \in K_i \) such that \( \tilde{g}(\Omega_1) = \Omega_2 \) and \( \tilde{g}(x_1) = x_2 \). Let \( S_i \subset \Omega_i \) be the level set of the characteristic function \( \Omega_i \xrightarrow{f_i} \mathbb{R} \) containing \( x_i \). Since \((K_i, u_i)\) is centered relative to \( E \), it follows that the tangent plane \( T_{x_i}S_i = E \subset V \).

Since the construction of the characteristic function is linearly invariant, it follows that \( \tilde{g}(S_1) = S_2 \). Moreover

\[
\tilde{g}(T_{x_1}S_1) = T_{x_2}S_2,
\]

that is, \( \tilde{g}(A) = E \) and \( g \in \text{Aff}(A) \) as desired. \(\square\)
4.3. Spaces of convex bodies in projective space

Let \( \mathcal{C}(\mathbb{P}) \) denote the set of all convex bodies in \( \mathbb{P} \), with the topology induced from the Hausdorff metric on the set of all closed subsets of \( \mathbb{P} \) (which is induced from the Fubini-Study metric on \( \mathbb{P} \)). Let

\[
\mathcal{C}^*(\mathbb{P}) = \{(K, x) \in \mathcal{C}(\mathbb{P}) \times \mathbb{P} \mid x \in \text{int}(K)\}
\]

be the corresponding set of pointed convex bodies, with a topology induced from the product topology on \( \mathcal{C}(\mathbb{P}) \times \mathbb{P} \). The collineation group \( G \) acts continuously on \( \mathcal{C}(\mathbb{P}) \) and on \( \mathcal{C}^*(\mathbb{P}) \). Recall that an action of a group \( \Gamma \) on a space \( X \) is syndetic if there exists a compact subset \( K \subset X \) such that \( \Gamma K = X \).

**Theorem 4.3.1 (Benzécri).** The collineation group \( G \) acts properly and syndetically on \( \mathcal{C}^*(\mathbb{P}) \). In particular the quotient \( \mathcal{C}^*(\mathbb{P})/G \) is a compact Hausdorff space.

While the quotient \( \mathcal{C}^*(\mathbb{P})/G \) is Hausdorff, the space of equivalence classes of convex bodies \( \mathcal{C}(\mathbb{P})/G \) is generally not Hausdorff. Some basic examples are the following. Suppose that \( \Omega \) is a properly convex domain whose boundary is not \( C^1 \) at a point \( x_1 \). Then \( \partial \Omega \) has a "corner" at \( x_1 \) and we may choose homogeneous coordinates so that \( x_1 = [1 : 0 : 0] \) and \( \Omega \) lies in the domain

\[
\Delta = \{[x : y : z] \in \mathbb{RP}^2 \mid x, y, z > 0\}
\]

in such a way that \( \partial \Omega \) is tangent to \( \partial \Delta \) at \( x_1 \). Under the one-parameter group of collineations defined by

\[
g_t = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^t \end{bmatrix}
\]

as \( t \to +\infty \), the domains \( g_t \Omega \) converge to \( \Delta \). It follows that the \( G \)-orbit of \( \Omega \) in \( \mathcal{C}(\mathbb{P}) \) is not closed and the equivalence class of \( \Omega \) is not a closed point in \( \mathcal{C}(\mathbb{P})/G \) unless \( \Omega \) was already a triangle.

Similarly suppose that \( \Omega \) is a properly convex domain which is not strictly convex, that is, its boundary contains a nontrivial line segment \( \sigma \). (We suppose that \( \sigma \) is a maximal line segment contained in \( \partial \Omega \).) As above, we may choose homogeneous coordinates so that \( \Omega \subset \Delta \) and such that \( \Omega \cap \Delta = \overline{\sigma} \) and \( \sigma \) lies on the line \( \{[x : y : 0] \mid x, y \in \mathbb{R}\} \). As \( t \to +\infty \) the image of \( \Omega \) under the collineation

\[
g_t = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{2t} \end{bmatrix}
\]
converges to a triangle region with vertices \(\{0 : 0 : 1\}\) \(\cup\) \(\partial\sigma\). As above, the equivalence class of \(\Omega\) in \(\mathfrak{C}(P)/G\) is not a closed point in \(\mathfrak{C}(P)/G\) unless \(\Omega\) is a triangle.

As a final example, consider a properly convex domain \(\Omega\) with \(C^1\) boundary such that there exists a point \(u \in \partial\Omega\) such that \(\partial\Omega\) is \(C^2\) at \(u\). In that case a conic \(C\) osculates \(\partial\Omega\) at \(u\). Choose homogeneous coordinates such that \(u = [1 : 0 : 0]\) and

\[
C = \{[x : y : z] \mid xy + z^2 = 0\}.
\]

Then as \(t \rightarrow +\infty\) the image of \(\Omega\) under the collineation

\[
g_t = \begin{bmatrix}
e^{-t} & 0 & 0 \\0 & e^t & 0 \\0 & 0 & 1
\end{bmatrix}
\]

converges to the convex region

\[
\{[x : y : z] \mid xy + z^2 < 0\}
\]

bounded by \(C\). As above, the equivalence class of \(\bar{\Omega}\) in \(\mathfrak{C}(P)/G\) is not a closed point in \(\mathfrak{C}(P)/G\) unless \(\partial\Omega\) is a conic.

In summary:

**Proposition 4.3.2.** Suppose \(\bar{\Omega} \subset \mathbb{RP}^2\) is a convex body whose equivalence class \([\bar{\Omega}]\) is a closed point in \(\mathfrak{C}(P)/G\). Suppose that \(\partial\Omega\) is neither a triangle nor a conic. Then \(\partial\Omega\) is a \(C^1\) strictly convex curve which is nowhere \(C^2\).

The forgetful map \(\mathfrak{C}_+(P) \xrightarrow{\pi_P} \mathfrak{C}(P)\) which forgets the point of a pointed convex body is induced from Cartesian projection \(\mathfrak{C}(P) \times P \rightarrow \mathfrak{C}(P)\).

**Theorem 4.3.3 (Benzécri).** Let \(\Omega \subset P\) is a properly convex domain such that there exists a subgroup \(\Gamma \subset \text{Aut}(\Omega)\) which acts syndetically on \(\Omega\). Then the corresponding point \([\bar{\Omega}]\) \(\in \mathfrak{C}(P)/G\) is closed.

In the following result, all but the continuous differentiability of the boundary in the following result was originally proved in Kuiper [Kp] using a somewhat different technique; the \(C^1\) statement is due to Benzécri [B2] as well as the proof given here.

**Corollary 4.3.4.** Suppose that \(M = \Omega/\Gamma\) is a convex \(\mathbb{RP}^2\)-manifold such that \(\chi(M) < 0\). Then either the \(\mathbb{RP}^2\)-structure on \(M\) is a hyperbolic structure or the boundary \(\partial\Omega\) of its universal covering is a \(C^1\) strictly convex curve which is nowhere \(C^2\).

**Proof.** Apply Proposition 4.3.2 to Theorem 4.3.3. \(\square\)
Proof of Theorem 4.3.3 assuming Theorem 4.3.1. Let $\Omega$ be a properly convex domain with an automorphism group $\Gamma \subset \text{Aff}(\Omega)$ acting syndetically on $\Omega$. It suffices to show that the $G$-orbit of $\{\Omega\}$ in $\mathcal{C}(\mathbb{P})$ is closed, which is equivalent to showing that the $G$-orbit of

$$\Pi^{-1}(\{\Omega\}) = \{\Omega\} \times \Omega$$

in $\mathcal{C}_*(\mathbb{P})$ is closed. This is equivalent to showing that the image of $\{\Omega\} \times \Omega \subset \mathcal{C}_*(\mathbb{P})$ under the quotient map $\mathcal{C}_*(\mathbb{P}) \to \mathcal{C}_*(\mathbb{P})/G$ is closed. Let $K \subset \Omega$ be a compact subset such that $\Gamma K = \Omega$; then $\{\Omega\} \times K$ and $\{\Omega\} \times \Omega$ have the same image in $\mathcal{C}_*(\mathbb{P})/G$ and hence in $\mathcal{C}_*(\mathbb{P})/G$. Hence it suffices to show that the image of $\{\Omega\} \times K$ in $\mathcal{C}_*(\mathbb{P})/G$ is closed. Since $K$ is compact and the composition

$$K \to \{\Omega\} \times K \to \{\Omega\} \times \Omega \subset \mathcal{C}_*(\mathbb{P}) \to \mathcal{C}_*(\mathbb{P})/G$$

is continuous, it follows that the image of $K$ in $\mathcal{C}_*(\mathbb{P})/G$ is closed. By Theorem 4.3.1, $\mathcal{C}_*(\mathbb{P})/G$ is Hausdorff and hence the image of $K$ in $\mathcal{C}_*(\mathbb{P})/G$ is closed, as desired. The proof of Theorem 4.3.3 (assuming Theorem 4.3.1) is now complete. □

Now we prove Theorem 4.3.1. Choose a fixed hyperplane $H_\infty \subset \mathbb{P}$ and let $E = \mathbb{P} - H_\infty$ be the corresponding affine patch and $\text{Aff}(E)$ the group of affine automorphisms of $E$. Let $\mathcal{C}(A) \subset \mathcal{C}(\mathbb{P})$ denote the set of convex bodies $K \subset E$, with the induced topology. (Note that the $\mathcal{C}(A)$ is a complete metric space with respect to the Hausdorff metric induced from the Euclidean metric on $E$ and we may use this metric to define the topology on $\mathcal{C}(A)$. The inclusion map $\mathcal{C}(A) \hookrightarrow \mathcal{C}(\mathbb{P})$ is continuous, although not uniformly continuous.) We define a map

$$\mathcal{C}(A) \hookrightarrow \mathcal{C}_*(\mathbb{P})$$

as follows. Let $K \in \mathcal{C}(A)$ be a convex body in affine space $A$; let $\iota(K)$ to be the pointed convex body

$$\iota(K) = (K, \text{centroid}(K)) \in \mathcal{C}_*(\mathbb{P});$$

clearly $\iota$ is equivariant respecting the embedding $\text{Aff}(A) \hookrightarrow G$.

4.3.1. Collineations and convex bodies.

Theorem 4.3.5. Let $A \subset \mathbb{P}$ be an affine patch in projective space. Then the map

$$\mathcal{C}(A) \hookrightarrow \mathcal{C}_*(\mathbb{P})$$

$$K \mapsto (K, \text{centroid}(K))$$
is equivariant with respect to the inclusion \( \text{Aff}(A) \longrightarrow G \) and the corresponding homomorphism of topological transformation groupoids

\[
\left( \mathfrak{C}(A), \text{Aff}(A) \right) \xrightarrow{\iota} \left( \mathfrak{C}^*(P), G \right)
\]

is an equivalence of groupoids.

**Proof.** The surjectivity of \( \mathfrak{C}(A)/\text{Aff}(A) \xrightarrow{\iota^*} \mathfrak{C}^*(P)/G \) follows immediately from Proposition 4.1.9 and the bijectivity of

\[
\text{Hom}(a, b) \xrightarrow{\iota^*} \text{Hom}(\iota(a), \iota(b))
\]

follows immediately from Proposition 4.2.2. □

Thus the proof of Proposition 4.2.3 reduces (via Lemma A.0.1 and Theorem 4.3.5) to the following:

**Theorem 4.3.6.** \( \text{Aff}(A) \) acts properly and syndetically on \( \mathfrak{C}(A) \).

Let \( \mathcal{E} \subset \mathfrak{C}(A) \) denote the subspace of ellipsoids in \( A \); the affine group \( \text{Aff}(A) \) acts transitively on \( \mathcal{E} \) with isotropy group the orthogonal group — in particular this action is proper. If \( K \in \mathfrak{C}(A) \) is a convex body, then there exists a unique ellipsoid \( \text{ell}(K) \in \mathcal{E} \) (the ellipsoid of inertia of \( K \)) such that for each affine map \( \psi : A \longrightarrow \mathbb{R} \) such that \( \psi(\text{centroid}(K)) = 0 \) the moments of inertia satisfy:

\[
\int_K \psi^2 \, dx = \int_{\text{ell}(K)} \psi^2 \, dx
\]

**Proposition 4.3.7.** Taking the ellipsoid-of-inertia of a convex body

\[
\mathfrak{C}(A) \xrightarrow{\text{ell}} \mathcal{E}
\]

defines an \( \text{Aff}(A) \)-invariant proper retraction of \( \mathfrak{C}(A) \) onto \( \mathcal{E} \).

**Proof of Theorem 4.3.6 assuming Proposition 4.3.7.**

Since \( \text{Aff}(A) \) acts properly and syndetically on \( \mathcal{E} \) and \( \text{ell} \) is a proper map, it follows that \( \text{Aff}(A) \) acts properly and syndetically on \( \mathfrak{C}(A) \). □

**Proof of Proposition 4.3.7.** \( \text{ell} \) is clearly affinely invariant and continuous. Since \( \text{Aff}(A) \) acts transitively on \( \mathcal{E} \), it suffices to show that a single fiber \( \text{ell}^{-1}(e) \) is compact for \( e \in \mathcal{E} \). We may assume that \( e \) is the unit sphere in \( \mathcal{E} \) centered at the origin 0. Since the collection of compact subsets of \( \mathcal{E} \) which lie between two compact balls is compact subset of \( \mathfrak{C}(A) \), Proposition 4.3.7 will follow from:

**Proposition 4.3.8.** For each \( n \) there exist constants \( 0 < r(n) < R(n) \) such that every convex body \( K \subset \mathbb{R}^n \) whose centroid is the origin and whose ellipsoid-of-inertia is the unit sphere satisfies

\[
B_{r(n)}(O) \subset K \subset B_{R(n)}(O).
\]
The proof of Proposition 4.3.8 is based on:

**Lemma 4.3.9.** Let \( K \subset E \) be a convex body with centroid \( O \). Suppose that \( l \) is a line through \( O \) which intersects \( \partial K \) in the points \( X, X' \). Then

\[
\frac{1}{n} \leq \frac{OX}{OX'} \leq n.
\]

**Proof.** Let \( \psi \in E^\ast \) be a linear functional such that \( \psi(X) = 0 \) and \( \psi^{-1}(1) \) is a supporting hyperplane for \( K \) at \( X' \); then necessarily \( 0 \leq \psi(x) \leq 1 \) for all \( x \in K \). We claim that

\[
\psi(O) \leq \frac{n}{n+1}.
\]

For \( t \in \mathbb{R} \) let

\[
E \xrightarrow{h_t} E \quad x \mapsto t(x - X) + X
\]

be the homothety fixing \( X \) having strength \( t \). We compare the linear functional \( \psi \) with the “polar coordinates” on \( K \) defined by the map

\[
[0, 1] \times \partial K \xrightarrow{F} K \quad (t, s) \mapsto h_t s
\]

which is bijective on \((0, 1] \times \partial K\) and collapses \( \{0\} \times \partial K \) onto \( X \). Thus there is a well-defined function \( K \xrightarrow{\mu} \mathbb{R} \) such that for each \( x \in K \), there exists \( x' \in \partial K \) such that \( x = F(t, x') \). Since \( 0 \leq \psi(F(t, s)) \leq 1 \), it follows that for \( x \in K \),

\[
0 \leq \psi(x) \leq t(x)
\]

Let \( \mu = \mu_K \) denote the probability measure supported on \( K \) defined by

\[
\mu(S) = \frac{\int_{S \cap K} dx}{\int_K dx}.
\]

There exists a measure \( \nu \) on \( \partial K \) such that for each measurable function \( f : E \to \mathbb{R} \)

\[
\int f(x) d\mu(x) = \int_{t=0}^{1} \int_{s \in \partial K} f(ts) t^{n-1} d\nu(s) dt,
\]

that is, \( F^* d\mu = t^{n-1} d\nu \wedge dt \).

The first moment of \( K \to [0, 1] \) is:

\[
\bar{t}(K) = \int_K t \, d\mu = \frac{\int_K t \, d\mu}{\int_K d\mu} = \frac{\int_0^1 t^n \int_{\partial K} d\nu \, dt}{\int_0^1 t^{n-1} \int_{\partial K} d\nu \, dt} = \frac{n}{n+1}
\]
and since the value of the affine function \( \psi \) on the centroid equals the first moment of \( \psi \) on \( K \), we have

\[
0 < \psi(O) = \int_K \psi(x) \, d\mu(x) < \int_K t(x) \, d\mu = \frac{n}{n+1}.
\]

Now the distance function on the line \( \overrightarrow{XX'} \) is affinely related to the linear functional \( \psi \), that is, there exists a constant \( c > 0 \) such that for \( x \in \overrightarrow{XX'} \) the distance \( Xx = c|\psi(x)| \); since \( \psi(X') = 1 \) it follows that

\[
\psi(x) = \frac{Xx}{XX'}
\]

and since \( OX + OX' = XX' \) it follows that

\[
\frac{OX'}{OX} = \frac{XX'}{OX} - 1 \geq \frac{n+1}{n} - 1 = \frac{1}{n}.
\]

This gives the second inequality of (16). The first inequality follows by reversing the roles of \( X, X' \). \( \square \)

**Proof of Proposition 4.3.8.** Let \( X \in \partial K \) be a point at minimum distance from the centroid \( O \); then there exists a supporting hyperplane \( H \) at \( x \) which is orthogonal to \( \overrightarrow{OX} \) and let \( E \xrightarrow{\psi} \mathbb{R} \) be the corresponding linear functional of unit length. Let \( a = \psi(X) > 0 \) and \( b = \psi(X') < 0 \); Proposition 4.3.7 implies \(-b < na\).

We claim that \( 0 < |\psi(x)| < na \) for all \( x \in K \). To this end let \( x \in K \); we may assume that \( \psi(x) > 0 \) since \(-na < \psi(X') \leq \psi(x)\). Furthermore we may assume that \( x \in \partial K \). Let \( z \in \partial K \) be the other point of intersection of \( \overrightarrow{Ox} \) with \( \partial K \); then \( \psi(z) < 0 \). Now

\[
\frac{1}{n} \leq \frac{Oz}{Ox} \leq n
\]

implies that

\[
\frac{1}{n} \leq \frac{|\psi(z)|}{|\psi(x)|} \leq n
\]

(since the linear functional \( \psi \) is affinely related to signed distance along \( \overrightarrow{Ox} \)). Since \( 0 > \psi(z) \geq -a \), it follows that \( |\psi(x)| \leq na \) as claimed.

Let \( w_n \) denote the moment of inertia of \( \psi \) for the unit sphere; then we have

\[
w_n = \int_K \psi^2 d\mu \leq \int_K n^2 a^2 d\mu = n^2 a^2
\]

whence \( a \geq \sqrt{w_n}/n \). Taking \( r(n) = \sqrt{w_n}/n \) we see that \( K \) contains the \( r(n) \)-ball centered at \( O \).
To obtain the upper bound, observe that if $C$ is a right circular cone with vertex $X$, altitude $h$ and base a sphere of radius $\rho$ and $C \to [0, h]$ is the altitudinal distance from the base, then the integral
\[
\int_C t^2 d\mu = \frac{2h^3 \rho^{n-1} v_{n-1}}{(n + 2)(n + 1)n}
\]
where $v_{n-1}$ denotes the $(n - 1)$-dimensional volume of the unit $(n - 1)$-ball. Let $X \in \partial K$ and $C$ be a right circular cone with vertex $X$ and base an $(n - 1)$-dimensional ball of radius $r(n)$. We have just seen that $K$ contains $B_{r(n)}(O)$; it follows that $K \supset C$. Let $K \to \mathbb{R}$ be the unit-length linear functional vanishing on the base of $C$; then
\[
t(X) = h = OX.
\]
Its second moment is
\[
w_n = \int_K t^2 d\mu \geq \int_C t^2 d\mu = \frac{2h^3 r(n)^{n-1} v_{n-1}}{(n + 2)(n + 1)n}
\]
and thus it follows that
\[
OX = h \leq R(n)
\]
where
\[
R(n) = \left(\frac{(n + 2)(n + 1)nw_n}{2r(n)^{n-1}v_{n-1}}\right)^{\frac{1}{3}}
\]
as desired. The proof is now complete. \hfill \square

The volume of the unit ball in $\mathbb{R}^n$ is given by
\[
v_n = \begin{cases} 
\pi^{n/2} / (n/2)! & \text{for } n \text{ even} \\
2^{(n+1)/2} \pi^{(n-1)/2} / (1 \cdot 3 \cdot 5 \cdots n) & \text{for } n \text{ odd}
\end{cases}
\]
and its moments of inertia are
\[
w_n = \begin{cases} 
v_n / (n + 2) & \text{for } n \text{ even} \\
2v_n / (n + 2) & \text{for } n \text{ odd}
\end{cases}
\]
Part 2

Geometric manifolds
CHAPTER 5

Geometric structures on manifolds

If $M$ is a manifold, we wish to impart to it (locally) affine and/or projective geometry. The corresponding global object is a geometric structure modeled on affine or projective geometry, or simply an affine structure or projective structure on $M$. (Such structures are also called “affinely flat structures,” “flat affine structures,” “flat projective structures,” etc. We will not be concerned with the more general “non-flat” structures here and hence refer to such structures as affine or projective structures.) For various reasons, it is useful to approach this subject from the more general point of view of locally homogeneous structures, that is, geometric structures modeled on a homogeneous space. In what follows $X$ will be a space with a geometry on it and $G$ is the group of transformations of $X$ which preserves this geometry. We shall consider manifolds $M$ having the same dimension as that of $X$: thus $M$ locally looks like $X$ — topologically — but we wish to model $M$ on $X$ geometrically. If $(G,X)$ is affine geometry (so that $X = \mathbb{R}^n$ and $G = \text{Aff}(\mathbb{R}^n)$) then a $(G,X)$-structure will be called an affine structure; if $(G,X)$ is projective geometry (so that $X = \mathbb{P}^n$ and $G = \text{Aut}(\mathbb{P}^n)$ the collineation group of $\mathbb{P}^n$) then an $(G,X)$-structure will be called a projective structure. An affine structure on a manifold is the same thing as a flat torsion-free affine connection, and a projective structure is the same thing as a flat normal projective connection (see Chern-Griffiths [29] Kobayashi [82] or Hermann [73] for the theory of projective connections). We shall refer to a projective structure modeled on $\mathbb{R}\mathbb{P}^n$ an $\mathbb{R}\mathbb{P}^n$-structure; a manifold with an $\mathbb{R}\mathbb{P}^n$-structure will be called an $\mathbb{R}\mathbb{P}^n$-manifold.

In many cases of interest, there may be a readily identifiable geometric object on $X$ whose stabilizer is $G$, and modeling a manifold on $(G,X)$ may be equivalent to a geometric object locally equivalent to the $G$-invariant geometric object on $X$. Perhaps the most important such object is a locally homogeneous Riemannian metric. For example if $X$ is a simply-connected Riemannian manifold of constant curvature $K$ and $G$ is its group of isometries, then locally modeling $M$ on $(G,X)$ is equivalent to giving $M$ a Riemannian metric of curvature $K$. 

75
This idea can be vastly extended, for example to cover indefinite metrics, locally homogeneous metrics whose curvature is not necessarily constant, etc.) In particular Riemannian metrics of constant curvature are special cases of \((G, X)-\text{structures}\) on manifolds.

Thurston [141] gives a detailed discussion of some of the pseudogroups defining structures on 3-manifolds.

5.1. Geometric atlases

Let \(G\) be a Lie group acting transitively on a manifold \(X\). Let \(U \subset X\) be an open set and let \(U \xrightarrow{f} X\) be a smooth map. We say that \(f\) is \(\text{locally-}G\) if for each component \(U_i \subset U\), there exists \(g_i \in G\) such that the restriction of \(g_i\) to \(U_i \subset X\) equals the restriction of \(f\) to \(U_i \subset U\). (Of course \(f\) will have to be a local diffeomorphism.) The collection of open subsets of \(X\), together with locally-\(G\) maps defines a pseudogroup upon which can model structures on manifolds as follows.

A \((G, X)-\text{atlas}\) on \(M\) is a pair \((\mathcal{U}, \Phi)\) where

\[
\mathcal{U} := \{U_\alpha \mid \alpha \in A\},
\]

is an open covering of \(M\) and

\[
\Phi = \{U_\alpha \xrightarrow{\phi_\alpha} X\}_{U_\alpha \in \mathcal{U}}
\]

is a collection of coordinate charts such that for each pair \((U_\alpha, U_\beta) \in \mathcal{U} \times \mathcal{U}\)

the restriction of \(\phi_\alpha \circ (\phi_\beta)^{-1}\) to \(\phi_\beta(U_\alpha \cap U_\beta)\) is locally-\(G\). An \((G, X)-\text{structure}\) on \(M\) is a maximal \((G, X)-\text{atlas}\) and an \((G, X)-\text{manifold}\) is a manifold together with an \((G, X)-\text{structure}\) on it.

An \((G, X)-\text{manifold}\) has an underlying real analytic structure, since the action of \(G\) on \(X\) is real analytic.

This notion of a map being \(\text{locally-}G\) has already been introduced for locally affine and locally projective maps.

Suppose that \(M\) and \(N\) are two \((G, X)-\text{manifolds}\) and \(M \xrightarrow{f} N\) is a map. Then \(f\) is an \((G, X)-\text{map}\) if for each pair of charts

\[
U_\alpha \xrightarrow{\phi_\alpha} X, \quad V_\beta \xrightarrow{\psi_\beta} X,
\]

for \(M\) and \(N\) respectively, the restriction

\[
\psi_\beta \circ f \circ \phi_\alpha^{-1} \bigg|_{\phi_\alpha(U_\alpha \cap f^{-1}(V_\beta))}
\]

is locally-\(G\). In particular we only consider \((G, X)-\text{maps}\) which are local diffeomorphisms. Clearly the set of \((G, X)-\text{automorphisms}\) \(M \xrightarrow{} M\)
forms a group, which we denote by \( \text{Aut}_{(G,X)}(M) \) or just \( \text{Aut}(M) \) when the context is clear.

**Exercise 5.1.1.** Let \( N \) be an \((G,X)\)-manifold and \( M \to N \) a local diffeomorphism.

- There is a unique \((G,X)\)-structure on \( M \) for which \( f \) is an \((G,X)\)-map.
- Every covering space of an \((G,X)\)-manifold has a canonical \((G,X)\)-structure.
- Conversely suppose \( M \) is an \((G,X)\)-manifold upon which a discrete subgroup \( \Gamma \subset \text{Aut}_{(G,X)}(M) \) acts properly and freely. Then \( M/\Gamma \) is an \((G,X)\)-manifold and the quotient mapping
  \[
  M \to M/\Gamma
  \]
  is a \((G,X)\)-covering space.

**5.1.1. The pseudogroup of local mappings.** The fundamental example of an \((G,X)\)-manifold is \( X \) itself. Evidently any open subset \( \Omega \subset X \) has an \((G,X)\)-structure (with only one chart—the inclusion \( \Omega \to X \)). Locally-\( G \) maps satisfy the following Unique Extension Property: If \( U \subset X \) is a connected nonempty open subset, and \( U \to X \) is locally-\( G \), then there exists a unique element \( g \in G \) restricted to each connected component of \( \phi_U(U \alpha \cap U \beta) \).

Here is another perspective on a \((G,X)\)-atlas. First regard \( M \) as a quotient space of the disjoint union

\[
\mathfrak{U} = \bigsqcup_{\alpha \in \Lambda} U_\alpha
\]

by the equivalence relation \( \sim \) defined by intersection of patches. A point \( u \in U_\alpha \cap U_\beta \) determines corresponding elements

\[
\begin{align*}
  u_\alpha & \in U_\alpha \subset \mathfrak{U} \\
  u_\beta & \in U_\beta \subset \mathfrak{U}
\end{align*}
\]

and we define the equivalence relation on \( \mathfrak{U} \) by: \( u_\alpha \sim u_\beta \).

Now the coordinate change

\[
\phi_\beta(U_\alpha \cap U_\beta) \xrightarrow{\phi_\alpha \circ (\phi_\beta)^{-1}} \phi_\alpha(U_\alpha \cap U_\beta)
\]

is locally-\( G \). By the unique extension property, it agrees with the action of a unique element of \( G \) restricted to each connected component of \( \phi_\beta(U_\alpha \cap U_\beta) \). Thus it corresponds to a locally constant map:

\[
U_\alpha \cap U_\beta \xrightarrow{g_{a\beta}} G
\]
We can alternatively define the \((G,X)\)-manifold \(M\) as the quotient of the disjoint union
\[ \Phi := \coprod_{\alpha \in A} \phi_\alpha(U_\alpha) \]
by the equivalence relation \(\sim_\Phi\) defined as:
\[ \phi_\alpha(u_\alpha) \sim_\Phi g_{\alpha\beta}(\phi_\beta(u_\beta)) \]
for \(u_\alpha \in U_\alpha \cap U_\beta\) notated as above. That \(\sim_\Phi\) is an equivalence relation follows from the cocycle identities
\[
\begin{align*}
g_{\alpha\alpha}(u_\alpha) &= 1 \\
g_{\alpha\beta}(u_\beta)g_{\beta\alpha}(u_\alpha) &= 1 \\
g_{\alpha\beta}(u_\beta)g_{\beta\gamma}(u_\gamma)g_{\gamma\alpha}(u_\alpha) &= 1
\end{align*}
\]
whenever \(u_\alpha \in U_\alpha, \ u_\beta \in U_\alpha \cap U_\beta, \ u_\gamma \in U_\alpha \cap U_\beta \cap U_\gamma\), respectively.

This rigidity property is a distinguishing feature of the kind of geometric structures considered here. However, many familiar pseudogroup structures lack this kind of rigidity:

**Exercise 5.1.2.** Show that the following pseudogroups do not satisfy the unique extension property:
- \(C^r\) local diffeomorphisms between open subsets of \(\mathbb{R}^n\), when \(r = 0, 1, \ldots, \infty, \omega\).
- Local biholomorphisms between open subsets of \(\mathbb{C}^n\).
- Smooth diffeomorphisms between open subsets of a domain \(\Omega \subset \mathbb{R}^n\) preserving an exterior differential form on \(\Omega\).

**5.1.2. \((G,X)\)-automorphisms.** Now we discuss the automorphisms of a structure locally modeled on \((G,X)\).

If \(\Omega \subset X\) is a domain, an \((G,X)\)-automorphism \(\Omega \xrightarrow{f} \Omega\) is the restriction of a unique element \(g \in G\) preserving \(\Omega\), that is:
\[
\text{Aut}_{(G,X)}(\Omega) \cong \text{Stab}_G(\Omega) = \{g \in G \mid g(\Omega) = \Omega\}
\]
Now suppose that \(M \xrightarrow{\phi} \Omega\) is a local diffeomorphism onto a domain \(\Omega \subset X\). There is a homomorphism
\[
\text{Aut}_{(G,X)}(M) \xrightarrow{\phi_*} \text{Aut}_{(G,X)}(\Omega)
\]
whose kernel consists of all maps \(M \xrightarrow{f} M\) making the diagram
\[
\begin{array}{ccc}
M & \xrightarrow{\phi} & \Omega \\
\downarrow f & & \downarrow \text{id} \\
M & \xrightarrow{\phi} & \Omega
\end{array}
\]
Exercise 5.1.3. Find examples where:

- \( \phi_* \) is surjective but not injective;
- \( \phi_* \) is injective but not surjective.

5.2. Development, holonomy

There is a useful globalization of the coordinate charts of a geometric structure in terms of the universal covering space and the fundamental group. The coordinate atlas \( \{ U_\alpha \}_{\alpha \in A} \) is replaced by a universal covering space \( \widetilde{M} \to M \) with its group of deck transformations \( \pi \). In the first approach, \( M \) is the quotient space of the disjoint union \( \coprod_{\alpha \in A} U_\alpha \), and in the second it is the quotient space of \( \widetilde{M} \) by the group action \( \pi \). The coordinate charts \( U_\alpha \xrightarrow{\psi_\alpha} X \) are replaced by a globally defined map \( \widetilde{M} \xrightarrow{\text{dev}} X \).

This process of development originated with Élie Cartan and generalizes the notion of a developable surface in \( \mathbb{E}^3 \). If \( S \hookrightarrow \mathbb{E}^3 \) is an embedded surface of zero Gaussian curvature, then for each \( p \in S \), the exponential map at \( p \) defines an isometry of a neighborhood of 0 in the tangent plane \( T_pS \), and corresponds to rolling the tangent plane \( A_p(S) \) on \( S \) without slipping. In particular every curve in \( S \) starting at \( p \) lifts to a curve in \( T_pS \) starting at \( 0 \in T_pS \). For a Euclidean manifold, this globalizes to a local isometry of the universal covering \( \widetilde{S} \to \mathbb{E}^2 \), called by Élie Cartan the development of the surface (along the curve). The metric structure is actually subordinate to the affine connection, as this notion of development really only involves the construction of parallel transport.

Later this was incorporated into the notion of a fiber space, as discussed in the 1950 conference [139]. The collection of coordinate changes of a \( (G, X) \)-manifold \( M \) defines a fiber bundle \( \mathcal{E}_M \to M \) with fiber \( X \) and structure group \( G \). The fiber over \( p \in M \) of the associated principal bundle

\[
\mathcal{P}_M \xrightarrow{\Pi_p} M
\]

consists of all possible germs of \( (G, X) \)-coordinate charts at \( p \). The fiber over \( p \in M \) of \( \mathcal{E}_M \) consists of all possible values of \( (G, X) \)-coordinate charts at \( p \). Assigning to the germ at \( p \) of a coordinate chart \( U \xrightarrow{\psi} X \) its value

\[
x = \psi(p) \in X
\]

defines a mapping

\[
(\mathcal{P}_M)_p \to (\mathcal{E}_M)_p.
\]
Working in a local chart, the fiber over a point in $(E_M)_p$ corresponding to $x \in X$ consists of all the different germs of coordinate charts $\psi$ taking $p \in M$ to $x \in X$. This mapping identifies with the quotient mapping of the natural action of the stabilizer $\text{Stab}(G, x) \subseteq G$ of $x \in X$ on the set of germs.

For Euclidean manifolds, $(P_M)_p$ consists of all affine orthonormal frames, that is, pairs $(x, F)$ where $x \in E^n$ is a point and $F$ is an orthonormal basis of the tangent space $T_x E^n \cong \mathbb{R}^n$. For an affine manifold, $(P_M)_p$ consists of all affine frames: pairs $(x, F)$ where now $F$ is any basis of $\mathbb{R}^n$.

5.2.1. Construction of the developing map. Let $M$ be an $(G, X)$-manifold. Choose a universal covering space

$$\tilde{M} \xrightarrow{\Pi} M$$

and let $\pi = \pi_1(M)$ be the corresponding fundamental group. The covering projection $\Pi$ induces an $(G, X)$-structure on $\tilde{M}$ upon which $\pi$ acts by $(G, X)$-automorphisms. The Unique Extension Property has the following important consequence.

**Proposition 5.2.1.** Let $M$ be a simply connected $(G, X)$-manifold. Then there exists a $(G, X)$-map $M \xrightarrow{f} X$.

It follows that the $(G, X)$-map $f$ completely determines the $(G, X)$-structure on $M$, that is, the geometric structure on a simply-connected manifold is “pulled back” from the model space $X$. The $(G, X)$-map $f$ is called a developing map for $M$ and enjoys the following uniqueness property. If $M \xrightarrow{f'} X$ is another $(G, X)$-map, then there exists an $(G, X)$-automorphism $\phi$ of $M$ and an element $g \in G$ such that

$$
\begin{array}{ccc}
M & \xrightarrow{f'} & X \\
\phi & & \downarrow g \\
M & \xrightarrow{f} & X
\end{array}
$$

**Proof of Proposition 5.2.1.** Choose a baspoint $x_0 \in M$ and a coordinate patch $U_0$ containing $x_0$. For $x \in M$, we define $f(x)$ as follows. Choose a path $\{x_t\}_{0 \leq t \leq 1}$ in $M$ from $x_0$ to $x = x_1$. Cover the path by coordinate patches $U_i$ (where $i = 0, \ldots, n$) such that $x_t \in U_i$ for $t \in (a_i, b_i)$ where

$$
a_0 < 0 < a_1 < b_0 < a_2 < b_1 < a_3 < b_2 < 
\cdots < a_{n-1} < b_{n-2} < a_n < b_{n-1} < 1 < b_n
$$
Let $U_i \xrightarrow{\psi_i} X$ be an $(G, X)$-chart and let $g_i \in G$ be the unique transformation such that $g_i \circ \psi_i$ and $\psi_{i-1}$ agree on the component of $U_i \cap U_{i-1}$ containing the curve $\{x_t\}_{a_i < t < b_i}$. Let $f(x) = g_1 g_2 \ldots g_{n-1} g_n \psi_n(x)$; we show that $f$ is indeed well-defined. The map $f$ does not change if the cover is refined. Suppose that a new coordinate patch $U'$ is “inserted between” $U_{i-1}$ and $U_i$. Let $\{x_t\}_{a' < t < b'}$ be the portion of the curve lying inside $U'$ so $a_{i-1} < a' < a_i < b_{i-1} < b' < b_i$. Let $U' \xrightarrow{\psi'} X$ be the corresponding coordinate chart and let $h_{i-1}, h_i \in G$ be the unique transformations such that $\psi_{i-1}$ agrees with $h_{i-1} \circ \psi'$ on the component of $U' \cap U_{i-1}$ containing $\{x_t\}_{a' < t < b_{i-1}}$ and $\psi'$ agrees with $h_i \circ \psi_i$ on the component of $U' \cap U_i$ containing $\{x_t\}_{a_i < t < b'}$. By the unique extension property $h_{i-1} h_i = g_i$ and it follows that the corresponding developing map

$$f(x) = g_1 g_2 \ldots g_{i-1} h_{i-1} h_i g_{i+1} \ldots g_{n-1} g_n \psi_n(x) = g_1 g_2 \ldots g_{i-1} g_i g_{i+1} \ldots g_{n-1} g_n \psi_n(x)$$

is unchanged. Thus the developing map as so defined is independent of the coordinate covering, since any two coordinate coverings have a common refinement.

Next we claim the developing map is independent of the choice of path. Since $M$ is simply connected, any two paths from $x_0$ to $x$ are homotopic. Every homotopy can be broken up into a succession of “small” homotopies, that is, homotopies such that there exists a partition $0 = c_0 < c_1 < \cdots < c_{m-1} < c_m = 1$ such that during the course of the homotopy the segment $\{x_t\}_{c_i < t < c_{i+1}}$ lies in a coordinate patch. It follows that the expression defining $f(x)$ is unchanged during each of the small homotopies, and hence during the entire homotopy. Thus $f$ is independent of the choice of path.

Since $f$ is a composition of a coordinate chart with a transformation $X \rightarrow X$ coming from $G$, it follows that $f$ is an $(G, X)$-map. The proof of Proposition 5.2.1 is complete. 

If $M$ is an arbitrary $(G, X)$-manifold, then we may apply Proposition 5.2.1 to a universal covering space $\tilde{M}$. We obtain the following basic result:

**Theorem 5.2.2 (Development Theorem).** Let $M$ be an $(G, X)$-manifold with universal covering space $\tilde{M} \xrightarrow{\pi} M$ and group of deck
transformations

\[ \pi = \pi_1(M) \subset \text{Aut}(\tilde{M} \xrightarrow{\Pi} M) \]

Then there exists a pair \((\text{dev}, \text{hol})\) consisting of an \((G, X)\)-map \text{dev} and a homomorphism \text{hol},

\[
\begin{align*}
\tilde{M} &\xrightarrow{\text{dev}} X \\
\pi &\xrightarrow{\text{hol}} G,
\end{align*}
\]

such that for each \(\gamma \in \pi\),

\[
\begin{array}{c}
\gamma \downarrow \\
\tilde{M} &\xrightarrow{\text{dev}} X \\
\end{array}
\quad
\begin{array}{c}
\gamma \downarrow \\
\tilde{M} &\xrightarrow{\text{dev}} X \\
\end{array}
\]

commutes. Furthermore if \((\text{dev}', \text{hol}')\) is another such pair, then then \(\exists g \in G\) such that \(\forall \gamma \in \pi\),

\[
\begin{align*}
\text{dev}' &= g \circ \text{dev} \\
\text{hol}'(\gamma) &= \text{Inn}(g) \circ \text{hol}(\gamma) : \\
\tilde{M} &\xrightarrow{\text{dev}} X \quad \xrightarrow{g} \quad X \\
\gamma \downarrow &\quad \text{hol}(\gamma) \quad \text{hol}'(\gamma) \quad \downarrow \quad \gamma \downarrow \\
\tilde{M} &\xrightarrow{\text{dev}} X \quad \xrightarrow{g} \quad X
\end{align*}
\]

We call such a pair \((\text{dev}, h)\) a development pair, and the homomorphism \(h\) the holonomy representation. (It is the holonomy of a flat connection on a principal \(G\)-bundle over \(M\) associated to the \((G, X)\)-structure.) The developing map globalizes the coordinate charts of the manifold and the holonomy representation globalizes the coordinate changes. In this generality the Development Theorem is due to C. Ehresmann \([44]\) in 1936.

5.2.2. Role of the holonomy group. The image of the holonomy representation is the “smallest” subgroup \(\Gamma \subset G\) such that \(M\) admits a \((\Gamma, X)\)-structure:

**Exercise 5.2.3.** Let \(M\) be an \((G, X)\)-manifold with development pair \((\text{dev}, h)\).

- Find a \((G, X)\)-atlas for \(M\) such that the coordinate changes \(g_{\alpha\beta}\) lie in \(\Gamma\).
• Suppose that $N \rightarrow M$ is a covering space. Show that there exists a $(G,X)$-map $N \rightarrow X$ if and only if the holonomy representation restricted to $\pi_1(N) \hookrightarrow \pi_1(M)$ is trivial.

Thus the holonomy covering space $\hat{M} \rightarrow M$ — the covering space of $M$ corresponding to the kernel of $h$ — is the “smallest” covering space of $M$ for which a developing map is “defined.”

The holonomy group

$$\text{hol}(\pi) = \Gamma \subset G$$

is the “smallest” subgroup of $G$ for which there is a compatible $(G,X)$-atlas, where the coordinate changes lie in $\Gamma$.

**Exercise 5.2.4.** Let $M$ be an $(G,X)$-manifold. Find a $(G,X)$-atlas such that all the coordinate changes are restrictions of transformations in $\Gamma$.

**Exercise 5.2.5.** Suppose that $(G,X)$ and $(G',X')$ represent a pair of geometries for which there exists a pair $(\Phi,\phi)$ as in §5.2.3. Show that if $M$ is a $(G,X)$-manifold with development pair $(\text{dev},h)$, then $(\Phi \circ \text{dev}, \phi \circ h)$ is a development pair for the induced $(X',G')$-structure on $M$.

### 5.2.3. Extending geometries.

A geometry may contain or refine another geometry. In this way one can pass from structures modeled on one geometry to structures modeled on a geometry containing it. Let $(X,G)$ and $(X',G')$ be homogeneous spaces and let $X \xrightarrow{\Phi} X'$ be a local diffeomorphism which is equivariant with respect to a homomorphism $\phi : G \rightarrow G'$ in the following sense: for each $g \in G$ the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\Phi} & X' \\
g \downarrow & & \downarrow \phi(g) \\
X & \xrightarrow{\Phi} & X'
\end{array}
$$

commutes. Hence locally-$G$ maps determine locally-$(X',G')$-maps and an $(G,X)$-structure on $M$ induces an $(X',G')$-structure on $M$ in the following way. Let $U_{\alpha} \xrightarrow{\psi_{\alpha}} X$ be an $(G,X)$-chart; the composition

$$U_{\alpha} \xrightarrow{\Phi \circ \psi_{\alpha}} X'$$

defines an $(X',G')$-chart.

**Exercise 5.2.6.** Explain the extension of geometries in terms of the development pair.
5.2.4. Simple applications of the developing map.

**Exercise 5.2.7.** Suppose that $M$ is a closed manifold with finite fundamental group.

- If $X$ is noncompact then $M$ admits no $(G,X)$-structure.
- If $X$ is compact and simply-connected show that every $(G,X)$-manifold is $(G,X)$-isomorphic to a quotient of $X$ by a finite subgroup of $G$.

(Hint: if $M$ and $N$ are manifolds of the same dimension, $M \xrightarrow{f} N$ is a local diffeomorphism and $M$ is closed, show that $f$ must be a covering space.)

As a consequence a closed affine manifold must have infinite fundamental group and every $\mathbb{RP}^n$-manifold with finite fundamental group is a quotient of $S^n$ by a finite group (and hence a spherical space form).

5.3. The graph of a geometric structure

This can be put in an even “more global” context using the fiber bundle associated to a $(G,X)$-structure. This is a fiber bundle $\mathcal{E}_M \to M$ with fiber $X$, structure group $G$ in the sense of Steenrod[135]. It plays a role analogous to the tangent bundle of a smooth manifold. It admits a flat structure, that is a foliation $\mathcal{F}$ transverse to the fibration, and a section $\mathcal{D}_M$ which is transverse to $\mathcal{F}$ as well as the fibration. The section $\mathcal{D}_M$ plays the role of the zero-section of the tangent bundle. Indeed, its normal bundle inside $\mathcal{E}_M$ is isomorphic to the tangent bundle $TM$ of $M$. It is obtained as the graph of the collection $\Phi$ of coordinate charts. The flat bundle $\mathcal{E}_M$ is the natural “home” in which $\mathcal{D}_M$ lives.

5.3.1. The tangent $(G,X)$-bundle. The total space $\mathcal{E}_M$ of this bundle is obtained from the disjoint union

$$\mathfrak{U}_X := \bigsqcup_{\alpha \in A} U_\alpha \times X$$

of trivial $X$-bundles.

Now suppose $U_\alpha, U_\beta \in \mathfrak{U}$ are coordinate patches. Introduce an equivalence relation $\sim_X$ on $\mathfrak{U}_X$ by:

$$(u_\alpha, x) \sim_X (u_\beta, g_{\alpha\beta}(u_\beta)x)$$

where $g_{\alpha\beta}$ is the cocycle introduced in (18). The cocycle identities (19) imply that $\sim_X$ is an equivalence relation. The projections

$$U_\alpha \times X \to U_\alpha$$

lifed from "historical"
are trivial $X$-bundles and define a trivial $X$-bundle
\[ \Lambda_X \to \Lambda \]
compatible with the equivalence relations $\sim_X, \sim$. The corresponding mapping of quotient spaces
\[ \begin{array}{ccc}
\mathcal{E}_M & \longrightarrow & \Lambda_X / \sim_X \\
\downarrow \Pi & & \\
M & \longrightarrow & \Lambda / \sim \\
\end{array} \]
is a locally trivial $X$-bundle with structure group $G$.

Furthermore the structure group is really $G$ with the discrete topology, since the transition functions
\[ U_\alpha \cap U_\beta \xrightarrow{g_{\alpha \beta}} G \]
are locally constant. This implies that the foliation of the total space $\mathcal{E}_M$ with local leaves (sometimes called plaques) $U_\alpha \times \{x\}$ piece together to define the leaves of a foliation $\mathcal{F}$ of $\mathcal{E}_M$. (Compare Steenrod [135, §13].)

**Exercise 5.3.1.**

- Show that for every leaf $L \subset \mathcal{E}_M$ of $\mathcal{F}$, the restriction $\Pi|_L$ is a covering space $L \to M$.
- If $M$ is simply connected, then $(\mathcal{E}_M, \mathcal{F})$ is trivial, that is, isomorphic to $M \times X$ with the trivial foliation, namely the one with leaves $M \times \{x\}$, where $x \in X$.

It follows that the flat $(G, X)$-bundle $(\mathcal{E}_M, \mathcal{F})$ arises from a representation $\pi_1(M) \xrightarrow{h} G$ as follows. The group $\pi_1(M)$ admits a (left-)action on the trivial bundle $\tilde{M} \times X$ by:
\[ (\tilde{p}, x) \mapsto (\tilde{p}\gamma^{-1}, h(\gamma)x) \]
where
\[ \tilde{M} \times \pi_1(M) \to \tilde{M} \]
\[ (\tilde{p}, \gamma) \mapsto \tilde{p}\gamma \]
denotes the (right-) action of $\pi_1(M)$ by deck transformations. Then $\mathcal{E}_M$ identifies as the quotient $(\tilde{M} \times X)/\pi_1(M)$, that is as the fiber product $\tilde{M} \times_h X$. Furthermore $h$ is unique up to the action of $\text{Inn}(G)$ by left-composition. We call $h \in \text{Hom}(\pi_1(M), G)$ the holonomy representation of the flat $(G, X)$-bundle $(\mathcal{E}_M, \mathcal{F})$. 
5.3.2. Developing sections. Just as $(\mathcal{E}_M, \mathcal{F})$ globalizes the coordinate changes, its transverse section $\mathcal{D}_M$ globalizes the coordinate atlas $\Phi$.

When $M$ is a single coordinate patch, then $\mathcal{E}_M$ is just the product $M \times X$ and $\mathcal{E}_M \rightarrow M$ is just the Cartesian projection $M \times X \rightarrow M$. A section of $\mathcal{E}_M \rightarrow M$ is just the graph of a map $M \xrightarrow{f} X$:

\[
\begin{align*}
M \xrightarrow{\text{graph}(f)} M \times X & \cong \mathcal{E}_M \\
p & \mapsto (p, f(p))
\end{align*}
\]

5.3.3. The associated principal bundle. The fiber over $p \in M$ of the associated principal bundle

\[
\begin{align*}
\mathcal{P}_M \xrightarrow{\Pi_p} M
\end{align*}
\]

consists of all possible germs of $(G, X)$-coordinate charts at $p$. The fiber over $p \in M$ of $\mathcal{E}_M$ consists of all possible values of $(G, X)$-coordinate charts at $p$. Assigning to the germ at $p$ of a coordinate chart $U \xrightarrow{\psi} X$ its value

\[
x = \psi(p) \in X
\]

defines a mapping

\[
(\mathcal{P}_M)_p \rightarrow (\mathcal{E}_M)_p.
\]

Working in a local chart, the fiber over a point in $(\mathcal{E}_M)_p$ corresponding to $x \in X$ consists of all the different germs of coordinate charts $\psi$ taking $p \in M$ to $x \in X$. This mapping identifies with the quotient mapping of the natural action of the stabilizer $\text{Stab}(G, x) \subset G$ of $x \in X$ on the set of germs.

For Euclidean manifolds, $(\mathcal{P}_M)_p$ consists of all affine orthonormal frames, that is, pairs $(x, F)$ where $x \in \mathbb{E}^n$ is a point and $F$ is an orthonormal basis of the tangent space $T_x\mathbb{E}^n \cong \mathbb{R}^n$. For an affine manifold, $(\mathcal{P}_M)_p$ consists of all affine frames: pairs $(x, F)$ where now $F$ is any basis of $\mathbb{R}^n$.

The coordinate atlas/developing map defines a section of $\mathcal{E}_M \rightarrow M$ which is transverse to the two complementary foliations of $\mathcal{E}_M$:

- As a section, it is necessarily transverse to the foliation of $\mathcal{E}_M$ by fibers;
- The nonsingularity of the coordinate charts/developing map implies this section is transverse to the horizontal foliation $\mathcal{F}_M$ of $\mathcal{E}_M$ defining the flat structure.
5.4. The classification of geometric 1-manifolds

The basic general question concerning geometric structures on manifolds is, given a topological manifold \( M \) and a geometry \((G, X)\), whether an \((G, X)\)-structure on \( M \) exists, and if so, to classify all \((G, X)\)-structures on \( M \). Ideally, one would like a deformation space, a topological space whose points correspond to isomorphism classes of \((G, X)\)-manifolds.

As an exercise to illustrate these general ideas, we classify geometric manifolds in dimension one. We consider the three geometries \( \mathbb{E}^1 \), \( \mathbb{A}^1 \), \( \mathbb{P}^1 \) in increasing order. Euclidean manifolds are affine manifolds, which in turn are projective manifolds. Thus we classify \( \mathbb{R} \mathbb{P}^1 \)-manifolds. (Compare Kuiper [95], Goldman [57].)

Let \( M \) be a connected 1-manifold. There are two cases:

- \( M \) is noncompact, in which case \( M \) is homeomorphic (diffeomorphic) to a line \((M \approx \mathbb{R})\);
- \( M \) is compact, in which case \( M \) homeomorphic (diffeomorphic) to a circle \((M \approx S^1)\).

In particular \( M \) is simply connected \( \iff \) \( M \) is noncompact and otherwise \( \pi_1(M) \cong \mathbb{Z} \).

5.4.1. Compact Euclidean 1-manifolds and flat tori. The cyclic group \( \mathbb{Z} \) acts by translations on \( \mathbb{E}^1 \cong \mathbb{R} \). The quotient

\[
E_1 := \mathbb{E}^1 / \mathbb{Z} \cong \mathbb{R} / \mathbb{Z}
\]

is a compact Euclidean 1-manifold. The Euclidean metric on \( \mathbb{R} \) induces a flat Riemannian structure on the quotient \( \mathbb{R} / \mathbb{Z} \) which has length 1.

More generally, choose \( \ell > 0 \). Then the quotient

\[
E_\ell := \mathbb{E}^1 / \ell \mathbb{Z} \cong \mathbb{R} / \ell \mathbb{Z}
\]

is a compact Euclidean 1-manifold which has length \( \ell \). Different choices of \( \ell \) determine different isometry classes of Euclidean 1-manifolds but \( E_1 \) is affinely isomorphic to \( E_\ell \) by the affine map \( x \mapsto \ell x \). In other words, if \( \ell \neq 1 \), then \( E_1 \) and \( E_\ell \) are inequivalent Euclidean manifolds but equivalent affine manifolds.

**Exercise 5.4.1.** Show that the total space of \( \mathcal{E}_M \) identifies with the quotient of \( \mathbb{R}^2 \) by the diagonally embedded \( \mathbb{Z} \) acting by translations:

\[
(x, y) \mapsto (x + n, y + n)
\]

for \( n \in \mathbb{Z} \), the fibration is induced by the projection \((x, y) \mapsto x\), the foliation induced by horizontal lines \( \mathbb{R} \times \{y\} \), and the developing section by the diagonal \( \Delta(x) := (x, x) \). When these structures are regarded as
\( \mathbb{RP}^1 \)-manifolds, \( \mathcal{E}_M \) acquires an extra (horizontal) closed leaf which is disjoint from the developing section.

These manifolds generalize to one of the most basic classes of closed geometric manifolds, namely the flat tori. Let \( \Lambda \subset \mathbb{R}^n \) be a lattice, that is the additive subgroup of \( \mathbb{R}^n \) generated by a basis. Then \( \Lambda \) acts by translations, so the quotient \( \mathbb{R}^n/\Lambda \) is a compact Euclidean manifold. Bieberbach proved that every compact Euclidean manifold is finitely covered by a flat torus.

**Exercise 5.4.2.** Since \( \Lambda \) is a normal subgroup of \( \mathbb{R}^n \), a flat torus is also an abelian Lie group. Show that this algebraic structure is compatible with the geometric structure: the Euclidean structure on \( \mathbb{R}^n/\Lambda \) is invariant under multiplications. (Since \( \mathbb{R}^n \) is commutative, left-multiplications and right-multiples coincide.

**5.4.2. Compact affine 1-manifolds and Hopf manifolds.** A compact affine manifold is either a Euclidean manifold as above, or given by the following construction. Let \( \lambda > 1 \) and consider the cyclic group \( \langle \lambda \rangle \cong \mathbb{Z} \) acting by homotheties on \( \mathbb{A}^1 \):

\[
x \mapsto \lambda^n x
\]

Then

\[
A_\lambda := \mathbb{R}^+ / \langle \lambda \rangle
\]

is a compact affine 1-manifold.

**Exercise 5.4.3.** Show that different values of \( \lambda \) yield inequivalent affine structures, and no \( A_\lambda \) is affinely equivalent to \( E_\ell \). However show that, for \( \lambda, \lambda' \) the developing maps for \( A_\lambda \) and \( A_{\lambda'} \) are topologically conjugate by a homeomorphism \( \mathbb{A}^1 \to \mathbb{A}^1 \) and the developing maps for \( A_\lambda \) and \( E_\ell \) are topologically semi-conjugate by a homeomorphism \( \mathbb{R}^+ \to \mathbb{R} \cong \mathbb{E}^1 \).

We call these latter affine 1-manifolds Hopf circles, since these are the 1-dimensional cases of Hopf manifolds discussed in §6.2.2 of Chapter 9.

**Exercise 5.4.4.** Show that these affine structures are invariant affine structures on the Lie group \( S^1 \), namely, that the translation on the group \( S^1 \) is affine. (Since \( S^1 \) is abelian, both left- and right-translation agree.)

These are the only examples of compact affine 1-manifolds, although there are projective manifolds which have the “same” holonomy homomorphisms, defined by grafting; see §5.4.4.
5.4.3. Classification of projective 1-manifolds. To simplify matters, we pass to the universal covering \( X = \mathbb{R}P^1 \), which is homeomorphic to \( \mathbb{R} \) and the corresponding covering group \( G = \text{PGL}(2, \mathbb{R}) \) which acts on \( X \). Suppose that \( M \) is a connected noncompact \( \mathbb{R}P^1 \)-manifold (and thus diffeomorphic to an open interval). Then a developing map

\[
M \approx \mathbb{R} \xrightarrow{\text{dev}} \mathbb{R} \approx X
\]

is necessarily an embedding of \( M \) onto an open interval in \( X \). Given two such embeddings

\[
M \xrightarrow{f} X, \quad M \xrightarrow{f'} X
\]

whose images are equal, then \( f' = j \circ f \) for a diffeomorphism \( M \xrightarrow{f} M \). Thus two \( \mathbb{R}P^1 \)-structures on \( M \) which have equal developing images are isomorphic. Thus the classification of \( \mathbb{R}P^1 \)-structures on \( M \) is reduced to the classification of \( G \)-equivalence classes of intervals \( J \subset X \). Choose a diffeomorphism

\[
X \approx \mathbb{R} \approx (-\infty, \infty);
\]

an interval in \( X \) is determined by its pair of endpoints in \([-\infty, \infty]\). Since \( G \) acts transitively on \( X \), an interval \( J \) is either bounded in \( X \) or projectively equivalent to \( X \) itself or one component of the complement of a point in \( X \). Suppose that \( J \) is bounded. Then either the endpoints of \( J \) project to the same point in \( \mathbb{R}P^1 \) or to different points. In the first case, let \( N > 0 \) denote the degree of the map

\[
J/\partial J \longrightarrow \mathbb{R}P^1
\]

induced by \( \text{dev} \); in the latter case choose an interval \( J^+ \) such that the restriction of the covering projection \( X \longrightarrow \mathbb{R}P^1 \) to \( J^+ \) is injective and the union \( J \cup J^+ \) is an interval in \( X \) whose endpoints project to the same point in \( \mathbb{R}P^1 \). Let \( N > 0 \) denote the degree of the restriction of the covering projection to \( J \cup J^+ \). Since \( G \) acts transitively on pairs of distinct points in \( \mathbb{R}P^1 \), it follows easily that bounded intervals in \( X \) are determined up to equivalence by \( G \) by the two discrete invariants: whether the endpoints project to the same point in \( \mathbb{R}P^1 \) and the positive integer \( N \). It follows that every \( (G,X) \)-structure on \( M \) is \( (G,X) \)-equivalent to one of the following types. We shall identify \( X \) with the real line and group of deck transformations of \( X \longrightarrow \mathbb{R}P^1 \) with the group of integer translations.

- A complete \( (G,X) \)-manifold (that is, \( M \xrightarrow{\text{dev}} X \) is a diffeomorphism);
- $M \xrightarrow{\text{dev}} X$ is a diffeomorphism onto one of two components of the complement of a point in $X$, for example, $(0, \infty)$.
- $\text{dev}$ is a diffeomorphism onto an interval $(0, N)$ where $N > 0$ is a positive integer;
- $\text{dev}$ is a diffeomorphism onto an interval $(0, N + \frac{1}{2})$.

Next consider the case that $M$ is a compact 1-manifold; choose a basepoint $x_0 \in M$. Let $\pi = \pi(M, x_0)$ be the corresponding fundamental group of $M$ and let $\gamma \in \pi$ be a generator. We claim that the conjugacy class of $h(\gamma) \in G$ completely determines the structure. Choose a lift $J$ of $M - \{x_0\}$ to $\tilde{M}$ which will serve as a fundamental domain for $\pi$. Then $J$ is an open interval in $\tilde{M}$ with endpoints $y_0$ and $y_1$. Choose a developing map $\tilde{M} \xrightarrow{\text{dev}} X$, a holonomy representation $\pi \longrightarrow G$; then $\text{dev}(y_1) = \text{hol}(\gamma)\text{dev}(y_0)$.

Now suppose that $\text{dev}'$ is a developing map for another structure with the same holonomy. By applying an element of $G$ we may assume that $\text{dev}(y_0) = \text{dev}'(y_0)$ and that $\text{dev}(y_1) = \text{dev}'(y_1)$. Furthermore there exists a diffeomorphism $J \xrightarrow{\phi} J$ such that $\text{dev}' = \phi \circ \text{dev}$; this diffeomorphism lifts to a diffeomorphism $\tilde{M} \longrightarrow \tilde{M}$ taking $\text{dev}$ to $\text{dev}'$. Conversely suppose that $\eta \in G$ is orientation-preserving (this means simply that $\eta$ lies in the identity component of $G$) and is not the identity. Then there exists $x_0 \in X$ which is not fixed by $\eta$; let $x_1 = \eta x_0$. There exists a diffeomorphism $J \longrightarrow X$ taking the endpoints $y_i$ of $J$ to $x_i$ for $i = 0, 1$. This diffeomorphism extends to a developing map $\tilde{M} \xrightarrow{\text{dev}} X$.

In summary:

**Theorem 5.4.5.** A compact $\mathbb{RP}^1$-manifold is either projectively equivalent to:
- A Hopf circle $\mathbb{R}^+ / \langle \lambda \rangle$;
- A Euclidean 1-manifold $\mathbb{R} / \mathbb{Z}$;
- A quotient of the universal covering of $\mathbb{RP}^1$ by a cyclic group.

The first two cases are the affine 1-manifolds.

**Exercise 5.4.6.** Determine all automorphisms of each of the above list of $\mathbb{RP}^1$-manifolds.

**Corollary 5.4.7.** Let $G^0$ denote the identity component of the universal covering group $G$ of $\text{PGL}(2, \mathbb{R})$. Let $M$ be a closed 1-manifold.
Then the set of isomorphism classes of $\mathbb{R}P^1$-structures on $M$ is in bijective correspondence with the set

$$G^0 - \{1\}/\text{Inn}(G)$$

of $G$-conjugacy classes in the set $G^0 - \{1\}$ of elements of $G^0$ not equal to the identity.

**Exercise 5.4.8.** Show that the quotient topology on $G^0 - \{1\}/\text{Inn}(G)$ is not Hausdorff.

**5.4.4. Grafting.** Describe the construction for RP1-manifolds, and how the surjective developments with hyperbolic/parabolic holonomy are grafts with the universal structure.
CHAPTER 6

Examples of Geometric Structures

This section introduces examples of geometric manifolds in dimensions greater than one. Just as the theory of Lie groups and their homogeneous spaces organized the abundance of classical geometries, we exploit this algebraicization to clarify the relationship between various geometric structures. We have already seen in §5.2.3 of Chapter 5 how one geometric structure induces another one, and we formalize this construction here.

We begin with the parallel structures in affine geometry. This generalizes the construction of Euclidean geometry as (flat) Riemannian geometry. From our viewpoint, a Euclidean structure is just a parallel Riemannian structure on an affine manifold. This is the first example of extending a geometry, where the model space $X$ is fixed (in this case an affine space) but the automorphism group $G$ is reduced or enlarged.

Another important case arises when one model space $X'$ embeds in the other model space as an open subset. The projective models of hyperbolic geometry and Euclidean geometry are fundamental examples. (Indeed the extension of Euclidean geometry passes through the embedding of affine geometry in projective geometry.)

Radiant affine structures (where the model space is the complement of a point in affine space) form another example. The fundamental example of a Hopf manifold is introduced here.

The chapter continues to discuss how these ideas naturally lead to various constructions of geometric structures, and, in some cases, their classification. For example, we discuss the classification of closed Euclidean manifolds (the Bieberbach theorems) from this viewpoint. One of the easiest constructions is the Cartesian product of affine structures. However they don't generalize directly (a Cartesian product of projective spaces does not have a projective structure. After discussing Cartesian products, we generalize this construction to several suspension or mapping torus constructions. For example the mapping torus of an affine automorphism of an affine $n$-manifold is an affine manifold of dimension $n + 1$ (parallel suspension). This provides a satisfactory classification of the compact complete affine manifolds in dimension
three: Every compact complete affine 3-manifold is finitely covered by a parallel suspension of an affine automorphism of a compact complete affine 2-manifold.

The mapping torus of a projective automorphism \( f \) of a projective \( n \)-manifold is an \((n + 1)\)-dimensional manifold with a radiant affine structure (radiant suspension). Hopf manifolds are a trivial case of this. Using the projective model for hyperbolic geometry and the fact that every closed orientable surface \( \Sigma \) of genus \( > 1 \) one obtains affine structures on products \( \Sigma \times S^1 \).

6.1. The hierarchy of geometries

There are many important examples of this correspondence, most of which occur when \( \Phi \) is an embedding. For example when \( \Phi \) is the identity map and \( G \subset G' \) is a subgroup, then every \((G, X)\)-structure is a fortiori an \((X', G')\)-structure. Thus every Euclidean structure is a similarity structure which in turn is an affine structure. Similarly every affine structure determines a projective structure, using the embedding

\[
(\mathbb{R}^n, \text{Aff}(\mathbb{R}^n)) \hookrightarrow (\mathbb{P}^n, \text{Proj}(\mathbb{P}^n))
\]
of affine geometry in projective geometry.

6.1.1. Projective structures from non-Euclidean geometry.

Using the Klein model of hyperbolic geometry

\[
(\mathbb{H}^n, \text{PO}(n, 1)) \hookrightarrow (\mathbb{P}^n, \text{Proj}(\mathbb{P}^n))
\]
every hyperbolic-geometry structure (that is, Riemannian metric of constant curvature -1) determines a projective structure. Using the inclusion of the projective orthogonal group \( \text{PO}(n + 1) \subset \text{PGL}(n + 1; \mathbb{R}) \) one sees that every elliptic-geometry structure (that is, Riemannian metric of constant curvature +1) determines a projective structure. Since every surface admits a metric of constant curvature, we obtain the following:

**Theorem 6.1.1.** Every surface admits an \( \mathbb{RP}^2 \)-structure.

6.1.2. Flat tori and Euclidean structures.

Recall that a flat torus is a Euclidean manifold of the form \( \mathbb{E}^n/\Gamma \), where \( \Gamma \) is a lattice of translations. We can regard flat tori as \((G, X)\)-manifolds where both \( X \) and \( G \) are the same vector space, and \( G \) is acting on \( X \) by translation. In fact, every closed \((G, X)\)-manifold is a flat torus.

Bieberbach’s structure theorem is essentially a qualitative structure theorem classifying closed Euclidean manifolds. It states that every closed Euclidean manifold is finitely covered by a flat torus. That is, given a closed Euclidean manifold \( M \), there is a flat torus \( N \) and a
finite subgroup $F \subset \text{Isom}(N)$ such that $F$ acts freely on $N$ and $M$ is isometric to the quotient manifold $N/F$.

From the viewpoint of enlarging and refining geometric structures, this result may be stated as follows. Corresponding to $F$ is a finite subgroup $\Phi \subset O(n)$, the linear holonomy group of $M$. Let $V\Phi$ be the subgroup of $\text{Isom}(\mathbb{E}^n)$ generated by the translation group $V$ and $\Phi$. Then Bieberbach’s structure theorem can be restated as follows:

**Theorem 6.1.2.** Every closed $(\text{Isom}(\mathbb{E}^n), \mathbb{E}^n)$ has a $(V\Phi, \mathbb{E}^n)$ structure for some finite subgroup $\Phi \subset O(n)$.

### 6.1.3. Hopf manifolds and radiant similarity manifolds

Here is another example of a refined geometric structure, which will arise in the classification of similarity structures on closed manifolds (Chapter 11, §11.4).

**Exercise 6.1.3.** Let $X = \mathbb{E}^n \setminus \{0\}$ and $G \subset \text{Sim}(\mathbb{E}^n)$ the stabilizer of $0$. Let $M$ be a compact $(G, X)$-manifold with holonomy group $\Gamma \subset G$.

- Prove that $G \cong \mathbb{R}^+ \times O(n)$.
- Find a $G$-invariant Riemannian metric $g_0$ on $X$.
- Suppose that $n > 2$. Prove that $M \cong \Gamma \backslash X$, and that $M$ admits a finite covering space isomorphic to a Hopf manifold.
- Suppose $n = 2$. Find an example where $M$ is not isomorphic to $\Gamma \backslash X$.

### 6.1.4. Enlarging and refining

Here is the general construction.

Suppose that $X \xrightarrow{\phi} X'$ is a universal covering space and $G$ is the group of lifts of transformations $X' \xrightarrow{g'} X'$ in $G'$ to $X$. Let $G \xrightarrow{\phi} G'$ be the corresponding homomorphism.

**Exercise 6.1.4.** Show that $(\Phi, \phi)$ induces an isomorphism between the categories of $(G, X)$-manifolds/maps and $(X', G')$-manifolds/maps.

For this reason we may always assume (when convenient) that our model space $X$ is simply-connected.

In many cases, we wish to consider maps between different manifolds with geometric structures modeled on different geometries. To this end we consider the following general situation. Let $(G, X)$ and $(X', G')$ be two homogeneous spaces representing different geometries and consider a family $\mathfrak{M}$ of maps $X \rightarrow X'$ such that if $f \in \mathfrak{M}, g \in G, g' \in G'$, then the composition

$$g' \circ f \circ g \in \mathfrak{M}.$$
If $U \subset X$ is a domain, a map $U \xrightarrow{f} X'$ is locally-$\mathcal{M}$ if for each component $U_i \subset U$ there exists $f_i \in \mathcal{M}$ such that the restriction of $f$ to $U_i \subset U$ equals the restriction of $f_i$ to $U_i \subset X$. Let $M$ be an $(G, X)$-manifold and $N$ an $(X', G')$-manifold. Suppose that $f : M \rightarrow N$ is a smooth map. We say that $f$ is an $\mathcal{M}$-map if for each pair of charts 

$$
U_\alpha \xrightarrow{\phi_\alpha} X \text{ (for } M) \\
V_\beta \xrightarrow{\psi_\beta} X \text{ (for } N)
$$

the restriction of the composition $\psi_\beta \circ f \circ \phi_\alpha^{-1}$ to $\phi_\alpha(U_\alpha \cap f^{-1}(V_\beta))$ is locally-$\mathcal{M}$.

The basic examples are affine and projective maps between affine and projective manifolds: For affine maps we take

$$(X, G) = (\mathbb{R}^m, \text{Aff}(\mathbb{R}^m)), \quad (X', G') = (\mathbb{R}^n, \text{Aff}(\mathbb{R}^n)), \quad \mathcal{M} = \text{aff}(\mathbb{R}^m, \mathbb{R}^n).$$

For example if $M, N$ are affine manifolds, and $M \times N$ is the product affine manifold (see §4.17), then the projections $M \times N \rightarrow M$ and $M \times N \rightarrow N$ are affine. Similarly if $x \in M$ and $y \in N$, the inclusions

$$
\{x\} \times N \hookrightarrow M \times N, \\
M \times \{y\} \hookrightarrow M \times N
$$

are each affine.

For projective maps we take

$$(X, G) = (\mathbb{P}^m, \text{Proj}(\mathbb{P}^m)), \quad (X', G') = (\mathbb{P}^n, \text{Proj}(\mathbb{P}^n)), \quad \mathcal{M} = \text{Proj}(\mathbb{P}^m, \mathbb{P}^n),$$

and $\mathcal{M} = \text{Proj}(\mathbb{P}^m, \mathbb{P}^n)$, the set of projective maps $\mathbb{P}^m \rightarrow \mathbb{P}^n$ (or more generally the collection of locally projective maps defined on open subsets of $\mathbb{P}^m$).

Here is a particularly striking application of projective mappings. If $M$ is an $\mathbb{RP}^n$-manifold it makes sense to speak of projective maps $I \rightarrow M$ and thus the Kobayashi pseudo-metric

$$
M \times M \xrightarrow{d^{Kob}} \mathbb{R}
$$

is defined. The following theorem combines results of Kobayashi [85] and Vey [146, 147], and is a kind of converse to §3.3.4:

**Theorem.** Let $M$ be a compact $\mathbb{RP}^n$-manifold and let $\widetilde{M}$ be its universal covering space. Then $d^{Kob}$ is a metric if and only if $\widetilde{M}$ is projectively isomorphic to a properly convex domain in $\mathbb{RP}^n$. 
6.2. Fibrations

One can also pull back geometric structures by fibrations of geometries as follows. Let \((G, X)\) be a homogeneous space and suppose that \(X' \xrightarrow{\Phi} X\) is a fibration with fiber \(F\) and that \(G' \xrightarrow{\phi} G\) is a homomorphism such that for each \(g' \in G'\) the diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{g'} & X' \\
\Phi \downarrow & & \downarrow \Phi \\
X' & \xrightarrow{\phi(g')} & X'
\end{array}
\]

commutes.

Suppose that \(M\) is an \((G, X)\)-manifold. Let \(\tilde{M} \rightarrow M\) be a universal covering with group of deck transformations \(\pi\) and \((\text{dev}, h)\) a development pair. Then the pullback \(\text{dev}^* \Phi\) is an \(F\)-fibration \(\tilde{M}'\) over \(\tilde{M}\) and the induced map \(M' \xrightarrow{\text{dev}'} X\) is a local diffeomorphism and thus a developing map for an \((G', X')\)-structure on \(\tilde{M}'\). We summarize these maps in the following commutative diagram:

\[
\begin{array}{ccc}
\tilde{M}' & \xrightarrow{\text{dev}'} & X' \\
\downarrow & & \downarrow \Phi \\
\tilde{M} & \xrightarrow{\text{dev}} & X
\end{array}
\]

Suppose that the holonomy representation \(\pi \xrightarrow{h} G\) lifts to \(\pi \xrightarrow{h'} G'\). (In general the question of whether \(h\) lifts will be detected by certain invariants in the cohomology of \(M\).) Then \(h'\) defines an extension of the action of \(\pi\) on \(\tilde{M}\) to \(\tilde{M}'\) by \((G', X')\)-automorphisms. Since the action of \(\pi\) on \(\tilde{M}'\) is proper and free, the quotient \(M' = \tilde{M}'/\pi\) is an \((G', X')\)-manifold. Moreover the fibration \(\tilde{M}' \rightarrow \tilde{M}\) descends to an \(F\)-fibration \(M' \rightarrow M\).

6.2.1. The sphere of directions. An important example is the following. Let \(G = \text{GL}(n+1; \mathbb{R})\) and \(X' = \mathbb{R}^{n+1} - \{0\}\). Let \(S^n = \mathbb{RP}^n\) denote the universal covering space of \(\mathbb{RP}^n\); for \(n > 1\) this is a two-fold covering space realized geometrically as the sphere of directions in \(\mathbb{R}^{n+1}\). Furthermore the group of lifts of \(\text{PGL}(n+1; \mathbb{R})\) to \(S^n\) equals the quotient \(\text{GL}(n+1; \mathbb{R})/\mathbb{R}^+ \cong \text{SL}^\pm(n+1; \mathbb{R})\).
with quotient map

\[ \mathbb{R}^{n+1} - \{0\} \xrightarrow{\varphi} S^n \]

a principal \( \mathbb{R}^+ \)-bundle.

### 6.2.2. Hopf manifolds.

The basic example of an incomplete affine structure on a closed manifold is a Hopf manifold. Consider the domain

\[ \Omega := \mathbb{R}^n - \{0\}. \]

The group \( \mathbb{R}^+ \) of homotheties (that is, scalar multiplications) acts on \( \Omega \) properly and freely with quotient the projective space \( \mathbb{R}\mathbb{P}^{n-1} \). Clearly the affine structure on \( \Omega \) is incomplete. If \( \lambda \in \mathbb{R} \) and \( \lambda > 1 \), then the cyclic group \( \langle \lambda \rangle \) is a discrete subgroup of \( \mathbb{R}^+ \) and the quotient \( \Omega / \langle \lambda \rangle \) is a compact incomplete affine manifold \( M \). We shall denote this manifold by \( \text{Hopf}^n_\lambda \). (A geodesic whose tangent vector “points” at the origin will be incomplete; on the manifold \( M \) the affinely parametrized geodesic will circle around with shorter and shorter period until in a finite amount of time will “run off” the manifold.) If \( n = 1 \), then \( M \) consists of two disjoint copies of the Hopf circle \( \mathbb{R}^+ / \langle \lambda \rangle \) — this manifold is an incomplete closed geodesic (and every incomplete closed geodesic is isomorphic to a Hopf circle). For \( n > 1 \), then \( M \) is connected and is diffeomorphic to the product \( S^1 \times S^{n-1} \). For \( n > 2 \) both the holonomy homomorphism and the developing map are injective.

If \( n = 2 \), then \( M \) is a torus whose holonomy homomorphism maps \( \pi_1(M) \cong \mathbb{Z} \oplus \mathbb{Z} \) onto the cyclic group \( \langle \lambda \rangle \). Note that \( \widetilde{M} \xrightarrow{\text{dev}} \mathbb{R}^2 \) is neither injective nor surjective, although it is a covering map onto its image. For \( k \geq 1 \) let \( \pi^{(k)} \subset \pi \) be the unique subgroup of index \( k \) which intersects \( \ker(h) \cong \mathbb{Z} \) in a subgroup of index \( k \). Let \( M^{(k)} \) denote the corresponding covering space of \( M \). Then \( M^{(k)} \) is another closed affine manifold diffeomorphic to a torus whose holonomy homomorphism is a surjection of \( \mathbb{Z} \oplus \mathbb{Z} \) onto \( \langle \lambda \rangle \).

**Exercise 6.2.1.** Show that for \( k \neq l \), the two affine manifolds \( M^{(k)} \) and \( M^{(l)} \) are not isomorphic. (Hint: consider the invariant defined as the least number of breaks of a broken geodesic representing a simple closed curve on \( M \) whose holonomy is trivial.) Thus two different affine structures on the same manifold can have the same holonomy homomorphism.

**Exercise 6.2.2.** Suppose that \( \lambda < -1 \). Then \( M = (\mathbb{R}^n - \{0\}) / \langle \lambda \rangle \) is an incomplete compact affine manifold doubly covered by \( \text{Hopf}^n_\lambda \). What is \( M \) topologically?
There is another point of view concerning Hopf manifolds in dimension two. Let \( M \) be a two-torus; we may explicitly realize \( M \) as a quotient \( \mathbb{C}/\Lambda \) where \( \Lambda \subset \mathbb{C} \) is a lattice. The complex exponential map \( \exp: \mathbb{C} \to \mathbb{C}^* \) is a universal covering space having the property that
\[
\exp \circ \tau(z) = e^z \cdot \exp
\]
where \( \tau(z) \) denotes translation by \( z \in \mathbb{C} \). For various choices of lattices \( \Lambda \), the exponential map
\[
\tilde{M} = \mathbb{C} \to \mathbb{C}^*
\]
is a developing map for a (complex) affine structure on \( M \) with holonomy homomorphism
\[
\pi \cong \Lambda \to \exp(\Lambda) \subset \text{Aff}(\mathbb{C})
\]
We denote this affine manifold by \( \exp(\mathbb{C}/\Lambda) \); it is an incomplete complex affine 1-manifold or equivalently an incomplete similarity 2-manifold. Every compact incomplete orientable similarity manifold is equivalent to an \( \exp(\mathbb{C}/\Lambda) \) for a unique lattice \( \Lambda \subset \mathbb{C} \). Taking \( \Lambda \subset \mathbb{C} \) to be the lattice generated by \( \log \lambda \) and \( 2\pi i \) we obtain the Hopf manifold \( \text{Hopf}_\lambda^2 \). More generally the lattice generated by \( \log \lambda \) and \( 2k\pi i \) corresponds to the \( k \)-fold covering space of \( \text{Hopf}_\lambda^2 \) described above. There are “fractional” covering spaces of the Hopf manifold obtained from the lattice generated by \( \log \lambda \) and \( 2\pi/n \) for \( n > 1 \); these manifolds admit \( n \)-fold covering spaces by \( \text{Hopf}_\lambda^2 \). The affine manifold \( M \) admits no closed geodesics if and only if \( \Lambda \cap \mathbb{R} = \{0\} \). Note that the exponential map defines an isomorphism \( \mathbb{C}/\Lambda \to M \) which is definitely not an isomorphism of affine manifolds.

Any \( \lambda > 1 \) generates a lattice inside the multiplicative group \( \mathbb{R}_+ \), which acts affinely on \( \mathbb{A}^1 \). The quotient \( \mathbb{R}_+/(\lambda) \) also defines an affine structure on \( M \), which is not a Euclidean structure since dilation by \( \lambda \) is not an isometry. Explicitly, take \( f \) to be a diffeomorphism onto the interval \([1, \lambda] \subset \mathbb{R} \approx \mathbb{A}^1 \), so that \( \text{dev} \) is a diffeomorphism of \( \tilde{M} \) onto \((0, \infty) = \mathbb{R}_+ \subset \mathbb{A}^1 \).

Like the preceding example, this affine structure is also bi-invariant with respect to the natural Lie group structure on \( \mathbb{R}_+/(\lambda) \).

Observe that, since the exponential map
\[
\mathbb{R} \to \mathbb{R}_+
\]
\[
x \mapsto e^x
\]
converts addition (translation) to multiplication (dilation), it defines a diffeomorphism between two quotients
\[
\mathbb{R}/\mathbb{1}\mathbb{Z} \to \mathbb{R}_+/(\lambda)
\]
where $l := \log(\lambda)$. This map also defines a (non-affine) analytic isomorphism between the corresponding Lie groups.

6.2.3. Lifting holonomy representations. Let $M$ be an $\mathbb{R}P^n$-manifold with development pair $(\text{dev}, h)$; we suppose that the holonomy representation lifts $\pi \xrightarrow{\tilde{h}} \text{PGL}(n+1, \mathbb{R})$ to $\pi \xrightarrow{\tilde{h}} \text{GL}(n+1; \mathbb{R})$.

Exercise 6.2.3. Find an example of a $\mathbb{R}P^n$-manifold whose holonomy representation does not lift to $\text{GL}(n+1; \mathbb{R})$.

The preceding construction then applies and we obtain a radiant affine structure on the total space $M'$ of a principal $\mathbb{R}^+$-bundle over $M$ with holonomy representation $\tilde{h}$. The radiant vector field $\rho_{M'}$ generates the (fiberwise) action of $\mathbb{R}^+$; this action of $\mathbb{R}^*$ on $M'$ is affine, given locally in affine coordinates by homotheties. (This construction is due to Benzécri [19] where the affine manifolds are called *variétés coniques affines*. He observes there that this construction defines an embedding of the category of $\mathbb{R}P^n$-manifolds into the category of $(n+1)$-dimensional affine manifolds.)

Since $\mathbb{R}^+$ is contractible, every principal $\mathbb{R}^+$-bundle is trivial (although there is in general no preferred trivialization). Choose any $\lambda > 1$; then the cyclic group $\langle \lambda \rangle \subset \mathbb{R}^+$ acts properly and freely on $M'$ by affine transformations. We denote the resulting affine manifold by $M'_\lambda$ and observe that it is homeomorphic to $M \times S^1$. (Alternatively, one may work directly with the Hopf manifold $\text{Hopf}_{n+1}^{\lambda}$ and its $\mathbb{R}^*$-fibration $\text{Hopf}_{n+1}^{\lambda} \rightarrow \mathbb{R}P^n$.) We thus obtain:

**Proposition 6.2.4** (Benzécri [19], §2.3.1). Suppose that $M$ is an $\mathbb{R}P^n$-manifold. Let $\lambda > 1$. Then $M \times S^1$ admits a radiant affine structure for which the trajectories of the radiant vector field are all closed geodesics each affinely isomorphic to the Hopf circle $\mathbb{R}^+/\langle \lambda \rangle$.

Since every (closed) surface admits an $\mathbb{R}P^2$-structure, we obtain:

**Corollary 6.2.5** (Benzécri [19]). Let $\Sigma$ be a closed surface. Then $\Sigma \times S^1$ admits an affine structure.

If $\Sigma$ is a closed hyperbolic surface, the affine structure on $M = \Sigma \times S^1$ can be described as follows. A developing map maps the universal covering of $M$ onto the convex cone

$$\Omega = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 - z^2 < 0, z > 0\}$$

which is invariant under the identity component $G$ of $\text{SO}(2, 1)$. The group $G \times \mathbb{R}^+$ acts transitively on $\Omega$ with isotropy group $\text{SO}(2)$. Choosing a hyperbolic structure on $\Sigma$ determines an isomorphism of $\pi_1(\Sigma)$
onto a discrete subgroup $\Gamma$ of $G$; then for each $\lambda > 1$, the group $\Gamma \times \langle \lambda \rangle$ acts properly and freely on $\Omega$ with quotient the compact affine 3-manifold $M$.

**Exercise 6.2.6.** A $\mathbb{CP}^n$-structure is a geometric structure modeled on complex projective space $\mathbb{CP}^n$ with coordinate changes locally from the projective group $\text{PGL}(n+1; \mathbb{C})$. If $M$ is a $\mathbb{CP}^n$-manifold, show that there is a $T^2$-bundle over $M$ which admits a complex affine structure and an $S^1$-bundle over $M$ which admits an $\mathbb{RP}^{2n+1}$-structure.

### 6.3. Suspensions

Before discussing Benzécri’s theorem and the classification of 2-dimensional affine manifolds, we describe several constructions for affine structures from affine structures and projective structures of lower dimension. Namely, let $\Sigma$ be a smooth manifold and $\Sigma \xrightarrow{f} \Sigma$ a diffeomorphism. The mapping torus of $f$ is defined to be the quotient $M = M_f(\Sigma)$ of the product $\Sigma \times \mathbb{R}$ by the $\mathbb{Z}$-action defined by

$$(x, t) \mapsto (f^{-n}x, t + n)$$

It follows that $dt$ defines a nonsingular closed 1-form $\omega$ on $M$ tangent to the fibration

$$M \xrightarrow{L} S^1 = \mathbb{R}/\mathbb{Z}.$$ 

Furthermore the vector field $\frac{\partial}{\partial t}$ on $\Sigma \times \mathbb{R}$ defines a vector field $S_f$ on $M$, the suspension of the diffeomorphism $\Sigma \xrightarrow{f} \Sigma$. The dynamics of $f$ is mirrored in the dynamics of $S_f$: there is a natural correspondence between the orbits of $f$ and the trajectories of $S_f$. The embedding $\Sigma \hookrightarrow \Sigma \times \{t\}$ is transverse to the vector field $S_f$ and each trajectory of $S_f$ meets $\Sigma$. Such a hypersurface is called a cross-section to the vector field. Given a cross-section $\Sigma$ to a flow $\{\xi_t\}_{t \in \mathbb{R}}$, then (after possibly reparametrizing $\{\xi_t\}_{t \in \mathbb{R}}$), the flow can be recovered as a suspension. Namely, given $x \in \Sigma$, let $f(x)$ equal $\xi_t(x)$ for the smallest $t > 0$ such that $\xi_t(x) \in \Sigma$, that is, the first-return map or Poincaré map for $\{\xi_t\}_{t \in \mathbb{R}}$ on $\Sigma$. For the theory of cross-sections to flows see Fried [51].

Suppose that $\mathcal{F}$ is a foliation of a manifold $M$; then $\mathcal{F}$ is locally defined by an atlas of smooth submersions $U \rightarrow \mathbb{R}^a$ for coordinate patches $U$. An $(G, X)$-atlas transverse to $\mathcal{F}$ is defined to be a collection of coordinate patches $U_\alpha$ and coordinate charts

$$U_\alpha \xrightarrow{\psi_\alpha} X.$$
such that for each pair \((U_\alpha, U_\beta)\) and each component \(C \subset U_\alpha \cap U_\beta\) there exists an element \(g_C \in G\) such that

\[ g_C \circ \psi_\alpha = \psi_\beta \]
on \(C\). An \((G, X)\)-structure transverse to \(\mathfrak{g}\) is a maximal \((G, X)\)-atlas transverse to \(\mathfrak{g}\). Consider an \((G, X)\)-structure transverse to \(\mathfrak{g}\); then an immersion \(\Sigma \xrightarrow{f} M\) which is transverse to \(\mathfrak{g}\) induces an \((G, X)\)-structure on \(\Sigma\).

A foliation \(\mathfrak{g}\) of an affine manifold is said to be affine if its leaves are parallel affine subspaces (that is, totally geodesic subspaces). It is easy to see that transverse to an affine foliation of an affine manifold is a natural affine structure. In particular if \(M\) is an affine manifold and \(\zeta\) is a parallel vector field on \(M\), then \(\zeta\) determines a one-dimensional affine foliation which thus has a transverse affine structure. Moreover if \(\Sigma\) is a cross-section to \(\zeta\), then \(\Sigma\) has a natural affine structure for which the Poincaré map \(\Sigma \xrightarrow{\phi} \Sigma\) is affine.

**Exercise 6.3.1.** Show that the Hopf manifold \(\text{Hopf}_n^\lambda\) has an affine foliation with one closed leaf if \(n > 1\) (two if \(n = 1\)) and its complement consists of two Reeb components.

**6.3.1. Parallel suspensions.** Let \(\Sigma\) be an affine manifold and \(f \in \text{Aff}(M)\) an automorphism. We shall define an affine manifold \(M\) with a parallel vector field \(S_f\) and cross-section \(\Sigma \xhookleftarrow M\) such that the corresponding Poincaré map is \(f\). (Compare §6.3.) We proceed as follows. Let \(\Sigma \times \mathbb{A}^1\) be the Cartesian product with the product affine structure and let \(\Sigma \times \mathbb{A}^1 \xrightarrow{\tilde{f}} \mathbb{A}^1\) be an affine coordinate on the second factor. Then the map

\[ \Sigma \times \mathbb{A}^1 \xrightarrow{\tilde{f}} \Sigma \times \mathbb{A}^1 \]

\[ (x, t) \mapsto (f^{-1}(x), t + 1) \]
is affine and generates a free proper \(\mathbb{Z}\)-action on \(\Sigma \times \mathbb{A}^1\), which \(t\)-covers the action of \(\mathbb{Z}\) on \(\mathbb{A}^1 \cong \mathbb{R}\) by translation. Let \(M\) be the corresponding quotient affine manifold. Then \(d/dt\) is a parallel vector field on \(\Sigma \times \mathbb{A}^1\) invariant under \(\tilde{f}\) and thus defines a parallel vector field \(S_f\) on \(M\). Similarly the parallel 1-form \(dt\) on \(\Sigma \times \mathbb{A}^1\) defines a parallel 1-form \(\omega_f\) on \(M\) for which \(\omega_f(S_f) = 1\). For each \(t \in \mathbb{A}^1/\mathbb{Z}\), the inclusion \(\Sigma \times \{t\} \xhookleftarrow M\) defines a cross-section to \(S_f\). We call \((M, S_f)\) the parallel suspension or affine mapping torus of the affine automorphism \((\Sigma, f)\).

**Exercise 6.3.2.** Suppose that \(N\) and \(\Sigma\) are affine manifolds and that

\[ \pi_1(\Sigma) \xrightarrow{\phi} \text{Aff}(N) \]
is an action of $\pi_1(\Sigma)$ on $N$ by affine automorphisms. The flat $N$-bundle over $\Sigma$ with holonomy $\phi$ is defined as the quotient of $\tilde{\Sigma} \times N$ by the diagonal action of $\pi_1(\Sigma)$ given by deck transformations on $\Sigma$ and by $\phi$ on $N$. Show that the total space $M$ is an affine manifold such that the fibration $M \to \Sigma$ is an affine map and the the flat structure (the foliation of $M$ induced by the foliation of $\tilde{\Sigma} \times N$ by leaves $\tilde{\Sigma} \times \{y\}$, for $y \in N$) is an affine foliation.

**6.3.2. Radiant supensions.** Now let $(M, \rho_M)$ be a radiant affine manifold of dimension $n+1$. Then there is an $\mathbb{RP}^n$-structure transverse to $\rho_M$. For in local affine coordinates the trajectories of $\rho_M$ are rays through the origin in $\mathbb{R}^{n+1}$ and the quotient projection maps coordinate patches submersively into $\mathbb{RP}^n$. In particular if $\Sigma$ is an $n$-manifold and $\Sigma \to M$ is transverse to $\rho_M$, then $f$ determines an $\mathbb{RP}^n$-structure on $\Sigma$.

**Proposition 6.3.3.** Let $\Sigma$ be a compact $\mathbb{RP}^n$-manifold and $f \in \text{Aut}(\Sigma)$ a projective automorphism. Then there exists a radiant affine manifold $(M, \rho_M)$ and a cross-section $\Sigma \to M$ to $\rho_M$ such that the Poincaré map for $f$ equals $f^{-1} \circ \phi \circ f$. In other words, the mapping torus of a projective automorphism of a compact $\mathbb{RP}^n$-manifold admits a radiant affine structure.

**Proof.** Let $S^n$ be the double covering of $\mathbb{RP}^n$ (realized as the sphere of directions in $\mathbb{R}^{n+1}$) and let

$$
\mathbb{R}^{n+1} \setminus \{0\} \xrightarrow{\Phi} S^n
$$

be the corresponding principal $\mathbb{R}^+$-fibration. Let $N$ be the principal $\mathbb{R}^+$-bundle over $M$ constructed in §6.2.1 and choose a section $M \xrightarrow{\sigma} N$. Let $\{\xi_t\}_{t \in \mathbb{R}}$ be the radiant flow on $N$ and denote by $\{\tilde{\xi}_t\}_{t \in \mathbb{R}}$ the radiant flow on $\tilde{N}$. Let $(\text{dev}, h)$ be a development pair; then $f$ lifts to an affine automorphism $\tilde{f}$ of $\tilde{M}$. Furthermore there exists a projective automorphism $g \in \text{GL}(n + 1; \mathbb{R})/\mathbb{R}^+$ of the sphere of directions $S^n$ such that

$$
\begin{array}{ccc}
\tilde{N} & \xrightarrow{\text{dev}'} & \mathbb{R}^{n+1} \setminus \{0\} \\
\downarrow j & & \downarrow g \\
\tilde{N} & \xrightarrow{\text{dev}'} & \mathbb{R}^{n+1} \setminus \{0\}
\end{array}
$$

commutes. Choose a compact set $K \subset \tilde{M}$ such that

$$
\pi_1(M) \cdot K = \tilde{M}.
$$
Let \( \tilde{K} \subset \tilde{N} \) be the image of \( K \) under a lift of \( \sigma \) to a section \( \tilde{M} \rightarrow \tilde{N} \). Then

\[
\tilde{K} \cap \tilde{J}_t(\tilde{K}) = \emptyset
\]

whenever \( t > t_0 \), for some \( t_0 \). It follows that the affine automorphism \( \xi_t \tilde{J} \) generates a free and proper affine \( \mathbb{Z} \)-action on \( N \) for \( t > t_0 \). We denote the quotient by \( M \). In terms of the trivialization of \( N \rightarrow M \) arising from \( \sigma \), it is clear that the quotient of \( N \) by this \( \mathbb{Z} \)-action is diffeomorphic to the mapping torus of \( f \). Furthermore the section \( \sigma \) defines a cross-section \( \Sigma \hookrightarrow M \) to \( \rho_M \) whose Poincaré map corresponds to \( f \). \( \square \)

We call the radiant affine manifold \((M, \rho_M)\) the radiant suspension of the pair \((\Sigma, f)\).

A natural question is whether every closed radiant affine manifold is a radiant suspension. A radiant affine manifold \((M, \xi)\) is a radiant suspension if and only if the flow \( \rho \) admits a cross-section. David Fried [49, 53] constructed a closed affine 6-manifold with diagonal holonomy whose radiant flow admits no cross-section. Choi [33] (using work of Barbot [7]) proves that every radiant affine 3-manifold is a radiant suspension, and therefore is either a Seifert 3-manifold covered by a product \( F \times S^1 \), where \( F \) is a closed surface, a nilmanifold or a hyperbolic torus bundle.

In dimensions 1 and 2 all closed radiant manifolds are radiant suspensions. When \( M \) is hyperbolic, that is, a quotient of a sharp convex cone (see Chapter 12), the existence of the Koszul 1-form implies that \( M \) is a radiant suspension.

In general affine automorphisms of affine manifolds can display quite complicated dynamics and thus the flows of parallel vector fields and radiant vector fields can be similarly complicated. For example, any element of \( \text{GL}(2; \mathbb{Z}) \) acts affinely on the flat torus \( \mathbb{R}^2/\mathbb{Z}^2 \); the most interesting of these are the hyperbolic elements of \( \text{GL}(2; \mathbb{Z}) \) which determine Anosov diffeomorphisms on the torus. Their suspensions thus determine Anosov flows on affine 3-manifolds which are generated by parallel or radiant vector fields. Indeed, it can be shown (Fried [52]) that every Anosov automorphism of a nilmanifold \( M \) can be made affine for some complete affine structure on \( M \).

As a simple example of this we consider the linear diffeomorphism of the two-torus \( T^2 = \mathbb{R}^2/\mathbb{Z}^2 \) defined by a hyperbolic element \( A \in \text{GL}(2; \mathbb{Z}) \). The parallel suspension of \( A \) is the complete affine 3-manifold \( \mathbb{R}^3/\Gamma \) where \( \Gamma \subset \text{Aff}(\mathbb{R}^3) \) consists of the affine transformations.
\[
\begin{bmatrix}
A^n & 0 & p \\
0 & 1 & n
\end{bmatrix}
\]

where \( n \in \mathbb{Z} \) and \( p \in \mathbb{Z}^2 \). Since \( A \) is conjugate in \( \text{SL}(2; \mathbb{R}) \) to a diagonal matrix with reciprocal eigenvalues, \( \Gamma \) is conjugate to a discrete cocompact subgroup of the subgroup of \( \text{Aff}(\mathbb{R}^3) \)

\[
G = \left\{ \begin{bmatrix}
e^u & 0 & 0 & s \\
0 & e^{-u} & 0 & t \\
0 & 0 & 1 & u
\end{bmatrix} \bigg| s, t, u \in \mathbb{R} \right\}
\]

which acts simply transitively. Since there are infinitely many conjugacy classes of hyperbolic elements in \( \text{SL}(2; \mathbb{Z}) \) (for example the matrices \( \begin{bmatrix} n+1 & n \\ 1 & 1 \end{bmatrix} \) for \( n > 1, n \in \mathbb{Z} \) are all non-conjugate), there are infinitely many isomorphism classes of discrete groups \( \Gamma \). Louis Auslander observed that there are infinitely many homotopy classes of compact complete affine 3-manifolds — in contrast to the theorem of Bieberbach that in each dimension there are only finitely many homotopy classes of compact flat Riemannian manifolds. Notice that each of these affine manifolds possesses a parallel Lorentz metric and hence is a flat Lorentz manifold. (Auslander-Markus [5]).

**Exercise 6.3.4.** Express the complete affine structures on the 2-torus as mapping tori of affine automorphisms of the complete affine manifold \( \mathbb{R}/\mathbb{Z} \).
CHAPTER 7

Classification

Given a topology $\Sigma$ and a geometry $(G, X)$, how does one determine the various ways of putting $(G, X)$-structures on $\Sigma$? This chapter discusses how to organize the geometric structures on a fixed topology. This is the general classification problem for $(G, X)$-structures.

7.1. Marking geometric structures

We begin with two more familiar and classical cases:

- The moduli space of flat tori;
- The classification of marked Riemann surfaces by Teichmüller space.

The latter is only analogous to our classification problem, but plays an important role, both historically and technically, in the study of locally homogeneous structures.

7.1.1. Marked Riemann surfaces. The prototype of this classification problem is the classification of Riemann surfaces of genus $g$. The Riemann moduli space is a space $\mathcal{M}_g$ whose points correspond to the biholomorphism classes of genus $g$ Riemann surfaces. It admits the structure of a quasiprojective complex algebraic variety. In particular it is a Hausdorff space, with a singular differentiable structure.

In general the set of $(G, X)$-structures on $\Sigma$ will not have such a nice structure. The natural space will in general not be Hausdorff, so we must expand our point of view. To this end, we introduce additional structures, called markings, such that the marked $(G, X)$-structures admit a more tractable classification. As before, the prototype for this classification is the Riemann moduli space $\mathcal{M}_g$, which can be understood as the quotient of the Teichmüller space $\mathcal{T}_g$ (comprising equivalence classes of marked Riemann surfaces of genus $g$) by the mapping class group $\text{Mod}_g$.

Here is the classical context for $\mathcal{T}_g$ and $\text{Mod}_g = \mathcal{T}_g/\text{Mod}_g$: The fixed topology is a closed orientable surface $\Sigma$ of genus $g$. A marked Riemann surface of genus $g$ is a pair $(M, f)$ where $M$ is a Riemann surface and $\Sigma \xrightarrow{f} M$ is a diffeomorphism. The Teichmüller space is
defined as the set of equivalence classes of marked Riemann surfaces of genus $g$, where two such marked Riemann surfaces $(M, f), (M', f')$ are equivalent if and only if there is a biholomorphism $M \xrightarrow{\phi} M'$ such that $\phi \circ f$ is isotopic to $\phi'$.

**Exercise 7.1.1.** Fix a Riemann surface $M$. The mapping class group

$$\text{Mod}_g := \pi_0(\text{Diff}(\Sigma))$$

acts simply transitively on the set of equivalence classes of marked Riemann surfaces $(M, f)$. Thus the Riemann moduli space $\text{Mod}_g$ is the quotient of the Teichmüller space $T_g$ by the mapping class group $\text{Mod}_g$.

**7.1.2. Moduli of flat tori.** Another common classification problem concerns flat tori. Recall (§5.4.1 a flat torus is a Euclidean manifold of the form $M^n := \mathbb{R}^n/\Lambda$, where $\Lambda \subset \mathbb{R}^n$ is a lattice. A marking of $M$ is just a basis of $\Lambda$. Clearly the set of marked flat $n$-tori is the set of bases of $\mathbb{R}^n$, which is a torsor for the group $\text{GL}(n, \mathbb{R})$. (The columns (respectively rows) of invertible $n \times n$ matrices are precisely bases of $\mathbb{R}^n$.)

**Exercise 7.1.2.** For $M = \mathbb{R}^n/\Lambda$ as above, compute the isometry group (respectively affine automorphism group) of $M$. Show that two invertible matrices $A, A' \in \text{GL}(n, \mathbb{R})$ define isometric marked flat tori if and only if $A'A^{-1} \in \text{O}(n)$. Show that all flat $n$-tori are affinely isomorphic.

The deformation space of marked flat tori identifies with the homogeneous space $\text{GL}(n, \mathbb{R})/\text{O}(n)$. The mapping class group $\text{Mod}(T^n)$ of the $n$-torus $T^n$ identifies with $\text{GL}(n, \mathbb{Z})$, which acts properly on the deformation space $\text{GL}(n, \mathbb{R})/\text{O}(n)$. The moduli space of flat tori in dimension $n$ identifies with the biquotient $\text{GL}(n, \mathbb{Z})\backslash\text{GL}(n, \mathbb{R})/\text{O}(n)$.

**7.1.3. Marked geometric manifolds.** Now we define the analogous construction for Ehresmann structures. As usual, we choose to work in the smooth category since $(G, X)$-manifolds carry natural smooth (in fact real analytic) structures, and the tools of differential topology are convenient. However, in general, there are many options, it may be more natural to consider homeomorphisms, or even homotopy equivalences, depending on the context. Since our primary interest in dimension two, where these notions yield equivalent theories, we do not discuss the alternative context.

**Definition 7.1.3.** Let $\Sigma$ be a smooth manifold. A marking of an $(G, X)$-manifold $M$ (with respect to $\Sigma$) is a diffeomorphism $\Sigma \xrightarrow{f} M$. 
A marked $(G, X)$-manifold is a pair $(M, f)$ where $f$ is a marking of $M$. Two marked $(G, X)$-manifolds $(f, M)$ and $(f', M')$ are equivalent if there exists an $(G, X)$-isomorphism $M \xrightarrow{\phi} M'$ such that $\phi \circ f \simeq \phi'$.

**7.1.4. The infinitesimal approach.** More useful for computations is another approach, where geometric structures are defined infinitesimally as structures on vector bundles associated to the tangent bundle. For example, a Euclidean manifold $M$ can be alternatively described as a Riemannian metric on $M$ with vanishing curvature tensor. Another example is defining a Riemann surface as a 2-manifold together with an almost complex structure, that is, a complex structure on its tangent bundle. A third example is defining an affine structure as a connection on the tangent bundle with vanishing curvature tensor. Projective structures and conformal structures can be defined in terms of projective connections and conformal connections, respectively.

In all of these cases, the underlying smooth structure is fixed, and the geometric structure is replaced by an infinitesimal object as above. The diffeomorphism group acts on this space, and the quotient by the full diffeomorphism group would serve as the moduli space. However, to avoid pathological quotient spaces, we prefer to quotient by the identity component of $\text{Diff}$. Alternatively define the deformation space of marked structures as the quotient of the space of the infinitesimal objects by the subgroup of $\text{Diff}$ consisting of diffeomorphisms isotopic to the identity.

The “infinitesimal objects” above are Cartan connections, to which we refer to Sharpe [127].

**7.2. The Ehresmann-Weil-Thurston holonomy principle**

Fundamental in the deformation theory of locally homogeneous (Ehresmann) structures is the following principle, first observed in this generality by Thurston [140]:

**Theorem 7.2.1.** Let $X$ be a manifold upon which a Lie group $G$ acts transitively. Let $M$ be a compact $(G, X)$-manifold with holonomy representation $\pi_1(M) \xrightarrow{h} G$.

1. Suppose that $h'$ is sufficiently near $h$ in the representation variety $\text{Hom}(\pi_1(M), G)$. Then there exists a (nearby) $(G, X)$-structure on $M$ with holonomy representation $h'$.
2. If $M'$ is a $(G, X)$-manifold near $M$ having the same holonomy $h$, then $M'$ is isomorphic to $M$ by an isomorphism isotopic to the identity.
Thurston sketches the intuitive ideas for Theorem 7.2.1 in his notes [140]. The first detailed proofs of this fact are Lok [104], Canary-Epstein-Green [26], and Goldman [61] (the proof in [61] was worked out with M. Hirsch, and were independently found by A. Haefliger). The ideas in these proofs may be traced to Ehresmann [45], although he didn’t express them in terms of moduli of structures. Corollary 7.4.1 was noted by Koszul [90], Chapter IV, §3, Theorem 3; compare also the discussion in Kapovich [80], Theorem 7.2.

7.3. Representation varieties

As this theorem concerns the topology of the space of holonomy representations, we first discuss the space $\text{Hom}(\pi, G)$ and its quotient $\text{Rep}(\pi, G)$. Good general references for this theory are:
- Lubotzky-Magid [106], Labourie [101] and Sikora [129]
- The infinitesimal theory, and its relation to cohomology, can be found in Raghunathan [125]. Explicit formulas using the free differential calculus of Fox [48], are described in Goldman [59].

7.4. Deformation spaces of geometric structures

Here the topology on marked $(G, X)$-manifolds is defined in terms of the atlases of coordinate charts, or equivalently in terms of developing maps, or developing sections. In particular one can define a deformation space $\text{Def}_{(G,X)}(\Sigma)$ whose points correspond to equivalence classes of marked $(G, X)$-structures on $\Sigma$. One might like to say the holonomy map

$$\text{Def}_{(G,X)}(\Sigma) \xrightarrow{\text{hol}} \text{Hom}(\pi_1(\Sigma), G)/\text{Inn}(G)$$

is a local homeomorphism, with respect to the quotient topology on $\text{Hom}(\pi_1(\Sigma), G)/\text{Inn}(G)$ induced from the classical topology on the $\mathbb{R}$-analytic set $\text{Hom}(\pi_1(\Sigma), G)$. In many cases this is true (see below) but misstated in [63]. However, Kapovich [79] and Baues [8] observed that this is not quite true, because local isotropy groups acting on $\text{Hom}(\pi_1(\Sigma), G)$ may not fix marked structures in the corresponding fibers.

In any case, these ideas have an important consequence:

**Corollary 7.4.1.** Let $M$ be a closed manifold. The set of holonomy representations of $(G, X)$-structures on $M$ is open in $\text{Hom}(\pi_1(M), G)$ (with respect to the classical topology).
7.5. AFFINE STRUCTURES AND CONNECTIONS

One can define a space of flat \((G, X)\)-bundles (defined by a fiber bundle \(\mathcal{E}_M\) having \(X\) as fiber and \(G\) as structure group) and the foliation \(\mathcal{F}\) transverse to the fibration \(\mathcal{E}_M \to M\). The foliation \(\mathcal{F}\) is equivalent to a reduction of the structure group of the bundle from \(G\) with the classical topology to \(G\) with the discrete topology. This set of flat \((G, X)\)-bundles over \(\Sigma\) identifies with the quotient of the \(\mathbb{R}\)-analytic set \(\text{Hom}(\pi_1(\Sigma), G)\) by the action of the group \(\text{Inn}(G)\) of inner automorphisms action by left-composition on homomorphisms \(\pi_1(\Sigma) \to G\).

Conversely, if two nearby structures on a compact manifold \(M\) have the same holonomy, they are equivalent. The \((G, X)\)-structures are topologized as follows. Let \(\Sigma \to M\) be a marked \((G, X)\)-manifold, that is, a diffeomorphism from a fixed model manifold \(\Sigma\) to a \((G, X)\)-manifold \(M\). Fix a universal covering \(\tilde{\Sigma} \to \Sigma\) and let \(\pi = \pi_1(\Sigma)\) be its group of deck transformations. Choose a holonomy homomorphism \(\pi \to G\) and a developing map \(\tilde{\Sigma} \xrightarrow{\text{dev}} X\).

In the nicest cases, this means that under the natural topology on flat \((G, X)\)-bundles \((X_h, \mathcal{F}_h)\) over \(M\), the holonomy map \(\text{hol}\) is a local homeomorphism. Indeed, for many important cases such as hyperbolic geometry (or when the structures correspond to geodesically complete affine connections), \(\text{hol}\) is actually an embedding.

7.5. Affine structures and connections

We shall be interested in putting affine geometry on a manifold, that is, finding a coordinate atlas on a manifold \(M\) such that the coordinate changes are locally affine. Such a structure will be called an affine structure on \(M\). We say that the manifold is modeled on an affine space \(E\) if its coordinate charts map into \(E\). Clearly an affine structure determines a differential structure on \(M\). A manifold with an affine structure will be called an affinely flat manifold, or just an affine manifold. If \(M, M'\) are affine manifolds (of possibly different dimensions) and \(M \xrightarrow{f} M'\) is a map, then \(f\) is affine if in local affine coordinates, \(f\) is locally affine. If \(G \subset \text{Aff}(E)\) then we recover more refined structures by requiring that the coordinate changes are locally restrictions of affine transformations from \(G\). For example if \(G\) is the group of Euclidean isometries, we obtain the notion of a Euclidean structure on \(M\).

Our locally homogeneous geometric structures (Ehresmann structures) can be characterized in terms of more general differential-geometric objects. Namely, affine structures on a smooth manifold \(M\) are flat torsion-free affine connections on \(M\). Specifically these are affine connections (connections on the tangent bundle \(\mathbb{T}M\) for which both the
curvature tensor and the torsion tensor vanish. Similarly Euclidean structures on $M$ are flat Riemannian metrics on $M$ (those for which the curvature tensor of the Levi-Civita connection vanishes). This notion is equivalent to an affine structure with parallel Riemannian structure, as described in §1.2.1.

If $M$ is a manifold, we denote the Lie algebra of vector fields on $M$ by $\text{Vec}(M)$. A vector field $\xi$ on an affine manifold is affine if in local coordinates $\xi$ appears as a vector field in $\text{aff}(E)$. We denote the space of affine vector fields on an affine manifold $M$ by $\text{aff}(M)$.

**Exercise 7.5.1.** Let $M$ be an affine manifold.

1. Show that $\text{aff}(M)$ is a subalgebra of the Lie algebra $\text{Vec}(M)$.
2. Show that the identity component of the affine automorphism group $\text{Aut}(M)$ has Lie algebra $\text{aff}(M)$.
3. If $\nabla$ is the flat affine connection corresponding to the affine structure on $M$, show that a vector field $\xi \in \text{Vec}(M)$ is affine if and only if

$$\nabla_{\xi} v = [\xi, v]$$

$\forall v \in \text{Vec}(M)$. 
CHAPTER 8

Completeness

In many important cases the developing map is a diffeomorphism \( \tilde{M} \rightarrow X \), or at least a covering map onto its image. In particular if \( \pi_1(X) = \{e\} \), such structures are quotient structures:

\[ M \cong \Gamma \backslash X \]

We also call such quotient structures tame. This chapter develops criteria for taming the developing map.

Many important geometric structures are modeled on homogeneous Riemannian manifolds. These structures determine Riemannian structures, which are locally homogeneous metric spaces. For these structures, completeness of the metric space will tame the developing map.

Although it is not completely necessary, this closely relates to geodesic completeness of the associated Levi-Civita connection. The key tool is the Hopf-Rinow theorem: Geodesic completeness (of the Levi-Civita connection) is equivalent to completeness of the associated metric space. In particular compact Riemannian manifolds are geodesically complete. Many Ehresmann structures have natural Riemannian structures whose completeness tames of the developing map. In particular such structures are quotient structures as above.

After giving some general remarks on the developing map, its relation to the exponential map (for affine connections), we describe all the complete affine structures on \( T^2 \). The chapter ends with a discussion of incomplete affine structures on \( T^2 \), and a general discussion of the most important incomplete examples, Hopf manifolds, which were introduced in §6.2.2 of Chapter 6.

8.1. Locally homogeneous Riemannian manifolds

Suppose \((G, X)\) is a Riemannian homogeneous space, that is, \(X\) possesses a \(G\)-invariant Riemannian metric \(g_X\). Equivalently, \(X = G/H\) where the isotropy group \(H\) is compact. Precisely, the image of the adjoint representation \(\text{Ad}(H) \subset \text{GL}(g)\) is compact.

Exercise 8.1.1. Prove that these two conditions on the homogeneous space \((G, X)\) are equivalent.
We use the following consequence of the Hopf-Rinow theorem from Riemannian geometry: Geodesic completeness of a Riemannian structure (the complete extendability of geodesics) is equivalent to the completeness of the corresponding metric space (convergence of Cauchy sequences). (Compare do Carmo [40], Kobayashi-Nomizu [86], Lee [102], Milnor [111], O’Neill [119], or Papadopoulos [121].)

Recall our standard notation from Chapter 5: $M$ is a $(G,X)$-manifold with universal covering space $\widetilde{M} \xrightarrow{\Pi} M$; denote by $\pi$ the associated fundamental group, and $(\text{dev},\text{hol})$ a development pair.

**Proposition 8.1.2.** Let $(G,X)$ be a Riemannian homogeneous space. Suppose that $X$ is simply connected. Then:

- $\widetilde{M} \xrightarrow{\text{dev}} X$ is a diffeomorphism;
- $\pi \xrightarrow{\text{hol}} G$ is an isomorphism of $\pi$ onto a cocompact discrete subgroup $\Gamma \subset G$.

**Proof of Proposition 8.1.2.** The Riemannian metric

$$g_{\widetilde{M}} = \text{dev}^* g_X$$

on $\widetilde{M}$ is invariant under the group of deck transformations $\pi_1(M)$ of $\widetilde{M}$ and hence there is a Riemannian metric $g_M$ on $M$ such that $\Pi^* g_M = g_{\widetilde{M}}$. Since $M$ is compact, the metric $g_M$ on $M$ is complete and so is the metric $g_{\widetilde{M}}$ on $\widetilde{M}$. By construction,

$$(\widetilde{M},g_{\widetilde{M}}) \xrightarrow{\text{dev}} (X,g_X)$$

is a local isometry. A local isometry from a complete Riemannian manifold into a Riemannian manifold is necessarily a covering map (Kobayashi-Nomizu [86]) so $\text{dev}$ is a covering map of $\widetilde{M}$ onto $X$. Since $X$ is simply connected, it follows that $\text{dev}$ is a diffeomorphism. Let $\Gamma \subset G$ denote the image of $h$. Since $\text{dev}$ is equivariant respecting $h$, the action of $\pi$ on $X$ given by $h$ is equivalent to the action of $\pi$ by deck transformations on $\widetilde{M}$. Thus $h$ is faithful and its image $\Gamma$ is a discrete subgroup of $G$ acting properly and freely on $X$. Furthermore $\text{dev}$ defines a diffeomorphism

$$M = \widetilde{M}/\pi \xrightarrow{} X/\Gamma.$$ 

Since $M$ is compact, it follows that $X/\Gamma$ is compact, and since the fibration $G \xrightarrow{} G/H = X$ is proper, the homogeneous space $\Gamma \setminus G$ is compact, that is, $\Gamma$ is cocompact in $G$. □
One may paraphrase the above result abstractly as follows. Let $(G, X)$ be a Riemannian homogeneous space. Then there is an equivalence of categories:

\[
\{ \text{Compact } (G, X)\text{-manifolds/maps} \} \iff \{ \text{Discrete cocompact subgroups of } G \text{ acting freely on } X \} \]

where the morphisms in the latter category are inclusions of subgroups composed with inner automorphisms of $G$).

**Exercise 8.1.3.** Let $G$ be a Lie group and let $X = G$ with the action of $G$ by left-translation. Prove the equivalence of categories:

\[
\{ (G, X)\text{-manifolds/maps} \} \iff \{ \text{Discrete subgroups of } G \} \]

We say that an $(G, X)$-manifold $M$ is complete if $\tilde{M} \xrightarrow{\text{dev}} X$ is a diffeomorphism (or a covering map if we don’t insist that $X$ be simply connected). An $(G, X)$-manifold $M$ is complete if and only if its universal covering $\tilde{M}$ is $(G, X)$-isomorphic to $X$, that is, if $M$ is isomorphic to the quotient $X/\Gamma$ (at least if $X$ is simply connected). Note that if $(G, X)$ is contained in $(G', X')$ in the sense of §5.2.3 and $X \neq X'$, then a complete $(G, X)$-manifold is never complete as an $(G', X')$-manifold.

8.2. Completeness and convexity of affine connections

A more traditional proof of Proposition 8.1.2 uses the theory of geodesics. Geodesics are curves with zero acceleration, where acceleration of a smooth curve is defined in terms of an affine connection, which is just a connection on the tangent bundle of a smooth manifold. Connections appear twice in our applications: first, as Levi-Civita connections for Riemannian homogeneous spaces, and second, for flat affine structures. These contexts meet in the setting of Euclidean manifolds.

After we briefly review the standard theory of affine connections and the geodesic flow, we discuss the theorem of Auslander-Markus characterizing complete affine structures. Then we discuss the closely related notion of geodesic convexity and prove Koszul’s theorem relating convexity to the developing map.

8.2.1. **Review of affine connections.** Suppose that $M$ is a smooth manifolds with an affine connection $\nabla$. Let $p \in M$ be a point and $v \in T_p M$ a tangent vector. Then

\[
\exists a, b \in \mathbb{R} \cup \{\pm \infty\} \]
such that

\[ -\infty \leq a < 0 < b \leq \infty \]

and a geodesic \( \gamma(t) \), defined for \( a < t < b \), with \( \gamma(0) = p \) and \( \gamma'(0) = \nu \). We call \((p, \nu)\) the initial conditions. Furthermore \( \gamma \) is unique in the sense that two such \( \gamma \) agree on their common interval of intersection. We may choose the interval \((a, b)\) to be maximal. When \( b = \infty \) (respectively \( a = -\infty \)), the geodesic is forwards complete, (respectively backwards complete). A geodesic is complete if and only if it is both forwards and backwards complete. In that case it is defined on all of \( \mathbb{R} \). We say \((M, \nabla)\) is geodesically complete if and only if every geodesic extends to a complete geodesic.

In general, an open subset \( \mathcal{E} \subset TM \) and a map \( \mathcal{E} \xrightarrow{\text{Exp}} M \) exist such that:

- \( \mathcal{E} \) contains the zero-section \( 0_M \) of \( TM \);
- For any \( p \in M \) and \( \nu \in \mathcal{T}_pM \),

\[
(a, b) \xrightarrow{} M \\
t \mapsto \text{Exp}(t\nu)
\]

is the maximal geodesic with initial conditions \((p, \nu)\).

For \( p \in M \), denote

\[
\mathcal{E}_p := \mathcal{E} \cap T_pM
\]

and \( \text{Exp}_p \) the restriction \( \text{Exp}|_{\mathcal{E}_p} \). Then \((M, \nabla)\) is geodesically complete if and only if \( \mathcal{E} = TM \). In that case

\[
(p, \nu) \xrightarrow{\Phi} \left(\text{Exp}_p(t\nu), \frac{d}{dt}\text{Exp}_p(t\nu)\right)
\]

defines a flow (that is, an additive \( \mathbb{R} \)-action) on \( TM \), called the geodesic flow of \((M, \nabla)\). The vector

\[
\frac{d}{dt}\text{Exp}_p(t\nu)
\]

is the image of \( \nu \) under parallel translation along the geodesic \( \text{Exp}_p|_{[0,t]} \).

**Definition 8.2.1.** Let \((M, \nabla)\) be a manifold with an affine connection, and let \( x, y \in M \). Then \( y \) is visible from \( x \) if and only if a geodesic joins \( x \) to \( y \). Equivalently, \( y \) lies in the image \( \text{Exp}_x(\mathcal{E}_x) \). Evidently \( y \) is visible from \( x \) if and only if \( x \) is visible from \( y \). We say that \( y \) is invisible from \( x \) if and only if \( y \) is not visible from \( x \).
In local coordinates on $\mathbb{T}M$, where $(x^1, \ldots, x^n)$ are local coordinates on $M$ and $(v^1, \ldots, v^n)$ are local coordinates on $T_pM$, the geodesic equations are:

$$\frac{d}{dt} x^k(t) = v^k(t)$$

$$\frac{d}{dt} v^i(t) = -\Gamma^k_{ij}(x) v^i(t) v^j(t)$$

(for $k = 1, \ldots, n$) and $\Gamma^k_{ij}(x)$ are the Christoffel symbols defined by:

$$\nabla_{\partial_i} (\partial_j) = \Gamma^k_{ij}(x) \partial_k.$$

In particular the geodesic flow corresponds to the vector field

$$v^k \frac{\partial}{\partial x^k} - \Gamma^k_{ij}(x)v^i v^j \frac{\partial}{\partial v^k}$$

on $\mathbb{T}M$. See Kobayshi-Nomizu [86], do Carmo [40] or O’Neill [119] for further details.

8.2.2. Geodesic completeness and the developing map. Recall from Chapter 1 that geodesics — curves in $\mathbb{A}$ with zero acceleration — are curves in Euclidean space travelling along straight lines at constant speed. Of course, in affine geometry, the speed doesn’t make sense, which is why we prefer to characterize geodesics by acceleration. A fundamental result of Auslander-Markus [4] is that geodesic completeness of affine manifolds is equivalent to the bijectivity of the developing map.

**Theorem 8.2.2 (Auslander-Markus [4]).** Let $M$ be an affine manifold, with a developing map $\mathbb{\widetilde{M}} \xrightarrow{\text{dev}} \mathbb{A}$. Then $\text{dev}$ is an isomorphism if and only if $M$ is geodesically complete. That is, the following two conditions are equivalent:

- $M$ is a quotient of affine space by a discrete subgroup $\Gamma \subset \text{Aff}(\mathbb{A})$ acting properly on $\mathbb{A}$;
- A particle on $M$ moving at constant speed in a straight line will continue indefinitely.

Clearly if $M$ is geodesically complete, so is its universal covering $\mathbb{\widetilde{M}}$. Hence we may assume that $M$ is simply connected. Let $p \in M$. If $M$ is complete, then $\text{Exp}_p$ is defined on all of $T_pM$ and

$$\begin{array}{ccc}
T_pM & \xrightarrow{\text{Ddev}} & T_{\text{dev}(p)} \mathbb{A} \\
\exp \downarrow & & \downarrow \exp \\
M & \xrightarrow{\text{dev}} & \mathbb{A}
\end{array}$$
commutes. Since the vertical arrows and the top horizontal arrows are bijective, $M \xrightarrow{\text{dev}} A$ is bijective.

The other direction follows as a corollary of the basic Theorem 8.2.3, proved below. First we make a few general remarks about the exponential map for flat affine manifolds.

**Theorem 8.2.3 (Koszul [91]).** Let $M$ be an affine manifold and $p \in M$. Suppose that the domain $\mathcal{E}_p \subset T_p M$ of the exponential map $\text{Exp}_p$ is convex. Then $\tilde{M} \xrightarrow{\text{dev}} A$ is a diffeomorphism of $\tilde{M}$ onto the open subset

$$\Omega_p := \text{Exp}_p(\mathcal{E}_p) \subset A$$

**Lemma 8.2.4.** Let $M$ be an affine manifold with developing map $M \xrightarrow{\text{dev}} A$. Let $p \in M$. Let $0_p \in \mathcal{E}_p \subset T_p M$ and

$$T_p M \xrightarrow{\text{Exp}_p} M$$

the exponential map. Then there exists a unique affine isomorphism $T_p M \xrightarrow{\tilde{\pi}} A$ whose restriction to the connected starshaped open subset $\mathcal{E}_p \subset T_p M$ equals the composition $\text{Exp}_p \circ \text{dev}$:

$$\mathcal{E}_p \xrightarrow{\text{Exp}_p} M \xrightarrow{\text{dev}} A$$

**Proof of Theorem 8.2.3.** Clearly it suffices to assume that $M$ is simply connected, so that $\tilde{M} \xrightarrow{\text{dev}} A$ is defined.

The key step is the following:

**Lemma 8.2.5.** The image $\text{Exp}_p(\mathcal{E}_p) = M$.

**Proof of Lemma 8.2.5.** $\mathcal{E}_p \subset T_p M$ is open and $\text{Exp}_p$ is an open map, so the image $\text{Exp}_p(\mathcal{E}_p)$ is open. Since $M$ is assumed to be connected, we show that $\text{Exp}_p(\mathcal{E}_p)$ is closed.

Let $q \in \overline{\text{Exp}_p(\mathcal{E}_p)} \subset M$. Since $M$ is simply connected, $\text{Exp}_p$ maps $\mathcal{E}_p$ bijectively onto $\overline{\text{Exp}_p(\mathcal{E}_p)}$. Since $q \in \overline{\text{Exp}_p(\mathcal{E}_p)}$, Thus there exists $v \in T_p M$ such that

$$\lim_{t \to 1} \text{Exp}_p(tv) = q.$$

Since $\mathcal{E}_p \subset T_p M$ is convex, $tv \in \mathcal{E}_p$ for $0 \leq t < 1$, so that $\text{Exp}(v)$ is defined, and equals $q$.\[\square\]

We return to the proof of Theorem 8.2.3.
The diagram

\[
\begin{array}{ccc}
T_pM & \xrightarrow{D_p\text{dev}} & T_{\text{dev}(p)}A \\
\uparrow & & \uparrow \\
\mathcal{E}_p & \xrightarrow{D_p\text{dev}} & (D_p\text{dev})(\mathcal{E}_p) \\
\Exp_p & \downarrow & \Exp_{\text{dev}(p)} \\
M & \xrightarrow{\text{dev}} & \Omega
\end{array}
\]

commutes, where the first vertical arrows are inclusions. By the previous argument (now applied to the subset \(\mathcal{E}_p \subset T_pM\)) the developing map \(\text{dev}\) is injective. However, Lemma 8.2.5 implies that \(\Exp_p(\mathcal{E}_p) = M\) and thus \(\text{dev}(M) = \Omega\).

**Exercise 8.2.6.** Suppose that \(X\) is simply connected. Let \(M\) be a closed \((G, X)\)-manifold with developing pair \((\text{dev}, \text{hol})\). Show that \(M\) is complete if and only if the holonomy representation \(\pi \xrightarrow{\text{hol}} G\) is an isomorphism of \(\pi\) onto a discrete subgroup of \(G\) which acts properly and freely on \(X\).

### 8.3. Complete affine structures on the 2-torus

The compact complete affine 1-manifold \(\mathbb{R}/\mathbb{Z}\) is unique up to affine isomorphism. Its Cartesian square \(\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}\) is a Euclidean structure on the two-torus, unique up to affine isomorphism. In this section we shall describe all other complete affine structures on the two-torus and show that they are parametrized by the plane \(\mathbb{R}^2\).

These structures were first discussed by Kuiper [96]; compare also Baues [10, 9] and Baues-Goldman [96].

We begin by considering the one-parameter family of (quadratic) diffeomorphisms of the affine plane \(A^2\) defined by

\[
\phi_r(x, y) = (x + ry^2, y)
\]

Since

\[
\phi_r \circ \phi_s = \phi_{r+s},
\]

\(\phi_r\) and \(\phi_{-r}\) are inverse maps. If \(u = (s, t) \in \mathbb{R}^2\) we denote translation by \(v\) as \(A \xrightarrow{\tau_v} A\). Conjugation of the translation \(\tau_vu\) by \(\phi_r\) yields the affine transformation

\[
\alpha_r(u) = \phi_r \circ \tau_u \circ \phi_{-r} = \begin{bmatrix} 1 & 2rt \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s + rt^2 \\ t \end{bmatrix}
\]

and

\[
\mathbb{R}^2 \xrightarrow{\alpha_r} \text{Aff}(A)
\]
defines a simply transitive affine action. (Compare [54], §1.19.) If \( \Lambda \subset \mathbb{R}^2 \) is a lattice, then \( A/\alpha_r(\Lambda) \) is a compact complete affine 2-manifold \( M = M(r; \Lambda) \) diffeomorphic to a 2-torus.

The parallel 1-form \( dy \) defines a parallel 1-form \( \eta \) on \( M \) and its cohomology class

\[
[\eta] \in H^1(M; \mathbb{R})
\]

is a well-defined invariant of the affine structure up to scalar multiplication. In general, \( M \) will have no closed geodesics. If \( \gamma \subset M \) is a closed geodesic, then it must be a trajectory of the vector field on \( M \) arising from the parallel vector field \( \partial / \partial x \) on \( A \); then \( \gamma \) is closed if and only if the intersection of the lattice \( \Lambda \subset \mathbb{R}^2 \) with the line \( \mathbb{R} \oplus \{0\} \subset \mathbb{R}^2 \) is nonzero.

To classify these manifolds, note that the normalizer of \( G_r = \alpha_r(\mathbb{R}^2) \) equals

\[
\left\{ \begin{bmatrix} \mu^2 & a \\ 0 & \mu \end{bmatrix} \bigg| \mu \in \mathbb{R}^*, a \in \mathbb{R} \right\} \cdot G_r
\]

which acts on \( G_r \) conjugating

\[
\alpha_r(s, t) \mapsto \alpha_r(\mu^2 s + at, \mu t)
\]

Let

\[
N = \left\{ \begin{bmatrix} \mu^2 & a \\ 0 & \mu \end{bmatrix} \bigg| \mu \in \mathbb{R}^*, a \in \mathbb{R} \right\};
\]

then the space of affine isomorphism classes of these tori may be identified with the homogeneous space \( \text{GL}(2, \mathbb{R})/N \) which is topologically \( \mathbb{R}^2 - \{0\} \). The groups \( G_r \) are all conjugate and as \( r \to 0 \), each representation \( \alpha_r|_\pi \) converges to an embedding of \( \pi \) as a lattice of translations \( \mathbb{R}^2 \to \mathbb{R}^2 \). It follows that the deformation space of complete affine structures on \( T^2 \) form a space which is the union of \( \mathbb{R}^2 - \{0\} \) with a point \( O \) (representing the Euclidean structure) which is in the closure of every other structure.
Figure 5. Tilings corresponding to some complete affine structures on the 2-torus. The first depicts a square Euclidean torus. The second and third pictures depict non-Riemannian deformations where the holonomy group contains horizontal translations.
Figure 6. Tilings corresponding to some complete affine structures on the 2-torus. The second picture depicts a complete non-Riemannian deformation where the affine holonomy contains no nontrivial horizontal translation. The corresponding torus contains no closed geodesics.
8.4. Examples of Incomplete Structures

Constructing incomplete geometric structures on noncompact manifolds $M$ is easy. Take any immersion $M \xrightarrow{f} X$ which is not bijective; then $f$ induces an $(G, X)$-structure on $M$. If $M$ is parallelizable, then such an immersion always exists (Hirsch [74]). More generally, let $\pi \xrightarrow{h} G$ be a representation; then as long as the associated flat $(G, X)$-bundle $E \rightarrow X$ possesses a section $M \xrightarrow{s} E$ whose normal bundle is isomorphic to $TM$, there exists an $(G, X)$-structure with holonomy $h$ (see Haefliger [70]).

It is harder to construct incomplete geometric structures on compact manifolds — indeed for certain geometries $(G, X)$, there exist closed manifolds for which every $(G, X)$-structure on $M$ is complete. As a trivial example, if $X$ is compact and $M$ is a closed manifold with finite fundamental group, then Theorem 5.2.2 implies every $(G, X)$-structure is complete. As a less trivial example, if $M$ is a closed manifold whose fundamental group contains a nilpotent subgroup of finite index and whose first Betti number equals one, then every affine structure on $M$ is complete (see Fried-Goldman-Hirsch [55]).

8.4.1. A manifold with only complete structures. Here is a simple example. Consider the group $\Gamma \subset \text{Isom}(E^3)$ generated by the three isometries

$$
A = \begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
\end{bmatrix}
$$

$$
B = \begin{bmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & -1 & 0 \\
\end{bmatrix}
$$

$$
C = \begin{bmatrix}
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
\end{bmatrix}
$$

and $\Gamma$ is a discrete group of Euclidean isometries which acts properly and freely on $\mathbb{R}^3$ with quotient a compact 3-manifold $M$. Furthermore there is a short exact sequence

$$
\mathbb{Z}^3 \cong \langle A^2, B^2, C^2 \rangle \hookrightarrow \Gamma \xrightarrow{\lambda} \mathbb{Z}/2 \oplus \mathbb{Z}/2
$$

**Exercise 8.4.1.** Prove that every affine structure on $M$ must be complete.
8.4.2. Geodesics on Hopf manifolds. These geodesically incomplete structures model incomplete closed geodesics on affine manifolds. Namely, the geodesic on $\mathbb{A}^1$ defined by
\[ t \mapsto 1 + t(\lambda^{-1} - 1) \]
begins at 1 and in time
\[ t_\infty := 1 + \lambda^{-1} + \lambda^{-2} + \cdots = (1 - \lambda^{-1})^{-1} > 0 \]
reaches 0. It defines a closed incomplete closed geodesic $p(t)$ on $M$ starting at $p(0) = p_0$. The lift
\[ (-\infty, t_\infty) \xrightarrow{\tilde{p}} \tilde{M} \]
satisfies
\[ \text{dev}(\tilde{p}(t)) = 1 + t(\lambda^{-1} - 1), \]
which uniquely specifies the geodesic $p(t)$ on $M$. It is a geodesic since its velocity
\[ p'(t) = (\lambda^{-1} - 1)\partial_x \]
is constant (parallel). However $p(t_n) = p_0$ for
\[ t_n := \frac{1 - \lambda^{-n}}{1 - \lambda^{-1}} = 1 + \lambda^{-1} + \cdots + \lambda^{1-n} \]
and as viewed in $M$, seems to go “faster and faster” through each cycle. By time $t_\infty = \lim_{n \to \infty} t_n$, it seems to “run off the manifold;” the geodesic is only defined for $t < t_\infty$. The apparent paradox is that $p(t)$ has zero acceleration: it would have “constant speed” if “speed” were only defined.

8.4.3. Radiant affine manifolds. A Hopf manifold is the prototypical example of a radiant affine manifold. Many properties of Hopf manifolds are shared by radiant structures. General properties of radiant affine structures are discussed in §11.3.

$M$ is said to be radiant if it satisfies the following equivalent conditions.

**Proposition 8.4.2.** Let $M$ be an affine manifold with development pair $(\text{dev}, h)$. The following conditions are equivalent:

- $h(\pi)$ fixes a point in $\mathbb{A}$ (by conjugation we may assume this fixed point is the origin 0);
- $M$ is isomorphic to a $(V, \text{GL}(V))$-manifold;
- $M$ possesses a radiant vector field.
If $\rho_M$ is a radiant vector field on $M$, we shall often refer to the pair $(M, \rho_M)$ as well as a radiant affine structure. A closed radiant affine manifold $M$ is always incomplete (§11.3) and the radiant vector field is always nonsingular. Therefore $\chi(M) = 0$. Furthermore the first Betti number of a closed radiant affine manifold is always positive.

### 8.5. Cartesian products

The following is due to Benzécri [19].

**Exercise 8.5.1** (Products of affine manifolds). Let $M^m, N^n$ be affine manifolds.

1. Show that the Cartesian product $M^m \times N^n$ has a natural affine structure.
2. Show that $M \times N$ is complete if and only if both $M$ and $N$ are complete.
3. Show that $M \times N$ is radiant if and only if both $M$ and $N$ are radiant.

For projective structures, the situation is somewhat different:

**Exercise 8.5.2.** On the other hand, find compact manifolds $M, N$ each of which has a projective structure but $M \times N$ does not admit a projective structure.

1. If $M_1, \ldots, M_r$ are manifolds with real projective structures, show that the Cartesian product $M_1 \times \cdots \times M_r \times T^{r-1}$ admits a projective structure.

### 8.6. Complete affine manifolds

This section describes the general theory of complete affine structures; compare §5.4.1 of Chapter 5 and §8.3 of Chapter 8 for the specific cases of the circle and the two-torus, respectively. The model for the classification is Bieberbach’s theorem that every closed Euclidean manifold $M$ is finitely covered by a flat torus: that is, $M$ is a quotient of $\mathbb{A}^n$ by a lattice of translations. For complete affine structures on closed manifolds, the conjectural picture replaces the simply transitive group of translations by a more general simply transitive group $G$ of affine transformations, such as the group

$$G = \alpha_r(\mathbb{R}^2) \subset \text{Aff}(\mathbb{A}^2)$$

of § 8.3. This statement had been claimed by Auslander [6] but the his proof was flawed. The ideas were clarified in Milnor’s wonderful paper [115] and Fried-Goldman [54], which classifies complete affine structures on closed 3-manifolds. (Compare §15.1 of Chapter 15.)
Milnor observed that Auslander’s false claim was equivalent to the amenability of the fundamental group. He asked whether the fundamental group of any complete manifold (possibly noncompact) must be amenable; this is equivalent by Tits’s theorem [143] to whether a two-generator free group can act properly and affinely.

In the late 1970’s, Margulis proved that such group actions do exist; compare §15.2 of Chapter 15.

8.6.1. The Bieberbach theorems. In 1911-1912 Bieberbach found a general group-theoretic criterion for such groups in arbitrary dimension. In modern parlance, $\Gamma$ is a lattice in $\text{Isom}(E^n)$, that is, a discrete cocompact subgroup. Furthermore $\text{Isom}(E^n)$ decomposes as a semidirect product $\mathbb{R}^n \rtimes O(n)$ where $\mathbb{R}^n$ is the vector space of translations. In particular every isometry $g$ is a composition of a translation $x \mapsto x + b$ by a vector $b \in \mathbb{R}^n$, with an orthogonal linear map $A \in O(n)$:

$$x \mapsto A x + b$$

We call $A$ the linear part of $g$ and denote it $L(g)$ and $b$ the translational part of $g$, and denote it $u(g)$. When $A$ is only required to be linear, then $g$ is affine. An affine automorphism is a Euclidean isometry if and only if its linear part lies in $O(n)$.

Bieberbach showed:
- $\Gamma \cap \mathbb{R}^n$ is a lattice $\Lambda \subset \mathbb{R}^n$;
- The quotient $\Gamma / \Lambda$ is a finite group, mapped isomorphically into $O(n)$.
- Any isomorphism $\Gamma_1 \rightarrow \Gamma_2$ between Euclidean crystallographic groups $\Gamma_1, \Gamma_2 \subset \text{Isom}(E^n)$ is induced by an affine automorphism $E^n \rightarrow E^n$.
- There are only finitely many isomorphism classes of crystallographic subgroups of $\text{Isom}(E^n)$.

A Euclidean manifold is a flat Riemannian manifold, that is, a Riemannian manifold of zero curvature. A Euclidean manifold is complete if the underlying metric space is complete, which by the Hopf-Rinow theorem, is equivalent to the condition of geodesic completeness.

A torsionfree Euclidean crystallographic group $\Gamma \subset \text{Isom}(E^n)$ acts freely on $E^n$ and the quotient $E^n / \Gamma$ is a complete Euclidean manifold. Conversely every complete Euclidean manifold is a quotient of $E^n$ by a crystallographic group. The geometric version of Bieberbach’s theorems is:

- Every compact complete Euclidean manifold is a quotient of a flat torus $E^n / \Lambda$ (where $\Lambda \subset \mathbb{R}^n$ is a lattice of translations by a finite group of isometries acting freely on $E^n / \Lambda$.)
• Any homotopy equivalence \( M_1 \rightarrow M_2 \) of compact complete Euclidean manifolds is homotopic to an affine diffeomorphism.

• There are only finitely many affine isomorphism classes of compact complete Euclidean manifolds in each dimension \( n \).

### 8.6.2. Complete affine solvmanifolds.

This gives a very satisfactory qualitative picture of compact Euclidean manifolds, or, essentially equivalently Euclidean crystallographic groups. Does a similar picture hold for affine crystallographic groups, that is, for discrete subgroups \( \Gamma \subset \text{Aff}(\mathbb{A}^n) \) which act properly on \( \mathbb{A}^n \)?

Auslander and Markus \([5]\) constructed examples of flat Lorentzian crystallographic groups \( \Gamma \) in dimension 3, for which all three Bieberbach theorems directly fail. In their examples, the quotient \( M^3 = \mathbb{A}^3/\Gamma \) is a flat Lorentzian manifold. Topologically these are all 2-torus bundles over \( S^1 \); conversely every torus bundle over the circle admits such a structure.

These examples arise from a more general construction: namely, \( \Gamma \) embeds as a lattice in a closed Lie subgroup \( G \subset \text{Aff}(\mathbb{A}) \) whose identity component \( G^0 \) acts simply transitively on \( \mathbb{A} \).

Furthermore \( \Gamma^0 := \Gamma \cap G^0 \) has finite index in \( \Gamma \), so the flat Lorentz manifold \( M^3 \) is finitely covered by the homogeneous space \( G^0/\Gamma^0 \). Necessarily \( G^0 \) is simply connected solvable. The group \( G^0 \) plays the role of the translation group \( \mathbb{R}^n \). and \( G \) is called the crystallographic hull in Fried-Goldman \([54]\).

Furthermore \( \Gamma^0 := \Gamma \cap G^0 \) has finite index in \( \Gamma \), so the flat Lorentz manifold \( M^3 \) is finitely covered by the homogeneous space \( G^0/\Gamma^0 \). Necessarily \( G^0 \) is simply connected solvable. The group \( G^0 \) plays the role of the translation group \( \mathbb{R}^n \). and \( G \) is called the crystallographic hull in Fried-Goldman \([54]\).

A weaker version of this construction is the syndetic hull, defined in \([54]\), but known to H. Zassenhaus, H. C. Wang and L. Auslander. (A good general reference for this theory is Raghunathan \([125]\)).

If \( \Gamma \subset \text{GL}(n) \) is a solvable group, then a syndetic hull for \( \Gamma \) is a subgroup \( G \) such that:

- \( \Gamma \subset G \subset A(\Gamma) \), where \( A(\Gamma) \subset \text{GL}(n) \) is the Zariski closure (algebraic hull) of \( \Gamma \) in \( \text{GL}(n) \);
- \( G \) is a closed subgroup having finitely many connected components;
- \( G/\Gamma \) is compact (although not necessarily Hausdorff).

The last condition is somewhat called syndetic, since “cocompact” sometimes refers to a subgroup whose coset space is compact and Hausdorff. (This terminology is due to Gottschalk and Hedlund \([68]\).)
Equivalently, $\Gamma \subset G$ is syndetic if and only if $\exists K \subset G$ which is compact and meets every left coset $g\Gamma$, for $g \in G$. In general syndetic hulls fail to be unique. For example, the lattice $\mathbb{Z}^3 \subset \text{Isom}(E^3)$ has infinitely many syndetic hulls.

If $M = \Gamma \setminus A$ is a complete affine manifold, then $\Gamma \subset \text{Aff}(A)$ is a discrete subgroup acting properly and freely on $A$. However, in the example above, $\langle A \rangle$ is a discrete subgroup which doesn’t act properly. A proper action of a discrete group is the usual notion of a \emph{properly discontinuous action}. If the action is also free (that is, no fixed points), then the quotient is a (Hausdorff) smooth manifold, and the quotient map $A \to \Gamma \setminus A$ is a covering space. A properly discontinuous action whose quotient is compact as well as Hausdorff is said to be \emph{crystallographic}, in analogy with the classical notion of a \emph{crystallographic group}: A \emph{Euclidean crystallographic group} is a discrete cocompact group of Euclidean isometries. Its quotient space is a Euclidean orbifold. Since such groups act isometrically on metric spaces, discreteness here does imply properness; this dramatically fails for more general discrete groups of \emph{affine transformations}.

L. Auslander [6] claimed to prove that the Euler characteristic vanishes for a compact complete affine manifold, but his proof was flawed. It rested upon the following question, which in [54], was demoted to a “conjecture,” and is now known as the “Auslander Conjecture”:

**Conjecture 8.6.1.** Let $M$ be a compact complete affine manifold. Then $\pi_1(M)$ is virtually polycyclic.

In that case the affine holonomy group $\Gamma \cong \pi_1(M)$ embeds in a closed Lie subgroup $G \subset \text{Aff}(A)$ satisfying:

- $G$ has finitely many connected components;
- The identity component $G^0$ acts simply transitively on $A$.

Then $M = \Gamma \setminus A$ admits a finite covering space $M^0 := \Gamma^0 \setminus A$ where $\Gamma^0 := \Gamma \cap G^0$.

The simply transitive action of $G^0$ define a complete \emph{left-invariant affine structure} on $G^0$ and the developing map is just the evaluation map of this action. Necessarily $G^0$ is a 1-connected solvable Lie group and $M^0$ is affinely isomorphic to the \emph{complete affine solvmanifold} $\Gamma^0 \setminus G^0$. In particular $\chi(M^0) = 0$ and thus $\chi(M) = 0$.

This theorem is the natural extension of Bieberbach’s theorems describing the structure of flat Riemannian (or Euclidean) manifolds; see Milnor [114] for an exposition of this theory and its historical importance. Every flat Riemannian manifold is finitely covered by a \emph{flat}
torus, the quotient of $A$ by a lattice of translations. In the more general case, $G^0$ plays the role of the group of translations of an affine space and the solvmanifold $M^0$ plays the role of the flat torus. The importance of Conjecture 8.6.1 is that it would provide a detailed and computable structure theory for compact complete affine manifolds.

Conjecture 8.6.1 was established in dimension 3 in Fried-Goldman [54]. The proof involves classifying the possible Zariski closures $A(L(\Gamma))$ of the linear holonomy group inside $GL(A)$. Goldman-Kamishima prove Conjecture 8.6.1 for flat Lorentz manifolds. Grunewald-Margulis Conjecture 8.6.1 when the Levi component of $L(\Gamma)$ lies in a real rank-one subgroup of $GL(A)$. See Tomanov Abels-Margulis-Soifer for further results. The conjecture is now known in all dimensions $\leq 6$ (Abels-Margulis-Soifer
Part 3

Affine and projective structures
CHAPTER 9

Affine structures on surfaces and the Euler characteristic

In this section we classify affine structures on closed 2-manifolds. This classification falls into two steps: first is the basic result of Benzécri that a closed surface admits an affine structure if and only if its Euler characteristic vanishes. From this it follows that the affine holonomy group of a closed affine 2-manifold is abelian and the second step uses simple algebraic methods to classify affine structures.

We observe that affine structures on noncompact surfaces have a much different theory. First of all, every orientable noncompact surface admits an immersion into $\mathbb{R}^2$ and such an immersion determines an affine structure with trivial holonomy. Immersions can be classified up to crude relation of regular homotopy, although the isotopy classification of immersions of noncompact surfaces seems forbiddingly complicated. Furthermore suppose $\pi \xrightarrow{h} \text{Aff}(E)$ is a homomorphism such that the character

$$\pi \xrightarrow{\text{det} \circ L \circ h} \mathbb{Z}/2$$

equals the first Stiefel-Whitney class. That is, suppose its kernel is the subgroup of $\pi$ corresponding to the orientable double covering of $M$. Then $M$ admits an affine structure with holonomy $h$. Classifying general geometric structures on noncompact manifolds without extra geometric hypotheses on noncompact manifolds seems hopeless under anything but the crudest equivalence relations.

9.1. Benzécri’s theorem on affine 2-manifolds

The following result was first proved in [18]. Shortly afterwards, Milnor [110] gave a more general proof, clarifying its homotopic-theoretic nature. For generalizations of Milnor’s result, see Benzécri [20], Gromov [69], Sullivan [137] and Smillie [130]. For an interpretation of this inequality in terms of hyperbolic geometry, see [57]. More recent developments are surveyed in [64].

**Theorem 9.1.1 (Benzécri 1955).** Let $M$ be a closed 2-dimensional affine manifold. Then $\chi(M) = 0$. 

133
proof. Replace $M$ by its orientable double covering to assume that $M$ is orientable. By the classification of surfaces, $M$ is diffeomorphic to a closed surface of genus $g \geq 0$. Since a simply connected closed manifold admits no affine structure, (§5.2.4), $M$ cannot be a 2-sphere and hence $g \neq 0$. We assume that $g > 1$ and obtain a contradiction.

9.1.1. The surface as an identification space. Begin with the topological model for $M$. There exists a decomposition of $M$ along $2g$ simple closed curves $a_1, b_1, \ldots, a_g, b_g$ which intersect in a single point $x_0 \in M$. (Compare Fig. 7.) The complement

\[ M \setminus \bigcup_{i=1}^{g}(a_i \cup b_i) \]

is the interior of a $4g$-gon $\Delta$ with edges

\[ a_1^+, a_1^-, b_1^+, b_1^-, \ldots, a_g^+, a_g^-, b_g^+, b_g^- \]

(Compare Fig. 8.) There exist maps $A_1, B_1, \ldots, A_g, B_g \in \pi$ defining indentifications:

\[ A_i(b_i^+) = b_i^- \]
\[ B_i(a_i^+) = a_i^- \]

for a quotient map $\Delta \rightarrow M$. A universal covering space is the quotient space of the product $\pi \times F$ by identifications defined by the generators $A_1, B_1, \ldots, A_g, B_g$. Fix a development pair $(\text{dev}, h)$.
Figure 7. Decomposing a genus $g = 2$ surface along $2g$ curves into a $4g$-gon. The single common intersection of the curves is a single point which decomposes into the $4g$ vertices of the polygon.

Figure 8. Identifying the edges of a $4g$-gon into a closed surface of genus $g$. The sides are paired into $2g$ curves, which meet at the single vertex.
9.1.2. The turning number. Let \( I = [t_0, t_1] \) be a closed interval. If \( I \to \mathbb{R}^2 \) is a smooth immersion, then its turning number \( \tau(f) \) is defined as the total angular displacement of its tangent vector (normalized by dividing by \( 2\pi \)). Explicitly, if \( f(t) = (x(t), y(t)) \), then

\[
\tau(f) = \frac{1}{2\pi} \int_{t_0}^{t_1} d\tan^{-1}\left(\frac{y'(t)}{x'(t)}\right) = \frac{1}{2\pi} \int_{t_0}^{t_1} \frac{x'(t)y''(t) - x''(t)y'(t)}{x'(t)^2 + y'(t)^2} dt.
\]

Extend \( \tau \) to piecewise smooth immersions as follows. Suppose that \( [t_0, t_N] \to \mathbb{R}^2 \) is an immersion which is smooth on subintervals \([a_i, a_{i+1}]\) where

\[
t_0 < t_1 < \cdots < t_{N-1} < t_N.
\]

Let \( f'_+(t_i) = \lim_{t \to t_i^+} f'(t) \) and \( f'_-(t_i) = \lim_{t \to t_i^-} f'(t) \) be the two tangent vectors to \( f \) at \( t_i \); define the total turning number of \( f \) by:

\[
\tau(f) := \tau^{\text{cont}}(f) + \tau^{\text{disc}}(f)
\]

where the continuous contribution is:

\[
\tau^{\text{cont}}(f) := \sum_{i=0}^{N-1} \left( \tau(f|_{[t_i, t_{i+1}]} \right)
\]

and the discrete contribution is:

\[
\tau^{\text{disc}}(f) := \frac{1}{2\pi} \sum_{i=0}^{N-1} \angle(f'_-(t_{i+1}), f'_+(t_{i+1}))
\]

where \( \angle(v_1, v_2) \) represents the positively measured angle between the vectors \( v_1, v_2 \).

Here are some other elementary properties of \( \tau \):

- Denote \(-f\) the immersion obtained by reversing the orientation on \( t \):

\[
(-f)(t) := f(t_0 + t_N - t)
\]

Then \( \tau(-f) = -\tau(f) \).

- If \( g \in \text{Isom}(\mathbb{E}^2) \) is an orientation-preserving Euclidean isometry, then \( \tau(f) = \tau(g \circ f) \).

- If \( f \) is an immersion of \( S^1 \), then \( \tau(f) \in \mathbb{Z} \).

- Furthermore, if an immersion \( \partial D^2 \to \mathbb{E}^2 \) extends to an orientation-preserving immersion \( D^2 \to \mathbb{E}^2 \), then \( \tau(f) = 1 \).

The Whitney-Graustein theorem asserts that immersions \( S^1 \to \mathbb{R}^2 \) \((i = 1, 2)\) are regularly homotopic if and only if \( \tau(f_1) = \tau(f_2) \), which implies the last remark.
Exercise 9.1.2. Suppose that $S$ is a compact oriented surface with boundary components $\partial_1 S, \ldots, \partial_k S$. Suppose that $S \xrightarrow{f} E^2$ is an orientation-preserving immersion. Then

$$\sum_{i=1}^k \tau(f|_{\partial_i S}) = \chi(S).$$

An elementary property relating turning number to affine transformations is the following:

Lemma 9.1.3. Suppose that $[a,b] \xrightarrow{f} \mathbb{R}^2$ is a smooth immersion and $\phi \in \text{Aff}^+(\mathbb{R}^2)$ is an orientation-preserving affine automorphism. Then

$$|\tau(f) - \tau(\phi \circ f)| < \frac{1}{2}$$

Proof. If $\psi$ is an orientation-preserving Euclidean isometry, then $\tau(f) = \tau(\psi \circ f)$; by composing $\phi$ with an isometry we may assume that $f(a) = (\phi \circ f)(a)$

$$f'(a) = \lambda(\phi \circ f)'(a)$$

for $\lambda > 0$. That is,

(21) \quad L(\phi)(f'(a)) = \lambda f'(a).

Suppose that $|\tau(f) - \tau(\phi \circ f)| \geq 1/2$. Since for $a \leq t \leq b$, the function

$$|\tau(f|_{[a,t]}) - \tau(\phi \circ f|_{[a,t]})|$$

is a continuous function of $t$ and equals 0 for $t = a$ and is $\geq 1/2$ for $t = b$. The intermediate value theorem implies that there exists $0 < t_0 \leq b$ such that

$$|\tau(f|_{[a,t_0]}) - \tau(\phi \circ f|_{[a,t_0]})| = 1/2.$$ 

Then the tangent vectors $f'(t_0)$ and $(\phi \circ f)'(t_0)$ have opposite direction, that is, there exists $\mu < 0$ such that

(22) \quad L(\phi)(f'(t_0)) = (\phi \circ f)'(t_0) = \mu f'(t_0).

Combining (21) with (22), the linear part $L(\phi)$ has eigenvalues $\lambda, \mu$ with $\lambda > 0 > \mu$. However $\phi$ preserves orientation, contradicting $\text{Det}(L(\phi)) = \lambda \mu < 0$. \(\square\)

We apply these ideas to the restriction of the developing map $\text{dev}$ to $\partial F$. Since $f := \text{dev}|_{\partial F}$ is the restriction of the immersion $\text{dev}|_{F}$ of the 2-disc,

$$1 = \tau(f) = \tau^{\text{disc}}(f) + \tau^{\text{cont}}(f)$$
where

\[ \tau^{\text{cont}}(f) = \sum_{i=1}^{g} \tau(\text{dev}|_{a_i^+}) + \tau(\text{dev}|_{a_i^-}) + \tau(\text{dev}|_{b_i^+}) + \tau(\text{dev}|_{b_i^-}) \]

\[ = \sum_{i=1}^{g} \tau(\text{dev}|_{a_i^+}) - \tau(h(B_i) \circ \text{dev}|_{a_i^+}) + \tau(\text{dev}|_{b_i^+}) - \tau(h(A_i) \circ \text{dev}|_{b_i^+}) \]

since \( h(B_i) \) identifies \( \text{dev}_{a_i^+} \) with \( -\text{dev}_{a_i^-} \) and \( h(A_i) \) identifies \( \text{dev}_{b_i^+} \) with \( -\text{dev}_{b_i^-} \). By Lemma 9.1.3, each

\[ |\tau(\text{dev}|_{a_i^+}) - \tau(h(B_i) \circ \text{dev}|_{a_i^+})| < \frac{1}{2} \]

\[ |\tau(\text{dev}|_{b_i^+}) - \tau(h(A_i) \circ \text{dev}|_{b_i^+})| < \frac{1}{2} \]

and thus

(23) \[ |\tau^{\text{cont}}(f)| < \sum_{i=1}^{g} \frac{1}{2} + \frac{1}{2} = g \]

Now we estimate the discrete contribution. The \( j \)-th vertex of \( \partial F \) contributes \( 1/2\pi \) of the angle

\[ \angle(f'(t_j), f'_+(t_j)) \]

which is supplementary to the \( i \)-th interior angle \( \alpha_j \) of the polygon \( \partial F \), as measured in a Euclidean metric defined in a coordinate patch around the point \( m_0 \in M \) corresponding to the vertices. Since the cone angle at \( m_0 \) (as measured in this local Euclidean metric) equals \( 2\pi \),

\[ \sum_{j=1}^{4} g\alpha_j = 2\pi. \]

Thus

\[ \tau^{\text{disc}}(f) = \frac{1}{2\pi} \sum_{j=1}^{4g} \angle(f'(t_j), f'_+(t_j)) \]

\[ = \frac{1}{2\pi} \sum_{j=1}^{4g} (\pi - \alpha_j) = 2g - 1 \]

Now

\[ \tau^{\text{cont}}(f) = \tau(f) - \tau^{\text{disc}} = 1 - (2g - 1) = 2 - 2g \]

but (23) implies \( 2g - 2 < g \), that is, \( g < 2 \) as desired. \( \square \)
Benzécri’s original proof uses a decomposition where all the sides of $F$ have the same tangent direction at $x_0$; thus all the $\alpha_j$ equal 0 except for one which equals $2\pi$.

9.1.3. The Milnor-Wood inequality. Shortly after Benzécri proved the above theorem, Milnor observed that this result follows from a more general theorem on flat vector bundles. Let $E$ be the 2-dimensional oriented vector bundle over $M$ whose total space is the quotient of $\tilde{M} \times \mathbb{R}^2$ by the diagonal action of $\pi$ by deck transformations on $\tilde{M}$ and via $L \circ h$ on $\mathbb{R}^2$, (that is, the flat vector bundle over $M$ associated to the linear holonomy representation.) This bundle has a natural flat structure, since the coordinate changes for this bundle are (locally) constant linear maps. Now an oriented $\mathbb{R}^2$-bundle $\xi$ over a space $M$ is classified by its Euler class
\[ \text{Euler}(\xi) \in H^2(M; \mathbb{Z}). \]

For $M$ a closed oriented surface, the orientation defines an isomorphism $H^2(M; \mathbb{Z}) \cong \mathbb{Z}$, and we henceforth identify these groups when the context is clear. If $\xi$ is an oriented $\mathbb{R}^2$-bundle over $M$ which admits a flat structure, Milnor [110] showed that
\[ |\text{Euler}(\xi)| < g. \]

Furthermore every integer in this range is realized by a flat oriented 2-plane bundle. If $M$ is an affine manifold, then the bundle $E$ is isomorphic to the tangent bundle $TM$ of $M$ and hence has Euler number
\[ \text{Euler}(TM) = 2 - 2g. \]

Thus the only closed orientable surface whose tangent bundle has a flat structure is a torus. Furthermore Milnor showed that any $\mathbb{R}^2$-bundle whose Euler number satisfies the above inequality has a flat connection.

Extensions of the Milnor-Wood inequality to higher dimensions have been proved by Benzécri [20], Smillie [130], Sullivan [137], Burger-Iozzi-Wienhard [24] and Bucher-Gelander [22].

9.2. Higher dimensions

The Euler Characteristic in higher dimensions In the early 1950’s Chern suggested that in general the Euler characteristic of a compact affine manifold must vanish. Based on the Chern-Weil theory of representing characteristic classes by curvature, several special cases of this conjecture can be solved: if $M$ is a compact complex affine manifold, then the Euler characteristic is the top Chern number and hence can be expressed in terms of curvature of the complex linear connection (which is zero). However, in general, for a real vector bundle, only the
Pontrjagin classes are polynomials in the curvature — indeed Milnor’s examples show that the Euler class cannot be expressed as a polynomial in the curvature of a linear connection (although it can be expressed as a polynomial in the curvature of an orthogonal connection).

This has been an extremely important impetus for research in this subject.

Deligne-Sullivan [39] proved a strong vanishing theorem for flat complex vector bundles. Namely, every flat complex vector bundle \( \xi \) over a finite complex \( M \) is virtually trivial: that is, there exists a finite covering space \( \hat{M} \rightarrow M \) such that \( f^*\xi \) is trivial. This immediately implies that \( \text{Euler}(\xi) = 0 \). Hirsch and Thurston [75] gave a very general criterion for vanishing of the Euler class of flat bundle with amenable holonomy; compare Goldman-Hirsch [65] for an elementary proof in the case of flat vector bundles.

For an ingenious argument proving the vanishing of the Euler characteristic for integral holonomy, see Sullivan [136].

Recently the vanishing of the Euler characteristic for closed affine manifolds with parallel volume has been proved by Bruno Klingler [81]. He uses the natural geometric structure on the total space of the tangent bundle \( TM \) of a compact affine manifold \( M \) which he calls a para-hypercomplex structure. Such a structure is an integrable reduction of the structure group to the split quaternions.

9.2.1. The Chern-Gauss-Bonnet Theorem. Most of the known special cases of the Chern-Sullivan conjecture follow from the Chern’s intrinsic generalization of the Gauss-Bonnet theorem [30] and the Chern-Weil theory of characteristic classes. This includes flat pseudo-Riemannian manifolds, flat complex manifolds, and complete affine manifolds (Kostant-Sullivan). Notable exceptions are Benzécri’s theorem for surfaces and the Kobayashi-Vey theorem for hyperbolic affine structures.

Chern’s theorem concerns an oriented orthogonal rank \( n \) vector bundle \( \xi \rightarrow M \) over an oriented closed \( n \)-dimensional manifold \( M \). That is, \( \xi \) is a smooth \( \mathbb{R}^n \)-bundle over \( M \) with an orthogonal connection \( \nabla \) and an orientation on the fibers. Let

\[
\text{Euler}(\xi) \in H^n(M, \mathbb{Z})
\]

denote the Euler class of the oriented \( \mathbb{R}^n \)-bundle \( \xi \). (Compare Milnor-Stasheff [116] and Steenrod [135].) The orthogonal connection \( \nabla \) determines an exterior \( n \)-form \( \text{Euler}(\nabla) \), the Euler form of \( \nabla \) on \( M \), such that

\[
\int_M \text{Euler}(\nabla) = \text{Euler}(\xi) \cdot [M]
\]
where \([M] \in H_n(M, \mathbb{Z})\) denotes the fundamental class of \(M\) arising from the orientation. The Euler form is a polynomial expression in the curvature of \(\nabla\) and vanishes if \(\nabla\) is flat. When \(\xi\) is the tangent bundle of \(M\), then

\[
\text{Euler}(\xi) \cdot [M] = \chi(M),
\]

the Euler characteristic of \(M\).

Milnor [110] showed that, over a closed oriented surface of genus \(g > 1\), every oriented \(\mathbb{R}^2\)-bundle \(E\) with \(|\text{Euler}(\xi)| < g\) admits a flat structure. That is, \(\xi\) admits a flat linear connection \(\nabla\), but if \(\text{Euler}(\xi) \neq 0\), then \(\nabla\) cannot be orthogonal.

We summarize some of the ideas in Chern's theorem, referring to Poor [124] (§3.56–3.73, pp. 138–49) for detailed proofs and discussion. According to Poor, this geometric approach is due to Gromoll.

A key point in this proof is the use of the associated principal \(\text{SO}(n)\)-bundle over \(M\), which is the bundle of positively oriented orthonormal frames. When \(n = 2\), this is the unit tangent bundle of \(M\) and is an \(S^1\)-bundle over \(M\). As discussed in Steenrod [135] and Milnor-Stasheff [116], the Euler class is really an invariant of the associated oriented \(S^{n-1}\)-bundle. The quotient of the total space by the antipodal map on the fiber is an \(\mathbb{R}P^{n-1}\)-bundle, which when \(n = 2\), identifies with a sphere bundle itself. (Compare Exercise 9.2.1.)

Let \(\xi\) be a smooth oriented real vector bundle of even rank \(n = 2m\) over an oriented smooth \(n\)-manifold \(M\) with an orthogonal connection \(\nabla\). Let \(\text{so}(\xi)\) be the vector bundle to \(\xi\) associated to the adjoint representation \(\text{SO}(2m) \rightarrow \text{Aut}(\text{so}(2m))\). The curvature tensor \(R(\nabla)\) is an \(\text{so}(\xi)\)-valued exterior 2-form on \(M\). The Pfaffian is an \(\text{Ad}\)-invariant polynomial mapping \(\text{so}(2m, \mathbb{R}) \xrightarrow{\text{Pfaff}} \mathbb{R}\) such that

\[
\text{Det}(A) = \text{Pfaff}(A)^2
\]

and \(\text{Pfaff}\) is a polynomial of degree \(m\). For example, for \(m = 1\), and

\[
A = \begin{bmatrix} 0 & -a \\ a & 0 \end{bmatrix},
\]

\(\text{Det}(A) = a^2\) and \(\text{Pfaff}(A) = a\). Applying the Pfaffian to \(R(\nabla)\) yields an exterior 2m-form \(\text{Pfaff}(R(\nabla))\) on \(M^{2m}\).

The Euler number of \(\xi\) (relative to the orientations of \(\xi\) and \(A\) can be computed by the Poincaré-Hopf theorem: Namely, let \(\eta\) be a section of \(\xi\) with isolated zeroes \(p_1, \ldots, p_k\). Find an open ball \(B_i\) containing \(p_i\) and a trivialization

\[
E_{B_i} \xrightarrow{\sim} B_i \times \mathbb{R}^{2m}.
\]
9. THE EULER CHARACTERISTIC

With respect to this trivialization, the restriction of $\eta$ to $B_i$ is the graph of a map $B_i \xrightarrow{f_i} \mathbb{R}^{2m}$ where $f_i(x) \neq 0$ if $x \neq p_i$. The degree of the smooth map

$$S^{2m-1} \approx \partial B_i \xrightarrow{f_i} S^{2m-1}$$

$$x \mapsto \frac{f_i(x)}{\|f_i(x)\|}$$

is independent of the trivialization, and is called the Poincaré-Hopf index $\text{ind}(\eta, p_i)$ of $\eta$ at $p_i$. The Euler number of $E$, defined as

$$\text{Euler}(\xi, \eta) := \sum_{i=1}^k \text{ind}(\eta, p_i) \in \mathbb{Z}$$

is independent of $\xi$ and the local trivializations of $\xi$.

The intrinsic Gauss-Bonnet theorem, due to Chern [30], states that there is a constant $c_m \in \mathbb{R}$ such that

$$\text{Euler}(\xi) = c_m \int_M \text{Pfaff}(R(\nabla)).$$

In particular if $\xi = TM$, and $R(\nabla) = 0$, then

$$\chi(M) = \text{Euler}(TM) = 0.$$

9.2.2. Smillie’s examples of flat tangent bundles. Two oriented 2-plane bundles over a space $M$ are isomorphic if their Euler classes in $H^2(M; \mathbb{Z})$ are equal. (Compare Milnor-Stasheff [116].) Milnor [110] showed that an oriented 2-plane bundle $\xi$ over an oriented surface of genus $g \geq 0$ admits a flat structure if and only if

$$|\text{Euler}(\xi)| < g.$$  

Suppose that $\xi$ is such a bundle which is nontrivial, that is, $\text{Euler}(\xi) \neq 0$. Then $\xi$ admits a connection $\nabla$ with curvature zero.

**Exercise 9.2.1.** Show that the complexification of such a bundle is trivial.

**Exercise 9.2.2.** Suppose that $F \longrightarrow \Sigma$ is an oriented $S^1$-bundle which admits a free proper action of the cyclic group $\mathbb{Z}/m\mathbb{Z}$ on the fibers so that $F' := F/(\mathbb{Z}/m\mathbb{Z})$ is an oriented $S^1$-bundle over $\Sigma$. Show that

$$m|\text{Euler}(F)$$

and

$$\text{Euler}(F') = \text{Euler}(F)/m.$$

Deduce that the Euler number of a flat $\mathbb{R}^2$-bundle over $\Sigma$ is always even.
Exercise 9.2.3. Show that the 3-sphere \( S^3 \) admits a flat affine connection (that is, a connection on its tangent bundle \( TM \) with vanishing curvature), but no flat affine connection with vanishing torsion.

Thus, in general, a manifold can have a flat tangent bundle even if it fails to have a flat affine structure. In this direction, Smillie [132] showed that Chern’s conjecture is false if one only requires that the curvature vanishes. We outline his (elementary) argument below.

First, he considers the class of stably parallelizable manifolds, that is, manifolds with stably trivial tangent bundles. Recall that a vector bundle \( E \to M \) is stably trivial if the Whitney sum \( E \oplus \mathbb{R}_M \) is trivial, where \( \mathbb{R}_M \to M \) denotes the trivial line bundle over \( M \). (Smillie uses the terminology “almost” instead of “stably” although we think that “stable” is more standard.)

Exercise 9.2.4. If \( \xi \to M \) is a stably trivial vector bundle, then \( \xi = f^*T S^n \), for some map \( M \xrightarrow{f} S^n \). Furthermore two stably trivial vector bundles \( \xi, \xi' \) are isomorphic if and only if

\[ \text{Euler}(\xi) = \text{Euler}(\xi') \in H^n(M; \mathbb{Z}). \]

An oriented 2-plane bundle \( \xi \) over a closed oriented surface is stably trivial if and only its Euler number \( \text{Euler}(\xi) \) is even (equivalently, if its second Stiefel-Whitney class \( w_2(\xi) = 0 \)).

Exercise 9.2.5. Let \( M \) be an orientable \( n \)-manifold. Show that the following conditions are equivalent:

- \( M \) is stably parallelizable;
- \( M \) immerses in \( \mathbb{R}^{n+1} \);
- For any point, the complement \( M\setminus\{x\} \) is parallelizable.

Deduce that the connected sum of two stably parallelizable manifolds is parallelizable.

Smillie constructs a 4-manifold \( N^4 \) with \( \chi(N) = 4 \), and a 6-manifold \( Q^6 \) with \( \chi(Q) = 8 \) such that both \( TN \) and \( TQ \) have flat structures. He begins with a closed orientable surface \( \Sigma_3 \) of genus 3 and the flat \( \text{SL}(2, \mathbb{R}) \)-bundle \( \xi \) over \( \Sigma_3 \) with \( \text{Euler}(\xi) = 2 \). (This bundle arises by lifting a Fuchsian representation

\[ \pi_1(\Sigma_3) \longrightarrow \text{PSL}(2, \mathbb{R}) \]

from \( \text{PSL}(2, \mathbb{R}) \) to \( \text{SL}(2, \mathbb{R}) \).) Then \( \xi \) is stably trivial and admits a flat structure.

The product 4-plane bundle \( \xi \times \xi \) over \( \Sigma_3 \times \Sigma_3 \) is also stably trivial and admits a flat structure. Furthermore its Euler number

\[ \text{Euler}(\xi \times \xi) = 2 + 2 = 4. \]
Let $P^4$ be a parallelizable 4-manifold and let $N$ be the connected sum of six copies of $P$ with $\Sigma_3 \times \Sigma_3$.

**Exercise 9.2.6.** Prove that $TN \cong \mu^*(\xi \times \xi)$ for some degree one map

$$N \xrightarrow{f} \Sigma_3 \times \Sigma_3.$$  

Deduce that $TN$ admits a flat structure and that $\chi(N) = 4$. Find a similar construction for a 6-manifold $Q^6$ with flat tangent bundle but $\chi(Q) = 8$. Find, for any even $n \geq 8$, an $n$-dimensional manifold with flat tangent bundle and positive Euler characteristic.

**9.2.3. The Kostant-Sullivan Theorem.** In 1960, L. Auslander published a false proof that the Euler characteristic of a closed complete affine manifold is zero. Of course, the difficulty is that the Euler characteristic can only be computed as a curvature integral for an orthogonal connection.

This difficulty was overcome by a clever trick by Kostant and Sullivan [87] who showed that the Euler characteristic of a compact complete affine manifold vanishes.

**Proposition 9.2.7 (Kostant-Sullivan [87]).** Let $M^{2n}$ be a compact affine manifold whose affine holonomy group acts freely on $A^{2n}$. Then $\chi(M) = 0$.

**Corollary 9.2.8 (Kostant-Sullivan [87]).** The Euler characteristic of a compact complete affine manifold vanishes.

The main lemma is the following elementary fact, which the authors attribute to Hirsch:

**Lemma 9.2.9.** Let $\Gamma \subset \text{Aff}(A)$ be a group of affine transformations which acts freely on $A$. Let $G$ denote the Zariski closure of the linear part $L(\Gamma)$ in $GL(V)$. Then every element $g \in G$ has 1 as an eigenvalue.

**Proof.** First we show that the linear part $L(\gamma)$ has 1 as an eigenvalue for every $\gamma \in \Gamma$. This condition is equivalent to the non-invertibility of $L(\gamma) - I$. Suppose otherwise; then $L(\gamma) - I$ is invertible. Writing

$$x \mapsto L(g)x + u(g),$$

where the vector $u(g)$ is the translational part $(u(g) = g(0))$ of $g$. the point

$$p := -(L(g) - I)^{-1}u(g)$$

is fixed by $\gamma$, contradicting our hypothesis that $g$ acts freely.

Non-invertibility of $L(\gamma) - I$ is equivalent to

$$\text{Det}(L(\gamma) - I) = 0,$$
9.2. HIGHER DIMENSIONS

evidently a polynomial condition on $\gamma$. Thus $L(g) - 1$ is non-invertible for every $g \in G$, as desired. \qed

**Proof of Proposition 9.2.7.** To show that $\chi(M) = 0$, we find an orthogonal connection $\nabla$ for which the Gauss-Bonnet integrand $Pfaff(R(\nabla)) = 0$. To this end, observe first that the tangent bundle $TM$ is associated to the linear holonomy $L(\Gamma)$, representation of $M$, and hence its structure group reduces from $Aff(A)$ to the algebraic hull $G$ of $L(\Gamma)$. Since $M$ is complete, its affine holonomy group $\Gamma$ acts freely and Lemma 9.2.9 implies that every element of $G$ has 1 as an eigenvalue.

Let $K \subset G$ be a maximal compact subgroup of $G$. (One can take $K$ to be the intersection of $G$ with a suitable conjugate of the orthogonal group $O(2m) \subset GL(2m, \mathbb{R})$.) A section of the $G/K$-bundle associated to the $G$-bundle corresponding to the tangent bundle always exists (since $G/K$ is contractible), and corresponds to a Riemannian metric on $M$. Let $\nabla$ be the corresponding Levi-Civita connection. Its curvature $R(\nabla)$ lies in the Lie algebra $k \subset so(2m)$.

Since every element of $G$ has 1 as an eigenvalue, every element of $K$ has 1 as an eigenvalue, and every element of $k$ has 0 as eigenvalue. That is, $\text{Det}(k) = 0$ for every $k \in k$. Since

$$Pfaff(k)^2 = \text{Det}(k) = 0,$$

the Gauss-Bonnet integrand $Pfaff(R(\nabla)) = 0$. Applying Chern’s intrinsic Gauss-Bonnet theorem (§9.2.1), $\chi(M) = 0$. \qed
CHAPTER 10

Affine structures on Lie groups and algebras

Let \( a \) be an associative algebra over the field of real numbers. We shall associate to \( a \) a Lie group \( G = G(a) \) with a bi-invariant affine structure. Conversely, if \( G \) is a Lie group with a bi-invariant affine structure, then we show that its Lie algebra \( g \) supports an associative multiplication \( g \times g \rightarrow g \) satisfying

\[
[X, Y] = XY - YX
\]

and that the corresponding Lie group with with bi-invariant affine structure is locally isomorphic to \( G \).

We begin by discussing invariant affine structures on Lie groups. If \( G \) is a Lie group and \( a \in G \), then the operations \( L_a \), \( R_b \) of left- and right-multiplication, respectively, are defined by

\[
ab = L_a(b) = R_b(a)
\]

Suppose that \( G \) is a Lie group with an affine structure. The affine structure is left-invariant (respectively right-invariant) if and only if the operations \( G \xrightarrow{L_a} G \) (respectively \( G \xrightarrow{R_b} G \)) are affine. An affine structure is bi-invariant if and only if it is both left-invariant and right-invariant.

Suppose that \( G \) is a Lie group with a left-invariant (respectively right-invariant, bi-invariant) affine structure. Let \( \tilde{G} \) be its universal covering group and

\[
\pi_1(G) \hookrightarrow \tilde{G} \twoheadrightarrow G
\]

the corresponding central extension. Then the induced affine structure on \( \tilde{G} \) is also left-invariant (respectively right-invariant, bi-invariant). Conversely, since \( \pi_1(G) \) is central in \( \tilde{G} \), every left-invariant (respectively right-invariant, bi-invariant) affine structure on \( \tilde{G} \) determines a left-invariant (respectively right-invariant, bi-invariant) affine structure on \( G \). Thus there is a bijection between left-invariant (respectively right-invariant, bi-invariant) affine structures on a Lie group and left-invariant (respectively right-invariant, bi-invariant) affine structures on...
any covering group. For this reason we shall mainly only consider simply connected Lie groups.

Suppose that \( G \) is a simply connected Lie group with a left-invariant affine structure and let \( G \xrightarrow{\text{dev}} A \) be a developing map. Corresponding to the affine action of \( G \) on itself by left-multiplications there is a homomorphism \( G \xrightarrow{\alpha} \text{Aff}(A) \) such that the diagram

\[
\begin{array}{ccc}
G & \xrightarrow{\text{dev}} & A \\
\downarrow{L_g} & & \downarrow{\alpha(g)} \\
G & \xrightarrow{\text{dev}} & A
\end{array}
\]

commutes for each \( g \in G \). We may assume that \( \text{dev} \) maps the identity element \( e \in G \) to an origin \( p_0 \in A \). Then (25) implies that the developing map is the translational part of the affine representation:

\[
\text{dev}(g) = (\text{dev} \circ L_g)(e) = (\alpha(g) \circ \text{dev})(e) = \alpha(g)(p_0)
\]

Furthermore since \( \text{dev} \) is open, it follows that the orbit \( \alpha(G)(p_0) \) equals the developing image and is open. Indeed the translational part, which is the differential of the evaluation map

\[
T_eG = g \longrightarrow V = T_{p_0}A
\]

is a linear isomorphism. Such an action will be called \textit{locally simply transitive}.

Conversely suppose that \( G \xrightarrow{\alpha} \text{Aff}(A) \) is an affine representation and \( O \subset A \) is an open orbit. Then for any point \( x_0 \in O \), the evaluation map

\[
g \mapsto \alpha(g)(x_0)
\]

defines a developing map for an affine structure on \( G \). Since

\[
\text{dev}(L_g h) = \alpha(gh)(x_0) = \alpha(g)\alpha(h)(x_0) = \alpha(g) \text{dev}(h)
\]

for \( g, h \in G \), this affine structure is left-invariant. Thus \textit{Lie groups} \( G \) with left-invariant affine structure correspond precisely to open orbits of \textit{locally simply transitive} affine representations \( G \longrightarrow \text{Aff}(A) \).

Now suppose that \( \mathfrak{a} \) is an associative algebra; we shall associate to \( \mathfrak{a} \) a Lie group with bi-invariant affine structure as follows. We formally adjoin to \( \mathfrak{a} \) a two-sided identity element \( 1 \) to construct an associative algebra \( \mathfrak{a} \oplus \mathbb{R}1 \). The the affine hyperplane \( A = \mathfrak{a} \times \{1\} \) in \( \mathfrak{a} \oplus \mathbb{R}1 \)
10.1. Koszul-Vinberg algebras

We seek the converse construction, namely to associate to an bi-invariant affine structure on a Lie group $G$ an associative algebra. This can be accomplished neatly as follows. Let $\mathfrak{g}$ be the Lie algebra of left-invariant vector fields on $G$ and let $\nabla$ be the flat torsionfree affine connection on $G$ corresponding to a left-invariant affine structure. Since the connection is left-invariant, for any two left-invariant vector fields $X, Y \in \mathfrak{g}$, the covariant derivative $\nabla_X Y \in \mathfrak{g}$ is left-invariant. It follows that covariant differentiation $(X, Y) \mapsto \nabla_X Y$ defines a bilinear multiplication $\mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ which we denote $(X, Y) \mapsto XY$.

Now the condition that $\nabla$ has zero torsion is
\begin{equation}
XY - YX = [X, Y]
\end{equation}
and the condition that $\nabla$ has zero curvature (using (26) is
\[X(YZ) - Y(XZ) = (XY - YX)Z\]
which is equivalent to the Koszul-Vinberg property
\begin{equation}
(XY)Z - X(YZ) = (YX)Z - Y(XZ).
\end{equation}
This is just the condition that the associator
\[ [X, Y, Z] := (XY)Z - X(YZ) \]
is symmetric in the first two variables $(X, Y)$. 

Exercise 10.1.1. Suppose that \( a \) is an \( \mathbb{R} \)-algebra, that is a vector space over \( \mathbb{R} \) with a bilinear mapping
\[
\begin{align*}
a \times a &\to a \\
(X,Y) &\mapsto XY.
\end{align*}
\]
If \( a \) is commutative, that is, \( XY = YX \), and satisfies (27), then \( a \) is associative: \( (XY)Z = X(YZ) \). (Hint: denoting the commutator in \( a \) by \( [X,Y] := XY - YX \), show that
\[
[X,Y,Z] - [X,Z,Y] + [Z,X,Y] = [XY,Z] + X[Z,Y] + [Z,X]Y
\]
for all \( X,Y,Z \in a \).)

10.1.1. Bi-invariant affine structures and associativity. Now suppose that \( \nabla \) is bi-invariant. Thus the right-multiplications are affine maps; it follows that the infinitesimal right-multiplications — the left-invariant vector fields — are affine vector fields. For a flat torsion-free affine connection a vector field \( Z \) is affine if and only if the second covariant differential \( \nabla \nabla Z \) vanishes. Now \( \nabla \nabla Z \) is the tensor field which associates to a pair of vector fields \( X,Y \) the vector field
\[
\nabla \nabla Z(X,Y) := \nabla_X (\nabla Z(Y)) - \nabla Z(\nabla_X Y)
\]
and if \( X,Y,Z \in \mathfrak{g} \) we obtain the associative law \( X(YZ) - (XY)Z = 0 \). One can check that these two constructions
\[
\{ \text{Associative algebras} \} \iff \{ \text{Bi-invariant affine structures on Lie groups} \}
\]
are mutually inverse.

10.1.2. Locally simply transitive affine actions. A (not necessarily associative) algebra whose multiplication satisfies (27) is said to be a Koszul-Vinberg algebra, or a left-symmetric algebra, (algèbre symétrique à gauche), or a pre-Lie algebra. Of course every associative algebra satisfies this property. If \( a \) is a Koszul-Vinberg algebra, then the operation
\[
[X,Y] = XY - YX
\]
is skew-symmetric and satisfies the Jacobi identity. Thus every Koszul-Vinberg algebra has an underlying Lie algebra structure. We denote this Lie algebra by \( \mathfrak{g} \). If \( \mathfrak{g} \) is a Lie algebra, then a Koszul-Vinberg operation satisfying (24) will be called an affine structure on \( \mathfrak{g} \). Let
a \xrightarrow{L} \text{End}(a) be the operation of left-multiplication defined by

\[ Y \xrightarrow{L(X)} XY \]

In terms of left-multiplication and the commutator operation defined in (24), a condition equivalent to (27) is:

\[ L([X,Y]) = [L(X), L(Y)] \]

that is, that \( g \xrightarrow{L} \text{End}(a) \) is a Lie algebra homomorphism. We denote by \( a_L \) the corresponding \( g \)-module. Furthermore the identity map \( g \xrightarrow{I} a_L \) defines a cocycle of the Lie algebra \( g \) with coefficients in the \( g \)-module \( a_L \):

\[ L(X)Y - L(Y)X = [X,Y] \]

Let \( A \) denote an affine space with associated vector space \( a \); then it follows from (29) and (30) that the map \( g \xrightarrow{\alpha} \text{aff}(A) \) defined by

\[ Y \xrightarrow{\alpha(X)} L(X)Y + X \]

is a Lie algebra homomorphism.

**Theorem 10.1.2.** There is an isomorphism between the categories of Koszul-Vinberg algebras and simply connected Lie groups with left-invariant affine structure. Under this isomorphism the associative algebras correspond to bi-invariant affine structures.

There is a large literature on Koszul-Vinberg algebras, under various names. We recommend Burde’s survey article [23], as well as works by Vinberg [148], Helmstetter [72] and Vey [146] and the references cited therein for more information.

One can translate geometric properties of a left-invariant affine structure on a Lie group \( G \) into algebraic properties of the corresponding Koszul-Vinberg algebra \( a \). For example, the following theorem is proved in Helmstetter [72] and indicates a kind of infinitesimal version of Markus’s conjecture relating geodesic completeness to parallel volume. For more discussion of this result and proofs, see Helmstetter [72] and Goldman-Hirsch [67].

**Theorem 10.1.3.** Let \( G \) be a simply connected Lie group with left-invariant affine structure. Let \( G \xrightarrow{\alpha} \text{Aff}(A) \) be the corresponding locally simply transitive affine action and \( a \) the corresponding Koszul-Vinberg algebra. Then the following conditions are equivalent:

- \( G \) is a complete affine manifold;
- \( \alpha \) is simply transitive;
• A volume form on $G$ is parallel if and only if it is right-invariant;
• For each $g \in G$,$$
\det L_\alpha(g) = \det \text{Ad}(g)^{-1},
$$
that is, the distortion of parallel volume by $\alpha$ equals the modular function of $G$;
• The subalgebra of $\text{End}(a)$ generated by right-multiplications $R_a : x \mapsto xa$ is nilpotent.

In a different direction, we may say that a left-invariant affine structure is radiant if and only if the affine representation $\alpha$ corresponding to left-multiplication has a fixed point, that is, is conjugate to a representation $G \rightarrow \text{GL}(V)$. Equivalently, $\alpha(G)$ preserves a radiant vector field on $A$. A left-invariant affine structure on $G$ is radiant if and only if the corresponding Koszul-Vinberg algebra has a right-identity, that is, an element $e \in a$ satisfying $ae = a$ for all $a \in a$.

Since the affine representation $G \rightarrow \text{Aff}(A)$ corresponds to left-multiplication, the associated Lie algebra representation, also denoted $\mathfrak{g} \rightarrow \text{aff}(A)$, maps $\mathfrak{g}$ into affine vector fields which correspond to the infinitesimal generators of left-multiplications, that is, to right-invariant vector fields. Thus with respect to a left-invariant affine structure on a Lie group $G$, the right-invariant vector fields are affine. Let $X_1, \ldots, X_n$ be a basis for the right-invariant vector fields; it follows that the exterior product
$$
\alpha(X_1) \wedge \cdots \wedge \alpha(X_n) = f(x) \; dx^1 \wedge \cdots \wedge dx^n
$$
for a polynomial $f \in \mathbb{R}[x^1, \ldots, x^n]$, called the characteristic polynomial of the left-invariant affine structure. In terms of the algebra $a$, we have
$$
f(X) = \det(R_{X\oplus 1})
$$
where $R_{X\oplus 1}$ denotes right-multiplication by $X \oplus 1$. By Helmstetter [72] and Goldman-Hirsch [67], the developing map is a covering space, mapping $G$ onto a connected component of complement of $f^{-1}(0)$. In particular the nonvanishing of $f$ is equivalent to completeness of the affine structure.

### 10.1.3. Two-dimensional commutative associative algebras.

Commutative associative algebras provide many examples of affine structures on closed 2-manifolds as follows.

Let $a$ be such an algebra and adjoin a two-sided identity element (denoted “1”) to form a new commutative associative algebra $a \oplus \mathbb{R}1$ with identity. The invertible elements in $a \oplus \mathbb{R}1$ form an open subset
closed under multiplication. The universal covering group $G$ of the group of invertible elements of the form
\[ a \oplus 1 \in a \oplus \mathbb{R}1 \]
acts locally simply transitively and affinely on the affine space
\[ A = a \oplus \{1\} \]
The Lie algebra of $G$ naturally identifies with the algebra $a$ and there is an exponential map $a \xrightarrow{\exp} G$ defined by the usual power series (in $a$). The corresponding evaluation map at 1 defines a developing map for an invariant affine structure on the vector group $a$.

Now let $\Lambda \subset a$ be a lattice. The quotient $a/\Lambda$ is a torus with an invariant affine structure. Some of these affine structures we have seen previously in other contexts. It is a simple algebraic exercise to classify 2-dimensional commutative associative algebras:

- $a = \mathbb{R}[x, y]$ where $x^2 = y^2 = xy = 0$. The corresponding affine representation is the action of $\mathbb{R}^2$ on the plane by translation and the corresponding affine structures on the torus are the Euclidean structure.
- $a = \mathbb{R}[x]$ where $x^3 = 0$. The corresponding affine representation is the simply transitive action discussed in §8.3. The corresponding affine structures are complete but non-Riemannian. These structures deform to the first one, where $a = \mathbb{R}[x, y]$ where $x^2 = ky$.

We describe this algebra by its multiplication table:

- $a = \mathbb{R}[x]$ where $x^3 = 0$. The corresponding affine representation is the simply transitive action discussed in The corresponding affine structures are complete but non-Riemannian.
- $a = \mathbb{R}[x, y]$ where $x^2 = xy = 0$ and $y^2 = y$. The algebra $a$ is the product of two 1-dimensional algebras, one corresponding to the complete structure and the other corresponding to the radiant structure. For various choices of $\Lambda$ one obtains parallel suspensions of Hopf circles. In these cases the developing image is a halfplane.
- $a = \mathbb{R}[x, y]$ where $x^2 = 0$ and $xy = x, y^2 = y$. Since $y$ is an identity element, the corresponding affine structure is radiant. For various choices of $\Lambda$ one obtains radiant suspensions of the complete affine 1-manifold $\mathbb{R}/\mathbb{Z}$. The developing image is a halfplane.
- $a = \mathbb{R}[x, y]$ where $x^2 = x, y^2 = y$ and $xy = 0$. This algebra is the product of two algebras corresponding to radiant
structures; this structure is radiant since $x + y$ is an identity element. Radiant suspensions of Hopf circles are examples of affine manifolds constructed in this way. The developing image is a quadrant in $\mathbb{R}^2$.

- $a = \mathbb{R}[x, y]$ where $x^2 = -y^2 = x$ and $xy = y$. In this case $a \cong \mathbb{C}$ and we obtain the complex affine 1-manifolds, in particular the Hopf manifolds are all obtained from this algebra. Clearly $x$ is the identity and these structures are all radiant. The developing image is the complement of a point in the plane.

Baues [10], surveys the theory of affine structures on the 2-torus. Kuiper [96] classified convex affine structures on $T^2$, including the complete case (see Chapter 8,§8.3). Nagano-Yagi [118] and Arrowsmith-Furness [2],[3] completed the classification. Projective structures on the 2-torus were classified by Goldman [56] and in higher dimensions by Benoist [12].
Figure 9. Some incomplete complex-affine structures on $T^2$
Figure 10. Some hyperbolic affine structures on $T^2$
Figure 11. Radiant affine structures on $T^2$ developing to a halfplane
Figure 12. Nonradiant affine structures on $T^2$ developing to a halfplane
Here are the multiplication tables for the 2-dimensional commutative associative algebras:

\[
\begin{array}{ccc}
X & Y \\
X & 0 & 0 \\
Y & 0 & 0 \\
\end{array}
\]

**Table 1.** Euclidean

\[
\begin{array}{ccc}
X & Y \\
X & 0 & 0 \\
Y & 0 & X \\
\end{array}
\]

**Table 2.** Complete Non-Riemannian structure

\[
\begin{array}{ccc}
X & Y \\
X & X & Y \\
Y & 0 & 0 \\
\end{array}
\]

**Table 3.** Radiant structure on halfplane

\[
\begin{array}{ccc}
X & Y \\
X & X & 0 \\
Y & 0 & 0 \\
\end{array}
\]

**Table 4.** Nonradiant halfplane

\[
\begin{array}{ccc}
X & Y \\
X & X & 0 \\
Y & 0 & Y \\
\end{array}
\]

**Table 5.** Hyperbolic affine structure on \(\mathbb{R}^2\)
10.1.4. Two-dimensional noncommutative associative algebras. Two-dimensional Lie algebras $\mathfrak{g}$ fall into two isomorphism types:

- $\mathfrak{g} \cong \mathbb{R}^2$ (abelian);
- $\mathfrak{g} \cong \text{aff}(1, \mathbb{R})$ the Lie algebra of affine vector fields on the affine line $\mathbb{A}^1$.

As we have just treated the abelian case, we turn now to the nonabelian case.

If $\mathfrak{g}$ is nonabelian, the corresponding Lie group is the group $G^0 = \text{Aff}^+ (1, \mathbb{R})$ of affine transformations of the line $\mathbb{A}^1$ with positive linear part.

Denote the elements of this group as $[y \ x]$ where $y > 0$ and $x \in \mathbb{R}$, with identity element $[1 \ 0]$ and group operation:

$$[y_1 \ x_1] [y_2 \ x_2] = [y_1 y_2 \ x_1 + y_1 x_2]$$

The exponential map is

$$\mathfrak{g} \xrightarrow{\exp} G$$

$$[s \ t] \mapsto [e^s \ e^{s-1} t] .$$

The affine representation

$$G \to \text{Aff}(2, \mathbb{R})$$

$$[y \ x] \mapsto \begin{bmatrix} y & 0 & x \\ 0 & 1 & y \end{bmatrix}$$

defines a simply transitive affine action on the upper halfplane $y > 0$.

We may identify the translations in $G^0$ with the $y$-axis and the positive homotheties with the positive $x$-axis. Thus the corresponding right-invariant vector fields (corresponding to infinitesimal left-multiplications by one-parameter subgroups) are based by a parallel vector field and a radiant vector field. $G^0$ identifies with right half-plane $\mathbb{R}^+ \times \mathbb{R}$, and acts on $\mathbb{A}^1$ by:

$$\mathbb{A}^1 \xrightarrow{L_{x,y}} \mathbb{A}^1$$

$$\xi \mapsto x\xi + y$$

Its Lie algebra $\mathfrak{g}$ corresponds to the algebra of matrices of the form

$$\begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix}$$
with matrix multiplication
\[
\begin{bmatrix}
a & b \\ 0 & 0 \\end{bmatrix}
\begin{bmatrix}
x & y \\ 0 & 0 \\end{bmatrix} =
\begin{bmatrix}
a x & a y + b \\ 0 & 0 \\end{bmatrix}.
\]

The pair of vector fields
\[ R = x \partial_x + y \partial_y, \partial_y \]
base the Lie algebra of right-invariant vector fields with multiplication table:

\[
\begin{array}{c|c}
R & \partial_y \\
\hline
R & R \\
\partial_y & 0 \\
\end{array}
\]

Table 6. Right-invariant vector fields on Aff(1, \mathbb{R})

The pair of vector fields
\[ x \partial_x, x \partial_y \]
base the Lie algebra of left-invariant vector fields with multiplication table:

\[
\begin{array}{c|c|c}
x \partial_x & x \partial_y & x \partial_y \\
\hline
x \partial_x & x \partial_y & 0 \\
x \partial_y & 0 & 0 \\
\end{array}
\]

Table 7. Left-invariant vector fields on Aff(1, \mathbb{R})

Dual to the above basis of left-invariant vector fields is the basis
\[ x^{-1} dx, x^{-1} dy \]
of left-invariant 1-forms, and \( x^{-2} dx \wedge dy \) defines the dual left-invariant area 2-form. Left-invariant Haar measure \( \mu_{\text{Left}} \) corresponds to its dual, the exterior bivector field \( x^2 \partial_x \wedge \partial_y \). Right-invariant Haar measure \( \mu_{\text{Right}} \) corresponds to the right-invariant exterior bivector field \( x \partial_x \wedge \partial_y \). Their ratio is the modular function
\[
G \xrightarrow{\Delta} \mathbb{R}_+ ,
\]
\[ g \mapsto \text{Det Ad}(g) \]
and
\[
\mu_{\text{Right}} = \Delta \mu_{\text{Left}},
\]
or, equivalently,
\[ \Delta(g) = \frac{(L_g)_* \mu_{\text{Right}}}{\mu_{\text{Right}}}. \]

**Exercise 10.1.5.** *Show that these two algebras comprise the only associative algebras whose underlying Lie algebra is \( \mathfrak{a} \).*

That is, we show the only multiplication tables are given by Table 6 and Table 7.

Here are some hints for Exercise 10.1.5: We choose a basis \( X, Y \) for \( \mathfrak{g} \) such that \([Y, X] = X\). Thus we want to show that this basis can be modified so that the associative algebra \( \mathfrak{a} \) has such a basis with the following multiplication tables:

\[
\begin{array}{ccc}
X & Y \\
X & 0 & 0 \\
Y & X & Y
\end{array}
\]

Table 8. Left-invariant vector fields

\[
\begin{array}{ccc}
X & Y \\
X & 0 & -X \\
Y & 0 & -Y
\end{array}
\]

Table 9. Right-invariant vector fields

Table 8 is just Table 7 rewritten using:
\[ X \leftrightarrow y\partial_x \]
\[ Y \leftrightarrow y\partial_y \]

and Table 9 is just Table 6 rewritten using:
\[ X \leftrightarrow \partial_x \]
\[ Y \leftrightarrow -R = -x\partial_x - y\partial_y \]

**10.2. Some locally simply transitive affine actions**

If \( G \) is a Lie group, then the kernel of the *modular homomorphism*

\[
G \xrightarrow{\text{Ad}} \text{Aut}(\mathfrak{g}) \xrightarrow{\text{Det}} \mathbb{R}^* \]

is a unimodular ideal, called the *unimodular kernel* in Milnor [113].
10.2. Simply transitive action of non-unimodular group.

The group of all
\[ g_{s,t} := \begin{bmatrix} 1 & 0 & s \\ 0 & e^s & t \\ 0 & 0 & 1 \end{bmatrix} = \exp \begin{bmatrix} 0 & 0 & s \\ 0 & 0 & t \end{bmatrix} \]
acts simply transitively on the affine plane.

\[ \{ \partial_x + y\partial_y, \partial_y \} \]
bases the Lie algebra of right-invariant vector fields, and
\[ X := \partial_x, \quad Y := e^x \partial_y \]
bases the Lie algebra of left-invariant vector fields, and multiplication given by Table 10.

<table>
<thead>
<tr>
<th>X</th>
<th>Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>Y</td>
</tr>
<tr>
<td>Y</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 10. 2-dimensional nonabelian algebra corresponding to complete structure

The dual basis of left-invariant 1-forms is:
\[ X^* := dx, \quad Y^* := e^x dy \]
and the 2-form \( X^* \wedge Y^* = e^x dx \wedge dy \) defines the left-invariant area form (Haar measure). The right-invariant area form is parallel.

10.2.2. An incomplete homogeneous flat Lorentzian structure. The group of all
\[ g_{s,t} := \begin{bmatrix} e^s & 0 & t \\ 0 & e^{-s} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \exp \begin{bmatrix} s & 0 & t \\ 0 & -s & 0 \end{bmatrix} \]
acts simply transitively on the upper halfplane \( y > 0 \).

\[ \{ \partial_x, x\partial_x - y\partial_y \} \]
bases the Lie algebra of right-invariant vector fields, and
\[ X := y^{-1}\partial_x, \quad Y := y\partial_y \]
bases the Lie algebra of left-invariant vector fields, with multiplication recorded in Table 11:

The dual basis of left-invariant 1-forms is:
\[ X^* := y \, dx, \quad Y^* := y^{-1} \, dy \]
and the symmetric 2-form $X^* \cdot Y^* = dx \cdot dy$ defines the left-invariant flat Lorentzian metric. This provides an example of a \textit{homogeneous} flat Lorentzian structure which is geodesically \textit{incomplete}.

**10.2.3. Right parabolic halfplane.** Let $f(x, y) := x - y^2/2$. Then the group of all

$$g_{s,t} := \begin{bmatrix} e^{2s} & 0 \\ 0 & e^s \end{bmatrix} \cdot \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} t^2/2 \\ t \end{bmatrix}$$

acts simply transitively on the right parabolic halfplane $f(x, y) > 0$ (and on the left parabolic halfplane $f(x, y) < 0$). The vector fields

$$2x \partial_x + y \partial_y, \quad y \partial_x + \partial_y$$

(corresponding, respectively, to $\partial_s, \partial_t$) base the Lie algebra of right-invariant vector fields. Their exterior product equals $2f(x, y)\partial_x \wedge \partial_y$.

To compute the left-invariant vector fields, consider a point $p = (x, y) \in A$ such that $f(x, y) \neq 0$. Then $g_{s,t}(p_0) = p$ where $p_0 = (1, 0)$, and

$$(s, t) = (f(x, y), y).$$

Then $L(g_{s,t})$ is the matrix

$$L(g_{s,t}) = \begin{bmatrix} f(x, y) & 0 \\ 0 & f(x, y)^{1/2} \end{bmatrix} \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix}$$

whose columns define the basis of left-invariant vector fields:

$$X := f(x, y)\partial_x$$
$$Y := f(x, y)^{1/2}(y\partial_x + \partial_y)$$

and with multiplication given by Table 12:
10.2.4. Parabolic cylinders. We can extend this structure to structures on a 3-dimensional solvable Lie group $G$, which admits compact quotients. These provide examples of compact convex incomplete affine 3-manifolds which are not properly convex, and nonradiant. Therefore Vey’s result that compact hyperbolic affine manifolds are radiant is sharp.

Further examples from the same group action give concave affine structures on these same 3-manifolds.

The function:

$$f : \mathbb{A}^3 \rightarrow \mathbb{R}$$

$$(x, y, z) \mapsto x - y^2/2$$

is invariant under the affine $\mathbb{R}^2$-action defined by:

$$\mathbb{R}^2 \xrightarrow{\mathbb{U}} \text{Aff}(\mathbb{A}^3)$$

$$(t, u) \mapsto \exp \begin{bmatrix} 0 & t & 0 \\ 0 & 0 & t \\ 0 & 0 & u \end{bmatrix} = \begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

Under the 1-parameter group of dilations

$$\mathbb{R} \xrightarrow{\delta} \text{Aff}(\mathbb{A}^3)$$

$$s \mapsto \exp \begin{bmatrix} 2s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & -s \end{bmatrix} = \begin{bmatrix} e^{2s} & 0 & 0 \\ 0 & e^s & 0 \\ 0 & 0 & e^{-s} \end{bmatrix}$$

the function $f$ scales as:

$$f \circ \delta(s) = e^{2s} f.$$
the Lie group $G$ has a basis of left-invariant vector fields
\[
X := f(x, y, z)\partial_x \\
Y := f(x, y, z)^{1/2}(y\partial_x + \partial_y) \\
Z := f(x, y, z)^{-1/2}\partial_z
\]
with multiplication recorded in Table 13. The dual basis of left-invariant 1-forms is:
\[
X^\ast := f(x, y, z)^{-1}(dx - ydy) = -d\log(f) \\
Y^\ast = f(x, y, z)^{-1/2}dy \\
Z^\ast = f(x, y, z)^{1/2}dz
\]
with bi-invariant volume form
\[
X^\ast \wedge Y^\ast \wedge Z^\ast = f(x, y, z)^{-1}dx \wedge dy \wedge dz.
\]
This example is due to Goldman [58] providing examples of non-conical convex domains covering compact affine manifolds.

<table>
<thead>
<tr>
<th></th>
<th>$X$</th>
<th>$Y$</th>
<th>$Z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X$</td>
<td>$X$</td>
<td>$Y/2$</td>
<td>$-Z/2$</td>
</tr>
<tr>
<td>$Y$</td>
<td>0</td>
<td>$X$</td>
<td>0</td>
</tr>
<tr>
<td>$Z$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 13. Algebra corresponding to parabolic 3-dimensional halfspaces

10.2.5. Nonradiant deformations of radiant halfspace quotients. Another example arises from radiant suspensions. Namely, consider the radiant affine representation
\[
\mathbb{R} \times \mathbb{R}^2 \xrightarrow{\rho_{\alpha}} \text{Aff}(\mathbb{A}^3)
\]
\[
(s; t, u) \mapsto e^{\alpha s} \exp \begin{bmatrix}
  s & 0 & t \\
  0 & -s & u \\
  0 & 0 & 0
\end{bmatrix} = \\
\begin{bmatrix}
  e^{(\alpha+1)s} & 0 & e^{\alpha s}t \\
  0 & e^{(\alpha-1)s} & e^{\alpha s}u \\
  0 & 0 & e^{\alpha s}
\end{bmatrix}
\]
depending on a parameter $\alpha \in \mathbb{R}$. When $\alpha \neq 0$, the action is locally simply transitive; the open orbits are the two halfspaces defined by
z > 0 and z < 0 respectively. The vector fields

\[ X := z^{(\alpha+1)/\alpha} \partial_x \]
\[ Y := z^{(\alpha-1)/\alpha} \partial_y \]
\[ Z := R = x \partial_x + y \partial_y + z \partial_z \]

correspond to a basis of left-invariant vector fields, with multiplication recorded in Table 14.

<table>
<thead>
<tr>
<th>X</th>
<th>Y</th>
<th>Z</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Y</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Z</td>
<td>((\alpha + 1)/\alpha)X)</td>
<td>((\alpha - 1)/\alpha)Y)</td>
</tr>
</tbody>
</table>

Table 14. A radiant suspension

When \(\alpha = \pm 1\), then this action admits nonradiant deformations. Namely let \(\beta \in \mathbb{R}\) be another parameter, and consider the case when \(\alpha = 1\). The affine representation

\[ \mathbb{R} \times \mathbb{R}^2 \xrightarrow{\rho^\beta} \text{Aff}(\mathbb{A}^3) \]

\[ (s; t, u) \mapsto \exp \begin{bmatrix} 2s & 0 & 0 \\ 0 & 0 & \beta s \\ 0 & s & 0 \end{bmatrix} \cdot \exp \begin{bmatrix} 0 & 0 & t \\ 0 & 0 & u \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} e^{2s} & 0 & e^{2st} \\ 0 & 1 & u \beta s \\ 0 & 0 & e^s \end{bmatrix} \]

maps

\[ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \xrightarrow{\rho^\beta} \begin{bmatrix} e^{2st} \\ u + \beta s \\ e^s \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \]

so the group element with coordinates \((s, t, u)\) corresponds to the point with coordinates

\[ x = e^{2st} \]
\[ y = u + \beta s \]
\[ z = e^s \]
with inverse transformation:

\[
\begin{align*}
s &= \log(z) \\
t &= z^{-2}x \\
u &= y - \beta \log(z)
\end{align*}
\]

Then the linear part \(L\rho^\lambda(s,t,u)\) corresponds to the matrix

\[
\begin{bmatrix}
z^2 & 0 & x \\
0 & 1 & y - \beta \log(z) \\
0 & 0 & z
\end{bmatrix}
\]

whose columns base the left-invariant vector fields:

\[
\begin{align*}
X &:= z^2 \partial_x \\
Y &:= \partial_y \\
Z &:= x \partial_x + (y - \beta \log(z)) \partial_y + z \partial_z
\end{align*}
\]

Table 15 records their covariant derivatives.

<table>
<thead>
<tr>
<th></th>
<th>X</th>
<th>Y</th>
<th>Z</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>0</td>
<td>0</td>
<td>X</td>
</tr>
<tr>
<td>Y</td>
<td>0</td>
<td>0</td>
<td>Y</td>
</tr>
<tr>
<td>Z</td>
<td>2X</td>
<td>0</td>
<td>Z - \beta Y</td>
</tr>
</tbody>
</table>

Table 15. Nonradiant Deformation
CHAPTER 11

Parallel volume and completeness

In 1961, L. Markus posed the following “research problem” among the exercises in lecture notes for a class in cosmology at the University of Minnesota:

**Question.** Let $M$ be a closed affine manifold. Then $M$ is geodesically complete if and only if $M$ has parallel volume.

An affine manifold $M$ has parallel volume if and only if it satisfies any of the following equivalent conditions:

- The orientable double-covering of $M$ admits a parallel volume form (in the sense of §1.2.2 of Chapter 1);
- $M$ admits a coordinate atlas where the coordinate changes are volume-preserving;
- $M$ admits a refined $(\text{SAff}(A), A)$-structure, where $\text{SAff}(A)$ denotes the subgroup $L^{-1}(\text{SL}_{\pm}(\mathbb{R}^n))$ of volume-preserving linear transformations;
- For each $\gamma \in \pi_1(M)$, the linear holonomy $L \circ h(\gamma)$ has determinant $\pm 1$.

11.1. The volume obstruction

**Exercise 11.1.1.** Prove the equivalence of the conditions stated in the introduction to Chapter 11.

The last condition suggests a topological interpretation. The composition of the linear holonomy representation $L \circ h$ with the logarithm of the absolute value of the determinant

$$\pi_1(M) \xrightarrow{L \circ h} \text{GL}(E) \xrightarrow{|\det|} \mathbb{R}^+ \xrightarrow{\log} \mathbb{R}$$

defines an additive homomorphism $\nu_M : \text{Hom}(\pi_1(M), \mathbb{R}) \cong H^1(M; \mathbb{R})$ which we call the volume obstruction. $M$ has parallel volume if and only if $\nu_M = 0$.

**Exercise 11.1.2.** Suppose that $M$ is a manifold with zero first Betti number. Then every affine structure on $M$ must have has parallel volume.

169
Helmstetter’s theorem [72] that a left-invariant affine structure on a Lie group is complete \(\iff\) right-invariant volume forms are parallel is an “infinitesimal version” of Markus’s conjecture. (Compare also Goldman-Hirsch [66].)

The plausibility of Markus’s question seems to be one of the main barriers in constructing examples of affine manifolds. A purely topological consequence of this conjecture is that a compact affine manifold \(M\) with zero first Betti number \(\beta_1(M)\) is covered by Euclidean space: in particular all of its higher homotopy groups vanish. Thus, if \(\beta_1(M) = 0\) there should be no such structure on a nontrivial connected sum in dimensions greater than two. (In fact no affine structure is presently known on a nontrivial connected sum.)

11.2. Smillie’s nonexistence theorem

**Theorem 11.2.1** (Smillie [133]). Let \(M\) be a closed affine manifold with parallel volume. Then the affine holonomy homomorphism cannot factor through a free group.

This theorem can be generalized much further — see Smillie [133] and Goldman-Hirsch [66].

**Corollary 11.2.2** (Smillie [133]). Let \(M\) be a closed manifold whose fundamental group is a free product of finite groups (for example, a connected sum of manifolds with finite fundamental group). Then \(M\) admits no affine structure.

**Proof of Corollary 11.2.2 assuming Theorem 11.2.1.**

Suppose \(M\) has an affine structure. Since \(\pi_1(M)\) is a free product of finite groups, the first Betti number of \(M\) is zero. Thus \(M\) has parallel volume. Furthermore if \(\pi_1(M)\) is a free product of finite groups, there exists a free subgroup \(\Gamma \subset \pi_1(M)\) of finite index. Let \(\hat{M}\) be the covering space with \(\pi_1(\hat{M}) = \Gamma\). Then the induced affine structure on \(\hat{M}\) also has parallel volume contradicting Theorem. \(\square\)

**Proof of Theorem 11.2.1.** Let \(M\) be a closed affine manifold modeled on an affine space \(E\), \(\hat{M} \to M\) a universal covering, and

\[
\left( \hat{M} \xrightarrow{\text{dev}} E, \pi \xrightarrow{\text{hol}} \text{Aff}(E) \right)
\]

a development pair. Suppose that \(M\) has parallel volume and that there is a free group \(\Pi\) through which the affine holonomy homomorphism \(h\) factors:

\[
\pi \xrightarrow{\phi} \Pi \xrightarrow{\text{hol}} \text{Aff}(E)
\]
Choose a graph $G$ with fundamental group $\Pi$; then there exists a map $f : M \to G$ inducing the homomorphism

$$\pi = \pi_1(M) \to \phi \to \pi_1(G) = \Pi.$$ 

By general position, there exist points $s_1, \ldots, s_k \in G$ such that $f$ is transverse to $s_i$ and the complement $G - \{s_1, \ldots, s_k\}$ is connected and simply connected. Let $H_i$ denote the inverse image $f^{-1}(s_i)$ and let $H = \bigcup_i H_i$ denote their disjoint union. Then $H$ is an oriented closed smooth hypersurface such that the complement $M - H \subset M$ has trivial holonomy. Let $M|H$ denote the manifold with boundary obtained by splitting $M$ along $H$; that is, $M|H$ has two boundary components $H^+_i, H^-_i$ for each $H_i$, and there exist diffeomorphisms $H^+_i \to H^-_i$ (generating $\Pi$) such that $M$ is the quotient of $M|H$ by the identifications $g_i$. There is a canonical diffeomorphism of $M - H$ with the interior of $M|H$.

Let $\omega_E$ be a parallel volume form on $E$; then there exists a parallel volume form $\omega_M$ on $M$ such that

$$\Pi^* \omega_M = \text{dev}^* \omega_E.$$ 

Since $H^n(E) = 0$, there exists an $(n-1)$-form $\eta$ on $E$ such that $d\eta = \omega_E$. For any immersion $S \to E$ of an oriented closed $(n-1)$-manifold $S$, the integral

$$\int_S f^* \eta$$

is independent of the choice of $\eta$ satisfying $d\eta = \omega_E$. Since $H^{n-1}(E)$, any other $\eta'$ must satisfy $\eta' = \eta + d\theta$ and

$$\int_S f^* \eta' - \int_S f^* \eta = \int_S d(f^* \theta) = 0.$$ 

Since $M - H$ has trivial holonomy there is a developing map

$$M - H \to E$$

and its restriction to $M - H$ extends to a developing map $M|H \to E$ such that

$$\text{dev}|_{H^+_i} = \bar{h}(g_i) \circ \text{dev}|_{H^-_i}$$

and the normal orientations of $H^+_i, H^-_i$ induced from $M|H$ are opposite. Since $h(g_i)$ preserves the volume form $\omega_E$,

$$d(h(g_i)^* \eta) = d(\eta) = \omega$$

and

$$\int_{H^+_i} \text{dev}^* \eta = \int_{H^+_i} \text{dev}^* h(g_i)^* \eta = -\int_{H^-_i} \text{dev}^* \eta.$$
since the normal orientations of $H_i^\pm$ are opposite. We now compute the $\omega_M$-volume of $M$:

$$\text{vol}(M) = \int_M \omega_M = \int_{\partial M} \text{dev}^* \omega_E$$

$$= \int_{\partial(M|H)} \eta = \sum_{i=1}^k \left( \int_{H_i^+} \eta + \int_{H_i^-} \eta \right) = 0,$$

a contradiction. \(\square\)

One basic method of finding a primitive $\eta$ for $\omega_E$ involves a radiant vector field $\rho$. Since $\rho$ expands volume, specifically,

$$d\iota_\rho \omega_E = n \omega_E,$$

and

$$\eta = \frac{1}{n} \iota_\rho \omega_E$$

is a primitive for $\omega_E$. An affine manifold is radiant if and only if it possesses a radiant vector field if and only if the affine structure comes from an $(E, \text{GL}(E))$-structure if and only if its affine holonomy has a fixed point in $E$. The following result generalizes the above theorem:

**Theorem 11.2.3 (Smillie)**. Let $M$ be a closed affine manifold with a parallel exterior differential $k$-form which has nontrivial de Rham cohomology class. Suppose $U$ is an open covering of $M$ such that for each $U \in U$, the affine structure induced on $U$ is radiant. Then $\dim U \geq k$; that is, there exist $k+1$ distinct open sets

$$U_1, \ldots, U_{k+1} \in U$$

such that the intersection

$$U_1 \cap \cdots \cap U_{k+1} \neq \emptyset.$$

(Equivalently the nerve of $U$ has dimension at least $k$.)

A published proof of this theorem can be found in Goldman-Hirsch [66].

Using these ideas, Carriére, d’Albo and Meignez [28] have proved that a nontrivial Seifert 3-manifold with hyperbolic base cannot have an affine structure with parallel volume. This implies that the 3-dimensional Brieskorn manifolds $M(p, q, r)$ with

$$p^{-1} + q^{-1} + r^{-1} < 1$$

admit no affine structure whatsoever. (Compare Milnor [112].)
11.3. Radiant affine structures

Radiant affine manifolds have many special properties, derived from the existence of a radiant vector field. If $M$ is a manifold with radiant affine structure modeled on an affine space $E$, let $(\text{dev}, h)$ be a development pair and $\rho_E$ a radiant vector field on $E$ invariant under $h(\pi)$, then there exists a (radiant) vector field $\rho_M$ on $M$ such that

$$\Pi^* \rho_M = \text{dev}^* \rho_E.$$ 

**Theorem 11.3.1.** Let $M$ be a compact radiant manifold.

- Then $M$ cannot have parallel volume. (In other words a compact manifold cannot support a $(\mathbb{R}^n, \text{SL}(n; \mathbb{R}))$-structure.) In particular the first Betti number of a closed radiant manifold is positive.

- The developing image $\text{dev}(\tilde{M})$ does not contain any stationary points of the affine holonomy. (Thus $M$ is incomplete.) In particular the radiant vector field $\rho_M$ is nonsingular and the Euler characteristic $\chi(M) = 0$.

**Proof.** Proof of (1) Let $\omega_E = dx^1 \wedge \cdots \wedge dx^n$ be a parallel volume form on $E$ and let $\omega_M$ be the corresponding parallel volume form on $M$, that is, $\Pi^* \omega_M = \text{dev}^* \omega_E$. Let $\eta_M$ denote the interior product

$$\eta_M = \frac{1}{n} \iota_{\rho_M} \omega_M$$

and it follows from

$$d\iota_{\rho_E} \omega_E = n \omega_E$$

that $d\eta_M = \omega_M$. But $\omega_M$ is a volume form on $M$ and

$$\text{vol}(M) = \int_M \omega_M = \int_M d\eta_M = 0$$

a contradiction. (Intuitively, the main idea in the proof above is that the radiant flow on $M$ expands the parallel volume uniformly. Thus by “conservation of volume” a compact manifold cannot support both a radiant vector field and a parallel volume form.)

**Proof.** Proof of (2) We may assume that

$$\rho_E = \sum_{i=1}^n x^i \frac{\partial}{\partial x_i}$$

and it will suffice to prove that $0 \not\in \text{dev}(\tilde{M})$. Since the only zero of $\rho_E$ is the origin $0 \in E$, the vector field $\rho_M$ is nonsingular on the complement of $F = \Pi(\text{dev}^{-1}(0))$. Since $\Pi$ and $\text{dev}$ are local diffeomorphisms and $0 \in E$ is discrete, it follows that $F \subset M$ is a discrete set; since $\text{dev}$
is continuous and 0 is $h(\pi)$-invariant, $F \subset M$ is closed. Hence $F$ is a finite subset of $M$.

Since $M$ is a closed manifold, $\rho_M$ is completely integrable and thus there is a flow

$$\{ M \overset{R_t}{\rightarrow} M \}_{t \in \mathbb{R}}$$

infinitesimally generated by $-\rho_M$. The flow lifts to $\{ \widetilde{M} \overset{\tilde{R}_t}{\rightarrow} \widetilde{M} \}_{t \in \mathbb{R}}$ on $\widetilde{M}$ which satisfies

$$\text{dev}(\tilde{R}_t x) = e^{-t}\text{dev}(x)$$

for $x \in \widetilde{M}, t \in \mathbb{R}$. Choose a neighborhood $U$ of $F$, each component of which develops to a small ball $B$ about 0 in $E$. Let $K \subset \widetilde{M}$ be a compact set such that the saturation $\pi(K) = \widetilde{M}$; then there exists $N >> 0$ such that

$$e^{-t}(\text{dev}(K)) \subset B$$

for $t \geq N$. Thus $\tilde{R}_t(K) \subset B$ for $t \geq N$. Therefore $U$ is an attractor for the flow of $\rho_M$ and that $M \overset{R_N}{\rightarrow} U$ deformation retracts the closed manifold $M$ onto $U$. Since a closed manifold is not homotopy-equivalent to a finite set, this contradiction shows that $0 \notin \text{dev}(\widetilde{M})$ as desired. \(\square\)

There is a large class of discrete groups $\Gamma$ for which every affine representation $\Gamma \rightarrow \text{Aff}(E)$ is conjugate to a representation factoring through $\text{SL}(E)$, that is,

$$\Gamma \rightarrow \text{SL}(E) \subset \text{Aff}(E).$$

For example finite groups have this property, and the above theorem gives an alternate proof that the holonomy of a compact affine manifold must be infinite. Another class of groups having this property are the Margulis-superrigid groups, that is, irreducible lattices $\Gamma$ in semisimple Lie groups $G$ of $\mathbb{R}$-rank greater than one (for example, $\text{SL}(n, \mathbb{Z})$ for $n > 2$). Margulis proved [107] that every unbounded finite-dimensional linear representation of $\Gamma$ extends to a representation of $G$. It then follows that the affine holonomy of a compact affine manifold cannot factor through a Margulis-superrigid group. However, since $\text{SL}(n; \mathbb{R})$ does admit a left-invariant $\mathbb{R}P^{n^2-1}$-structure, it follows that if $\Gamma \subset \text{SL}(n; \mathbb{R})$ is a torsion-free cocompact lattice, then there exists a compact affine manifold with holonomy group $\Gamma \times \mathbb{Z}$ although $\Gamma$ itself is not the holonomy group of an affine structure.
11.4. Fried's classification of closed similarity manifolds

Fried [50] gives a sharp classification of closed similarity manifolds; this was announced earlier by Kuiper [93], although the proof contains a gap. Later Reischer and Vaisman [144] proved this, using a completely different set of ideas. Miner [117] extended Fried’s theorem to manifolds modeled on the Heisenberg group and its group of similarity transformations. Later, in §14.3, the ideas in Fried’s proof are related to Thurston’s parametrization of $\mathbb{CP}^1$-structures and the Kulkarni-Pinkall theory of flat conformal structures [98, 99].

11.4.1. Completeness versus radiance. Fried’s theorem is a prototype of a theorem about geometric structures on closed manifolds. Here $X = \mathbb{E}^n$ and $G = \text{Sim}(\mathbb{E}^n)$. Namely, Fried shows that a $(G,X)$-structure on a closed manifold $M$ must reduce to one of two special types, corresponding to subgeometries $(G',X') \hookrightarrow (G,X)$. Specifically, a closed similarity manifold must be one of the following two types:

- A Euclidean manifold where $X' = X = \mathbb{E}^n$ and $G' = \text{Isom}(\mathbb{E}^n) \hookrightarrow \text{Sim}(\mathbb{E}^n)$. This is precisely the case when the underlying affine structure on $M$ is complete;

- A finite quotient of a Hopf manifold where $X' = \mathbb{E}^n \setminus \{0\}$ and $G' = \text{Sim}_0(\mathbb{E}^n) \hookrightarrow \text{Sim}(\mathbb{E}^n)$, the group of linear similarity (or conformal) transformations of $\mathbb{E}^n$. The is precisely the case when the underlying affine structures is incomplete.

The complete case is easy to handle, since in that case $\Gamma$ acts freely, and any similarity transformation which is not isometric must fix a point. When $M$ is incomplete, very little can be said in general, and the compactness hypothesis must be crucially used.

**Exercise 11.4.1.** Prove that a complete similarity manifold is a Euclidean manifold, and diffeomorphic to a finite quotient of a product $\mathbb{T}^r \times \mathbb{E}^{n-r}$.

The other extreme — radiant similarity manifolds — were discussed in §6.1.3 of Chapter 6. The recurrence of an incomplete geodesic on a compact manifold guarantees a divergent sequence in the affine holonomy group $\Gamma$. This holonomy sequence converges to a singular projective transformation $\phi$ as in §2.5. The condition that $\Gamma \subset \text{Sim}(\mathbb{E}^n)$ strongly restricts $\phi$; in particular it has “rank one” in that most points approach a single point, which Fried shows must lie in $\mathbb{E}^n$. From this he deduces radiance, and finds that the structure is modeled on $\mathbb{E}^n \setminus \{0\}$. 

11.4.2. **Canonical metrics and incompleteness.** Choose a Euclidean metric \( g_E \) on \( \mathbb{E}^n \); the pullback \( \text{dev}^* g_E \) is a Euclidean metric on \( \tilde{M} \). Unless \( M \) is a Euclidean manifold, this metric is not invariant under \( \pi \), rather it transforms by the scale factor homomorphism: 
\[
\pi : \mathbb{R}^+ \to \mathbb{R}^+; \quad \lambda \mapsto \lambda \cdot \gamma.
\]
(34) 
\[
\gamma^* (\text{dev}^* g_E) = \lambda(\gamma) \text{dev}^* g_E.
\]
defined in §1.2.1 of Chapter 1.

**Exercise 11.4.2.** Relate the scale factor \( \lambda \) to the volume obstruction \( \nu_M \).

Unless \( M \) is Euclidean, then \( \text{dev}^* g_E \) is incomplete. Thus we assume that \( (\tilde{M}, \text{dev}^* g_E) \) is an incomplete Euclidean manifold with distance function \( \tilde{M} \times \tilde{M} \to \mathbb{R}, \) non-bijective developing map \( \tilde{M} \to \mathbb{E} \) and nontrivial scale factor homomorphism \( \pi_1(M) \to \mathbb{R}^+ \).

We begin with some general facts about an incomplete Euclidean manifold \( (N,d) \) which we apply to the case when \( N \) is the universal covering \( \tilde{M} \) of an incomplete similarity manifold \( M \).

For any metric space \( (X,d) \) and \( x \in X, r > 0 \), define the **open ball** with center \( x \) and radius \( r \) as:
\[
B_r(x) := \{ y \in X \mid d(x,y) < r \}.
\]
Partially order the open balls in a metric space by inclusion.

**Exercise 11.4.3.** Let \( N \) be a Euclidean manifold with developing map \( N \to \mathbb{E} \) and \( B \subset N \) be an open set. The following conditions are equivalent:

- \( B \) is an open ball in \( N \), that is, \( \exists c \in N, r > 0 \) such that \( B = B_r(c) \).
- \( B \) develops to an open ball in \( \mathbb{E} \), that is, \( \exists c \in \mathbb{E}, r > 0 \) such that the restriction \( \text{dev}|_B \) is a diffeomorphism \( B \to B_r(c) \subset \mathbb{E} \);
- \( B \) is the exponential image of a metric ball in the tangent space \( T_c N \), that is, \( \exists c \in N, r > 0 \) such that the restriction \( \text{Exp}|_{B_r(0,e)} \) is a diffeomorphism \( B_r(0,e) \to B \).

Under these conditions, the maps
\[
B_r(0,e) \xrightarrow{\text{Exp}_e} B \xrightarrow{\text{dev}} B_r(\text{dev}(c))
\]
are isometries with respect to the restrictions of the Euclidean metrics on \( T_c N, N \) and \( \mathbb{E} \), respectively.

**Definition 11.4.4.** A maximal ball in \( M \) is an open ball \( B \subset M \) which is maximal among open balls with respect to inclusion.
Exercise 11.4.5. A Euclidean manifold is complete if and only if no ball is maximal.

Exercise 11.4.6. Suppose \( N \) is an incomplete Euclidean manifold.
- Let \( B \subset N \) be a maximal ball and let \( c \) be its center. Then \( B \) is maximal among open balls centered at \( c \).
- Every open ball lies in a maximal ball.
- Every \( x \in N \) is the center of a unique maximal ball \( \mathcal{B}(x) \).
- For each \( x \), not every point on \( \partial \mathcal{B}(x) \) is visible from \( x \).

Definition 11.4.7. Let \( N \) be an incomplete Euclidean manifold. For each \( x \in N \), let \( R(x) < \infty \) be the radius of the maximal ball \( \mathcal{B}(x) \subset N \) centered at \( x \).

Exercise 11.4.8. The maximal ball \( \mathcal{B}(x) = \mathcal{B}_{R(x)}(x) \). Moreover \( R(x) \) is the supremum of \( r \) such that \( \mathcal{B}_r(0_x) \subset \mathcal{E}_x \), where \( \mathcal{E}_x \subset T_x N \) denotes the domain of \( \text{Exp}_x \) defined in §8.2 of Chapter 8.

Let \( (N,d) \) be an incomplete Euclidean manifold with developing map \( N \xrightarrow{\text{dev}} E \) and \( N \xrightarrow{R} \mathbb{R}^+ \) as above.

Lemma 11.4.9. The function \( R \) is Lipschitz:

\[
|R(x) - R(y)| \leq d(x,y)
\]

if \( x, y \in N \) are sufficiently close. In particular \( R \) is continuous.

Proof. Suppose that \( x \in N \) and \( \epsilon \) such that \( \epsilon < \sup \{R(x), R(y)\} \).
Choose \( r < R(x) \) so that \( \mathcal{B}_r(0_x) \subset \mathcal{E}_x \). First we show that if \( d(x, y) < \epsilon \), then

\[
r < d(x, y) + R(y)
\]

Choose \( u \in \mathcal{B}(x) \) such that \( d(x, u) = r \). Suppose that \( d(x, y) < \epsilon \).
Then closed ball \( \overline{\mathcal{B}_r(x)} \) lies in the convex set \( \mathcal{B}(x) \) which also contains \( y \). Thus \( u \in \partial \mathcal{B}_r(x) \) is visible from \( y \), whence

\[
d(y, u) < R(y).
\]

Thus

\[
r = d(x, u) \leq d(x, y) + d(y, u) < d(x, y) + R(y),
\]

proving (36). Taking the supremum over \( r \) yields:

\[
R(x) < d(x, y) + R(y),
\]

so \( R(x) - R(y) < d(x, y) \) if \( d(x, y) < \epsilon \). Similarly, symmetry of \( d \) implies that \( R(x) - R(y) < d(x, y) \) if \( d(x, y) < \epsilon \) which implies (35). \( \square \)
We return to the case that $M$ is an incomplete similarity manifold. Apply the previous discussion to the universal covering $N = \tilde{M}$ with its incomplete Euclidean structure defined by $\text{dev}^* g_E$. If $\gamma \in \pi_1(M)$ is a deck transformation $\tilde{M} \to \tilde{M}$, then $\gamma(B_{R(\tilde{p})})$ is a maximal ball at $\gamma(\tilde{p})$, so:

$$R(\gamma\tilde{p}) = \lambda(\gamma)R(\tilde{p})$$

This leads to a natural conformal Riemannian structure on $\tilde{M}$ which descends to a conformal Riemannian structure on $M$. This will be the canonical Riemannian structure on a radiant similarity manifold, and imply that if $M$ is closed, is the product metric on a finite covering space which is a Hopf manifold. (34) and (37) together imply that the Riemannian metric $\tilde{g}$ on $\tilde{M}$ defined by:

$$\tilde{g}(\tilde{p}) := R(\tilde{p})^{-1}\text{dev}^* g_E$$

is $\pi_1(M)$-invariant. Therefore $\tilde{g}$ passes down to a Riemannian metric $g_M$ on $M$.

The Riemannian structure $g_M$ has the property that its unit ball is maximal inside the domain $E$ of the exponential map $\text{Exp}$. When $M$ is closed, even more is true:

**Proposition 11.4.10.** Let $M$ be a closed incomplete similarity manifold. Then $\exists \xi \in \text{Vec}(M)$ such that:

- $\|\xi\|_{g_M} = 1$;
- The halfspace

$$\tilde{H}_p := \{v \in T_p M \mid g(v, \xi) < 1\}$$

lies in $E_p$.

In particular each $p \in M$ has a natural half-space neighborhood

$$H_p := \text{Exp}_p(\tilde{H}_p).$$

By analyzing $H_p$, we shall prove that $\xi$ is a radiant vector field and $M$ is (covered by) a Hopf manifold.

**11.4.3. Incomplete geodesics are recurrent.** Fried makes a detailed analysis of an incomplete geodesic

$$[0, 1) \to M$$

$$t \mapsto p(t),$$

that is, $p(t) = \text{Exp}_{p(0)}(tv)$, where $v \in T_{p(0)} M$ but, for $t \geq 0$,

$$tv \in E_{p(0)} \iff t < 1$$
where $E$ is defined as in Theorem 8.2.3. Since $M$ is compact, the path $p(t)$, for $0 \leq t < 1$, accumulates, that is, a sequence $p(t_n)$ converges for some sequence $t_n \nearrow 1$ as $n \to +\infty$. Denote

$$p' := \lim_{n \to +\infty} p(t_n)$$

Next, we use the recurrence of $p(t)$ to obtain a holonomy sequence converging to a singular projective transformation as in §2.5. To that end, we pass to a covering space and use a developing map. A convenient model is to use $p' \in M$ as a basepoint, and the total space of the universal covering $\tilde{M}$ as the space of relative homotopy classes of paths $[0, T] \to M$ with $\gamma(0) = p'$:

$$\tilde{M} \xrightarrow{\Pi} M$$

$$[\gamma] \to \gamma(T)$$

The constant path defines a basepoint $\tilde{p}' \in \tilde{M}$ with $\Pi(\tilde{p}') = p'$.

Choose a coordinate patch $U' \ni p'$. Let $\tilde{U}' \subset \tilde{M}$ be the component of $\Pi^{-1}(U')$ containing $\tilde{p}'$. Choose a developing map $\tilde{M} \xrightarrow{\text{dev}} E^n$ such that $\text{dev}|_{\tilde{U}'}$ is injective.

11.4.4. Degenerate similarities. Recall from Exercise 2.5.5 of Chapter 2, that a sequence of similarity transformations accumulates to either:

- The zero affine transformation (undefined at the ideal hyperplane, otherwise constant);
- A singular projective transformation of rank one, taking values at an ideal point.

Proof of Proposition 11.4.10:

Rule out the second case for the holonomy sequence of an incomplete geodesic.
CHAPTER 12

Hyperbolicity

Based on the theory of intrinsic metrics on complex manifolds due to Carathéodory and Kobayashi, Kobayashi and Vey developed the corresponding theory of intrinsic metrics on affine and projective manifolds. We only consider the Kobayashi metric, and refer to Kobayashi for a description of the Carathédory construction.

Namely, an affine (respectively projective) manifold is hyperbolic if the Kobayashi pseudo-metric $d^{\text{Kob}}$ is a metric, that is, if $d^{\text{Kob}}(x, y) > 0$ for $x \neq y$. Combining Kobayashi and Vey, a compact affine manifold $M$ is hyperbolic if and only if it is a quotient of a properly convex cone; a compact projective manifold is hyperbolic if and only if it is a quotient of a properly convex domain in projective space.

Hyperbolicity in this sense is precisely the condition that no geodesic in $M$ is complete both directions. In this sense these are the completely incomplete affine manifolds. Such manifolds are always radiant, and by Theorem 11.3.1 have Euler characteristic zero. Furthermore the developing map is a diffeomorphism. Hence compact affine manifolds with pathological development must contain both complete and incomplete geodesics.

12.1. Benoist’s theory of divisible convex sets


12.2. Vey’s semisimplicity theorem

The following theorem is due to Vey [147].

12.3. Kobayashi hyperbolicity

Now we discuss intrinsic metrics on affine and projective manifolds. The case of domains was discussed in §3.3.

The opposite of geodesic completeness is *hyperbolicity* in the sense of Vey [146] and Kobayashi [85, 83], which is equivalent to the following
12. HYPERBOLICITY

Notion: Say that an affine manifold $M$ is **completely incomplete** if there exists no affine map $\mathbb{R} \rightarrow M$, that is, $M$ admits no complete geodesic. Similarly, an $\mathbb{RP}^n$-manifold is **completely incomplete** if there exists no projective map $\mathbb{R} \rightarrow M$. As noted by the author (see Kobayashi [85]), the combined results of Kobayashi [85], Wu [151], and Vey [147] imply:

**Theorem.** Let $M$ be a closed hyperbolic affine manifold. Then $M$ is a quotient of a properly convex cone.

In particular $M$ is radiant. Moreover $M$ fibers over $S^1$ (which implies that $\chi(M) = 0$ and $b_1(M) > 0$.

For projective manifolds, taking the radiant suspension of a hyperbolic projective structure yields a radiant affine structure, which one easily sees is hyperbolic. Applying the above theorem implies that $M$ is a quotient of a properly convex cone.

This striking characterization of hyperbolicity uses **intrinsic metrics** in the category of affine and projective manifolds, developed by Vey [146] and Kobayashi [85, 83]. Their constructions were inspired by the intrinsic metrics of Carathéodory and Kobayashi in the category of complex manifolds.

Denote by $I$ the open unit interval $(-1,1)$ and

$$g_I := \frac{4}{(1-u^2)^2} du^2 = \left(\frac{2 du}{1-u^2}\right)^2$$

its **Poincaré metric.** Since

$$\frac{2 du}{1-u^2} = d\tanh^{-1}(u/2)$$

the natural parameter is $u = 2 \tanh(s)$ where $s \in \mathbb{R}$ is arc length.

For projective manifolds $M$, one defines a “universal” pseudo-metric $M \times M \xrightarrow{d_M} \mathbb{R}$ such that affine (respectively projective) maps $I \to M$ are distance non-increasing with respect to $g_I$.

The definition of $d_M$ enforces the triangle inequality by taking the infimum of $g_I$-distances over sequences $x_0 = x, x_1, \ldots, x_m = y$ where $x_i$ and $x_{i+1}$ are “close” in the following sense: there are projective maps $I \xrightarrow{f_i} M$ such that $x_i = f_i(a_i)$ and $x_{i+1} = f_i(b_i)$ for $-1 < a_i < b_i < 1$. Then define $d_M(x,y)$ as the infimum over all such sequences $(f_i, a_i, b_i)$ of

$$\sum_{i=0}^{m-1} d_I(a_i, b_i)$$

where $d_I$ is the distance function on the Riemannian 1-manifold $(I, g_{[-1,1]})$. 
That is, \( d_M(x, y) \) is the infimum of
\[
\int_a^b f(\gamma'(t)) \, dt
\]
over all piecewise \( C^1 \) paths \([a, b] \xrightarrow{\gamma} M\) with \( \gamma(a) = x, \gamma(b) = y \).

This function has an infinitesimal form, defined by a nonnegative upper-semicontinuous function \( TM \xrightarrow{\phi} \mathbb{R} \). For affine manifolds, completeness is equivalent to \( f \equiv 0 \).

Following Kobayashi and Vey, \( M \) is projectively hyperbolic if and only if \( d_M \) is a metric, that is, if \( d_M(x, y) > 0 \) for \( x \neq y \). Then \( d_M \) is a Finsler metric and equals the Hilbert metric on the convex domain \( \tilde{M} \).

When \( M \) is affine, then Vey [147] proves that \( M \) is a quotient of a properly convex cone. In that case there is (in addition to the Hilbert metric), a natural Riemannian metric introduced by Vinberg [148], Koszul [89, 88, 92, 91] and Vesentini [145]. In particular Koszul and Vinberg observe that this Riemannian structure is the covariant differential \( \nabla \omega \) of a closed 1-form \( \omega \). In particular \( \omega \) is everywhere nonzero, so by Tischler [142], \( M \) fibers over \( S^1 \).

This implies Koszul’s beautiful theorem [92] that the holonomy mapping \( \text{hol} \) (described in Chapter 7, §7.4) embeds the space of convex structures onto an open subset of the representation variety. This has recently been extended to noncompact manifolds by Cooper-Long-Tillmann [37].

12.3.1. Hessian manifolds. Hyperbolic affine manifolds are closely related to Hessian manifolds. If \( \omega \) is a closed 1-form, then its covariant differential \( \nabla \omega \) is a symmetric 2-form. Since closed forms are locally exact, \( \omega = df \) for some function; in that case \( \nabla \omega \) equals the Hessian \( d^2f \). Koszul [92] showed that hyperbolicity is equivalent to the existence of a closed 1-form \( \omega \) whose covariant differential \( \nabla \omega \) is positive definite, that is, a Riemannian metric. More generally, Shima [128] considered Riemannian metrics on an affine manifold which are locally Hessians of functions, and proved that such a closed Hessian manifold is a quotient of a convex domain, thus generalizing Koszul’s result.

12.3.2. Completely incomplete manifolds. We discuss these ideas in the context of affine structures on 3-dimensional hyperbolic torus bundles.

12.3.2.1. Hyperbolic torus bundles. Although the class of affine structures on closed 3-manifolds with nilpotent holonomy are understood, the general case of solvable holonomy remains mysterious. However,
Serge Dupont [43] completely classifies affine structures on 3-manifolds with solvable fundamental group. (Compare also Dupont [42], in the volume [77].) In terms of the Thurston geometrization, these are the geometric 3-manifolds modeled on $\text{Sol}$, that is, 3-manifolds finitely covered by hyperbolic torus bundles: mapping tori (suspensions) of hyperbolic elements of $\text{GL}(2,\mathbb{Z})$. Dupont shows that all such structures arise from left-invariant affine structures on the corresponding Lie group $G$, which is the semidirect product of $\mathbb{R}^2$ by $\mathbb{R}$, where $\mathbb{R}$ acts on $\mathbb{R}^2$ as a unimodular hyperbolic one-parameter subgroup (explicitly, $G$ is isomorphic to the identity component in the group of Lorentz isometries of the Minkowski plane).

Two structures are particularly interesting for the behavior of geodesics in light of the results of Vey [147]. A properly convex domain $\Omega \in \mathbb{A}^n$ is said to be divisible if $\Omega$ admits a discrete group $\Gamma$ of projective automorphisms acting properly on $\Omega$ such that $\Omega/\Gamma$ is compact. (Equivalently, the quotient space $\Omega/\Gamma$ by a discrete subgroup $\Gamma \subset \text{Aut}(\Omega)$ is compact and Hausdorff.) Vey proved that a divisible domain is a cone. However, dropping the properness of the action of $\Gamma$ on $\Omega$ allows counterexamples: the parabolic cylinder

$$\Omega := \{(x, y) \in \mathbb{A}^2 \mid y > x^2\}$$

is a properly convex domain which is not a cone, but admits a group $\Gamma$ of automorphisms such that $\Omega/\Gamma$ is compact but not Hausdorff.

Now take the product $\Omega \times \mathbb{R} \subset \mathbb{A}^3$. The author [58] found a discrete subgroup $\Gamma \subset \text{Aff}(\mathbb{A}^3)$ acting properly on $\Omega \times \mathbb{R}$ such that:

- The quotient $M = (\Omega \times \mathbb{R})/\Gamma$ is a hyperbolic torus bundle (and in particular compact and Hausdorff);
- $\Omega \times \mathbb{R}$ is not a cone.

Clearly $\Omega \times \mathbb{R}$ is not properly convex, showing that Vey’s result is sharp. The Kobayashi pseudometric degenerates along a 1-dimensional foliation of $M$, and defines the hyperbolic structure transverse to this foliation discussed by Thurston [140], Chapter 4.
CHAPTER 13

Projective structures on surfaces

$\mathbb{RP}^2$-manifolds are relatively well understood, due to intense activity in recent years. Aside from the two structures with finite fundamental group ($\mathbb{RP}^2$ itself, and its double cover $S^2$), this class of geometric structures includes affine structures on surfaces, some new $\mathbb{RP}^2$-structures on tori (first analyzed by Sullivan-Thurston [138], Smillie [131] and the author [56] in 1976–1977), as well as convex structures (which are hyperbolic in the sense of Kobayashi and Vey). The deformation space of convex structures was calculated by the author [62] in 1985. Suhyoung Choi [31, 32, 32] classified all structures with Euler characteristic $\chi < 0$ in terms of a grafting construction; compare Choi-Goldman [34] for a summary. Somewhat curiously, the diversity of affine structures on the torus gives a much less clean classification for $\chi = 0$ than the geometrically and analytically more interesting case when $\chi < 0$.

A new feature for these structures is pathological developing maps, even on $T^2$. That is, even for closed $\mathbb{RP}^2$-manifolds, the developing map may fail to be a covering space onto its image. For closed surfaces of genus $> 1$, a typical developing map will be a surjection onto all of $\mathbb{RP}^2$. Their radiant suspensions are affine 3-manifolds whose developing maps surject to $\mathbb{R}^3 \setminus \{0\}$, but are not covering spaces (indeed the 3-manifolds are mapping tori of periodic automorphisms of surfaces with $\chi < 0$, and are thus aspherical).

13.1. Pathological developing maps

13.2. Generalized Fenchel-Nielsen twist flows

13.3. Bulging deformations

A convex $\mathbb{RP}^2$-manifold is a representation of a surface $S$ as a quotient $\Omega / \Gamma$, where $\Omega \subset \mathbb{RP}^2$ is a convex domain and $\Gamma \subset \text{SL}(3,\mathbb{R})$ is a discrete group of collineations acting properly on $\Omega$. We shall describe a construction of deformations of such structures based on Thurston’s earthquake deformations for hyperbolic surfaces and quakebend deformations for $\mathbb{CP}^1$-manifolds.
In general if $\Omega/\Gamma$ is a convex $\mathbb{R}P^2$-manifold which is a closed surface $S$ with $\chi(S)$, then either $\partial \Omega$ is a conic, or $\partial \Omega$ is a $C^1$ convex curve (Benzécri [18] which is not $C^2$ (Kuiper [97]). In fact its derivative is
Hölder continuous with Hölder exponent strictly between 1 and 2. Figure 14 depicts such a domain tiled by the $(3, 3, 4)$-triangle tesselation.

Figure 14. A convex domain tiled by triangles
This drawing actually arises from Lie algebras (see Kac-Vinberg [149]). Namely the Cartan matrix
\[
C = \begin{bmatrix}
2 & -1 & -1 \\
-2 & 2 & -1 \\
-1 & -1 & 2
\end{bmatrix}
\]
determines a group of reflections as follows. For \( i = 1, 2, 3 \) let \( E_{ii} \) denote the elementary matrix having entry 1 in the \( i \)-th diagonal slot. Then, for \( i = 1, 2, 3 \), the reflections
\[
\rho_i = I - E_{ii}C
\]
generate a discrete subgroup of \( SL(3, \mathbb{Z}) \) which acts properly on the convex domain depicted in (and appears on the cover of the November 2002 Notices of the American Mathematical Society). This group is the Weyl group of a hyperbolic Kac-Moody Lie algebra.

Compare also Long-Reid-Thistlethwaite [105], where this example is embedded in a one-parameter family of subgroups of \( SL(3, \mathbb{Z}) \).

We describe here a general construction of such convex domains as limits of \textit{piecewise conic} curves.

If \( \Omega/\Gamma \) is a convex \( \mathbb{RP}^2 \)-manifold homeomorphic to a closed esurface \( S \) with \( \chi(S) < 0 \), then every element \( \gamma \in \Gamma \) is \textit{positive hyperbolic}, that is, conjugate in \( SL(3, \mathbb{R}) \) to a diagonal matrix of the form
\[
\delta = \begin{bmatrix}
e^s & 0 & 0 \\
0 & e^t & 0 \\
0 & 0 & e^{s-t}
\end{bmatrix},
\]
where \( s > t > -s - t \). Its centralizer is the \textit{maximal \( \mathbb{R} \)-split torus} \( A \) consisting of all diagonal matrices in \( SL(3, \mathbb{R}) \). It is isomorphic to a Cartesian product \( \mathbb{R}^* \times \mathbb{R}^* \) and has four connected components. Its identity component \( A^+ \) consists of diagonal matrices with positive entries.

The \textit{roots} are linear functionals on its Lie algebra \( \mathfrak{a} \), the \textit{Cartan subalgebra}. Namely, \( \mathfrak{a} \) consists of diagonal matrices
\[
(39) \quad \mathfrak{a} = \begin{bmatrix}
a_1 & 0 & 0 \\
0 & a_2 & 0 \\
0 & 0 & a_3
\end{bmatrix}
\]
where \( a_1 + a_2 + a_3 = 0 \). The roots are the six linear functionals on \( \mathfrak{a} \) defined by
\[
\mathfrak{a} \overset{\alpha_{ij}}{\longrightarrow} a_i - a_j
\]
where \( 1 \leq i \neq j \leq 3 \). Evidently \( \alpha_{ji} = -\alpha_{ij} \).
13.3. BULGING DEFORMATIONS 189

Writing \( a(s, t) \) for the diagonal matrix (39) with
\[
a_1 = s, \quad a_1 = 2, \quad a_3 = -s - t,
\]
the roots are the linear functionals defined by
\[
\begin{align*}
a(s, t) &\mapsto \alpha_{12} \quad s - t \\
a(s, t) &\mapsto \alpha_{21} \quad t - s \\
a(s, t) &\mapsto \alpha_{23} \quad t - (s + t) = s + 2t \\
a(s, t) &\mapsto \alpha_{32} \quad (s + t) - t = -s - 2t \\
a(s, t) &\mapsto \alpha_{31} \quad (s + t) - s = -2s - t \\
a(s, t) &\mapsto \alpha_{13} \quad s - (s + t) = 2s + t
\end{align*}
\]
which we write as
\[
\begin{align*}
\alpha_{12} &= \begin{bmatrix} 1 & -1 \end{bmatrix} \\
\alpha_{21} &= \begin{bmatrix} -1 & 1 \end{bmatrix} \\
\alpha_{23} &= \begin{bmatrix} 1 & 2 \end{bmatrix} \\
\alpha_{32} &= \begin{bmatrix} -1 & -2 \end{bmatrix} \\
\alpha_{31} &= \begin{bmatrix} -2 & -1 \end{bmatrix} \\
\alpha_{13} &= \begin{bmatrix} 2 & 1 \end{bmatrix}
\end{align*}
\]

The *Weyl group* is generated by reflections in the roots and in this case is just the symmetric group, consisting of permutations of the three variables \( a_1, a_2, a_3 \) in \( a \) (as in (39)). A fundamental domain is the *Weyl chamber* consisting of all \( a \) satisfying \( \alpha_{12} > 0 \) and \( \alpha_{23} > 0 \). This corresponds to the ordering of the roots where \( \alpha_{12} > \alpha_{23} \) are the *positive simple roots*. In other words, the roots are totally ordered by:
the rule
\[
\alpha_{13} > \alpha_{12} > \alpha_{23} > 0 > \alpha_{32} > \alpha_{21} > \alpha_{21} > \alpha_{31}.
\]
In terms of the parametrization of \( a \) by \( a(s, t) \), the Weyl chamber equals
\[
\{ a(s, t) \mid s \geq t \geq -\frac{1}{2}s \}.
\]

The *trace form* on \( \text{SL}(3, \mathbb{R}) \) defines the inner product \( \langle , \rangle \) with associated quadratic form
\[
\text{tr}(a(s, t)^2) = 2(s^2 + st + t^2) = 2|s + \omega t|^2
\]
where
\[
\omega = \frac{1}{2} + \frac{\sqrt{-3}}{2} = e^{\pi i/3}
\]
is the primitive sixth root of 1.

The elements of $\text{SL}(3, \mathbb{R})$ which dual to the roots (via the inner product $\langle , \rangle$) are the root vectors:

\[
\begin{align*}
  h_{12} &= a(1, -1), \\
  h_{21} &= a(-1, 1), \\
  h_{23} &= a(0, 1), \\
  h_{32} &= a(0, -1), \\
  h_{31} &= a(-1, 0), \\
  h_{13} &= a(1, 0)
\end{align*}
\]

The Weyl chamber consists of all

\[
a(s, \lambda s) = \begin{bmatrix}
  s & 0 & 0 \\
  0 & \lambda s & 0 \\
  0 & 0 & -(1 + \lambda)s
\end{bmatrix}
\]

where $1 \geq \lambda \geq -\frac{1}{2}$. Its boundary consists of the rays generated by the singular elements

\[
a(1, 1) = h_{13} + h_{23} = h_{12} + 2h_{23}
\]

and

\[
a(2, -1) = h_{12} + h_{13} = 2h_{12} + h_{23}.
\]

The sum of the simple positive roots is the element

\[
a(1, 0) = h_{13} = h_{12} + h_{23}
\]

which generates the one-parameter subgroup

\[
H_t := \exp (a(t, 0)) = \begin{bmatrix} e^t & 0 & 0 \\
  0 & 1 & 0 \\
  0 & 0 & e^{-t} \end{bmatrix}.
\]

The orbits of $\mathbb{A}^+$ on $\mathbb{RP}^2$ are the four open 2-simplices defined by the homogeneous coordinates, their (six) edges and their (three) vertices. The orbits of $H_t$ are arcs of conics depicted in Figure 15.

Associated to any measured geodesic lamination $\lambda$ on a hyperbolic surface $S$ is bulging deformation as an $\mathbb{RP}^2$-surface. Namely, one applies a one-parameter group of collineations

\[
\begin{bmatrix}
  1 & 0 & 0 \\
  0 & e^t & 0 \\
  0 & 0 & 1
\end{bmatrix}
\]
13.4. FOCK-GONCHAROV COORDINATES

Figure 15. Conics tangent to a triangle

to the coordinates on either side of a leaf. This extends Thurston’s earthquake deformations (the analog of Fenchel-Nielsen twist deformations along possibly infinite geodesic laminations), and the bending deformations in $\text{PSL}(2, \mathbb{C})$.

In general, if $S$ is a convex $\mathbb{R}P^2$-manifold, then deformations are determined by a geodesic lamination with a transverse measure taking values in the Weyl chamber of $\text{SL}(3, \mathbb{R})$. When $S$ is itself a hyperbolic surface, all the deformations in the singular directions become earthquakes and deform $\partial \tilde{S}$ trivially (just as in $\text{PSL}(2, \mathbb{C})$).

Figure 16. Deforming a conic

13.4. Fock-Goncharov coordinates

In their paper [46], Fock and Goncharov develop an ambitious program for studying surface group representations into split $\mathbb{R}$-forms,
and develop natural coordinates on certain components discovered by Hitchin [76]. and studied by Labourie [100].
13.6. Choi's convex decomposition theorem

A version when $G = \text{SL}(3, \mathbb{R})$ is developed in Fock-Goncharov [47], giving coordinates on the deformation space of convex $\mathbb{R}P^2$-structures. Compare also Ovsienko-Tabachnikov [120].

13.5. Affine spheres and Labourie-Loftin parametrization

13.6. Choi’s convex decomposition theorem
CHAPTER 14

Complex-projective structures

From the general viewpoint of locally homogeneous geometric structures, \( \mathbb{C}P^1 \)-manifolds occupy a central role. Historically these objects arose from the applying the theory of second-order holomorphic linear differential equations to conformal mapping of plane domains. Theoretically these objects seem to be fundamental in so many homogeneous spaces extend the geometry of \( \mathbb{C}P^1 \). Furthermore \( \mathbb{C}P^1 \)-manifolds play a fundamental role in the theory of hyperbolic 3-manifolds and classical Kleinian groups.

A \( \mathbb{C}P^1 \)-manifold has the underlying structure as a Riemann surface. Starting from a Riemann surface \( M \), a compatible \( \mathbb{C}P^1 \)-structure is a (holomorphic) projective structure on the Riemann surface \( M \). Remarkably, projective structures on a Riemann surface \( M \) admit an extraordinarily clean classification: the deformation space of projective structures on a fixed Riemann surface \( M \) is a complex affine space whose underlying vector space is the space \( H^0(M; \kappa^2) \) of holomorphic quadratic differentials on \( M \). However, the geometry of the developing map can become extremely complicated, despite this clean determination of the deformation space.

An alternate parametrization of \( \mathbb{C}P^1(\Sigma) \) is due to Thurston, involves locally convex developments into hyperbolic 3-manifolds. In this case \( \mathbb{C}P^1(\Sigma) \) identifies with the product of the space \( \mathcal{F}(\Sigma) \) of marked hyperbolic structures on \( \Sigma \) and the Thurston cone \( \mathcal{ML}(\Sigma) \) of measured geodesic laminations on \( \Sigma \). This generalizes more directly to higher dimensions, and more closely relates to topological constructions with the developing map, such as grafting.

The grafting construction was first developed by Hejhal \([71]\) and Maskit \([109]\) (Theorem 5) and Sullivan-Thurston \([138]\). As for \( \mathbb{R}P^2 \)-surfaces studied in the previous chapter, this construction yields pathological developing maps, typically local homeomorphisms from the universal covering space onto all of \( \mathbb{C}P^1 \).

We will only touch on the subject, which has a vast and ever-expanding literature. We refer to the excellent survey article of Dumas \([41]\) for more details.
14. COMPLEX-PROJECTIVE STRUCTURES

14.1. Schwarzian derivative

14.2. Affine structures and the complex exponential map

14.3. Thurston parametrization

We briefly summarize some of the unpublished work of Thurston and its extension by Kulkarni and Pinkall [98, 99].

14.4. Fuchsian holonomy

A related idea is the classification of projective structures with Fuchsian holonomy [60]. This is the converse to the grafting construction, whereby grafting is the only construction yielding geometric structures with the same holonomy. Recall that a Fuchsian representation of a surface group $\pi$ is an embedding of $\pi$ as a discrete subgroup of the group $\mathrm{PGL}(2,\mathbb{R}) \cong \text{Isom}(\mathbb{H}^2)$. Equivalently, $\rho$ is the holonomy representation of a hyperbolic structure on a surface $\Sigma$ with $\pi_1(\Sigma) \cong \pi$.

The main result is:

**Theorem 14.4.1.** Let $M$ be a closed $\mathbb{C}P^1$-manifold whose holonomy representation $\pi_1(M) \rightarrow \mathrm{PSL}(2,\mathbb{C})$ is a composition

$$\pi_1(M) \rightarrow_{\rho_0} \mathrm{PGL}(2,\mathbb{R}) \hookrightarrow \mathrm{PSL}(2,\mathbb{C})$$

where $\rho_0$ is Fuchsian representation. Let $M_0$ be a hyperbolic structure with holonomy representation $\rho_0$, regarded as a $\mathbb{C}P^1$-manifold. Then there is a unique multicurve $S \subset M_0$ such that $M$ is obtained from $M_0$ by grafting along $S$.

However the proof contained a gap, which I first learned from M. Kapovich, who pointed me to the paper of Kuiper [94]. The proof was later fixed by Choi-Lee [35]. Here we give a correct proof, based on Kulkarni-Pinkall [98] (Theorem 4.2), communicated to me by Daniele Alessandrini.

14.4.1. Normality domains. The key definition is that of a point of normality, due to Kulkarni-Pinkall [98]. Let $G$ be a group acting on a space $X$ strongly effectively. A point $x \in X$ is a point of normality with respect to $G$ if and only if $x$ admits an open neighborhood $U$ such the set of restrictions

$$G|_U := \{g|_U \mid g \in G\}$$

is a compact subset of $\text{Map}(U,X)$ with respect to the compact-open topology on $\text{Map}(U,X)$. Denote the open subset of $X$ comprising points of normality by $\mathfrak{N}(G,X)$. Clearly $\mathfrak{N}(G,X)$ is a $G$-invariant open subset of $X$. 
14.4. FUCHSIAN HOLONOMY

We use the following standard notation. Let $M$ be a connected smooth manifold. Let $\tilde{M} \xrightarrow{\Pi} M$ be a universal covering space with covering group $\pi = \pi_1(M)$. Give $M$ a $(G,X)$-structure and let $(\text{dev}, \rho)$ be a developing pair:

- $\tilde{M} \xrightarrow{\text{dev}} X$ denotes the developing map;
- $\pi \xrightarrow{\rho} X$ denotes the holonomy representation.

Let $\Gamma := \rho(\pi) \subset G$ denote the holonomy group.

This could be an exercise in Part 2

**Proposition 14.4.2.** Suppose $\Omega \subset X$ is a $\Gamma$-invariant open subset.

- $\text{dev}^{-1}(\Omega)$ is a a $\pi$-invariant open subset of $\tilde{M}$;
- Its image $M_\Omega := \Pi(\text{dev}^{-1}(\Omega))$ is an open subset of $M$;
- Let $W \subset M_\Omega$. Then each connected component $\tilde{W} \subset \Pi^{-1}(M_\Omega) \subset \tilde{M}$ is a connected component of $\text{dev}^{-1}(\Omega)$ for which the restriction $\tilde{W} \xrightarrow{\text{dev}|_{\tilde{W}}} \Omega$ is a covering space.

**Theorem 14.4.3 (Kulkarni-Pinkall [98], Theorem 4.2).** Let $M$ be a closed $(G,X)$-manifold. Suppose that $\Omega = \mathfrak{M}(\Gamma, X)$ is the normality domain as above. Then for each component $W \subset M_\Omega$, and each component $\tilde{W}$, the restriction $\tilde{W} \xrightarrow{\text{dev}|_{\tilde{W}}} \Omega$ is a covering space. In particular $\text{dev}|_{\tilde{W}}$ is onto.

We break the proof into a sequence of lemmas. We show that $\text{dev}|_{\tilde{W}}$ satisfies the path-lifting criterion for covering spaces: every path in $\Omega$ lifts to a path in $\tilde{W}$. Let $[0,1] \xrightarrow{\gamma} \Omega$ be a path in $\Omega$ and choose a point $\tilde{w}_0 \in \Pi^{-1}(\gamma(0)) \cap \tilde{W}$.

We seek a path $[0,1] \xrightarrow{\tilde{c}} \tilde{W}$ satisfying:

- $\tilde{c}(0) = \tilde{w}_0$;
- $\text{dev} \circ \tilde{c} = \gamma$. 
Since dev is a local homeomorphism, the set of \( t \in [0,1] \) such that
\[
\begin{array}{c}
\gamma_{[0,t]} \\
\longrightarrow
\end{array}
\Omega
\] lifts to
\[
\begin{array}{c}
[0,t] \\
\gamma_{[0,t]} \\
\longrightarrow
\tilde{W}
\end{array}
\]
is open. Furthermore, since extensions of lifts are unique, it is a connected open neighborhood of 0 in \([0,1]\). By reparametrizing \( \tilde{c} \), we may assume that \( \tilde{c} \) is defined on \([0,1)\). It suffices to show that \( \tilde{c} \) can be lifted to \([0,1]\).

Let \( c = \Pi \circ \tilde{c} \) be the curve in \( M \). Since \( M \) is compact and
\[
[0,1) \xrightarrow{\xi} \Pi(W) \subset M,
\]
c accumulates in \( M \). That is, there exists a sequence \( t_n \in [0,1) \) with \( t_n \nrightarrow 1 \) and \( z \in M \) such that
\[
\lim_{n \to \infty} c(t_n) = z.
\]
Employ \( z \) as the basepoint in \( M \). Fix the corresponding universal covering space \( \tilde{M} \xrightarrow{\Pi} M \), where \( \tilde{M} \) comprises relative homotopy classes of paths in \( M \) starting at \( z \) and covering group \( \pi_1(M,z) \). Choose a developing map \( \tilde{M} \xrightarrow{\text{dev}} X \).

Let \( U \ni z \) be a coordinate patch in \( M \) such that the restriction \( \Pi^{-1}(U) \) is a homeomorphism. Choose paths \( \alpha_n \) in \( U \) from \( z \) to \( c(t_n) \). The concatenation
\[
\alpha_n^{-1} \ast \gamma_{[t_1,t_n]} \ast \alpha_1
\]
is a based loop in \( M \) having relative homotopy class
\[
\beta_n \in \pi_1(M,z).
\]
Fix \( \tilde{U} \) to be the component of \( \Pi^{-1}(U) \) containing \( \tilde{c}(t_1) \). Let \( \tilde{z} \) be the unique element of \( \tilde{U} \cap \Pi^{-1}(z) \).

**Lemma 14.4.4.**
\[
\lim_{n \to \infty} \rho(\beta_n)^{-1} \gamma(t_n) = \text{dev}(\tilde{z}).
\]

**Proof.** Each \( c(t_n) \in \tilde{U} \), and therefore \( \tilde{c}(t_n) \in \beta_n \tilde{U} \). Hence
\[
\beta_n^{-1} \tilde{c}(t_n) \in \tilde{U}.
\]
Since \( c(t_n) \xrightarrow{} z \) and the restriction \( \Pi|_{\tilde{U}} \) is bijective,
\[
\lim_{n \to \infty} \beta_n^{-1} \tilde{c}(t_n) = \tilde{z}.
\]
Applying the continuous map dev, the conclusion follows. \( \square \)
Now we apply the condition of normality to the images
\[
\rho(\beta_n)^{-1} \circ \gamma
\]
of the curve \([0, 1] \xrightarrow{\gamma} \Omega\). By the definition of \(\Omega\), these images form a precompact sequence in \(\text{Map}([0, 1], X)\). After passing to a subsequence, we may assume that \(\rho(\beta_n)^{-1} \circ \gamma\) converges uniformly to a continuous map \([0, 1] \xrightarrow{\delta} X\):

\[
\rho(\beta_n)^{-1} \circ \gamma \rightrightarrows \delta
\]

(40) explain the notation somewhere.

**Lemma 14.4.5.** For \(N\) sufficiently large,
\[
\delta(1) \in \text{dev}(\beta_N(\tilde{U}))
\]
and
\[
\lim_{t \to 1} \text{dev}(\tilde{c}(t)) = \delta(1).
\]

**Proof.** The uniform convergence of \(\rho(\beta_n^{-1}) \circ \gamma\) implies that
\[
\rho(\beta_N)^{-1}(\gamma([t_N, 1])) \subset \text{dev}(\tilde{U}).
\]
Thus
\[
\gamma([t_N, 1]) \subset \rho(\beta_N)\text{dev}(\tilde{U}) = \text{dev}(\beta_N \tilde{U}).
\]
Applying Lemma 14.4.4 to the uniform convergence in (40) implies convergence to \(\delta(1)\). \(\Box\)

Thus \(\tilde{c}(t)\) lies in \(\text{dev}^{-1}(\beta_N \tilde{U})\) for \(t\) sufficiently near 1. Since \(\text{dev} \circ \tilde{c} = \gamma\) on \([t_N, 1]\) and \(\text{dev}|_{\beta_N \tilde{U}}\) is a homeomorphism,
\[
\tilde{\gamma}(t) := \begin{cases} 
\tilde{c}(t) & \text{for } t < 1 \\
(\text{dev}|_{\beta_N \tilde{U}})^{-1}(\gamma(1)) & \text{for } t = 1
\end{cases}
\]
is a continuous lift of \(\gamma\) to \([0, 1]\), as desired.

### 14.5. Higher dimensions: flat conformal and spherical CR-structures

These structures generalize to \((G, X)\)-structures where \(G\) is a semisimple Lie group and \(X = G/P\), where \(P \subset G\) is a parabolic subgroup. The simplest generalization occurs when \(G = \text{SO}(n+1, 1)\) and \(X = S^n\). The conformal automorphisms of \(S^n\) are just Möbius transformations. In this case \(X\) is the model space for **conformal (Euclidean) geometry** and a \((G, X)\)-structure is a **flat conformal structure**, that is, a conformal equivalence class of conformally flat Riemannian metrics.
A key point in this identification is the famous result of Liouville that in dimensions $> 2$, a conformal map from a nonempty connected domain in $S^n$ is the restriction of a unique Möbius transformation of $S^n$.

Furthermore this is the boundary structure for hyperbolic structures in dimension $n+1$, since $S^n = \partial (H)^{n+1}$ and the group of isometries of $H^{n+1}$ restricts to the group of conformal automorphisms of $S^n$.

The boundary structure for complex hyperbolic geometry is a spherical CR-structure, where $G = PU(n, 1)$ and $X = \partial H^n_\mathbb{C} \approx S^{2n-1}$. 
CHAPTER 15

Geometric structures on 3-manifolds

In this final chapter we collect a few results on Ehresmann structures on closed 3-manifolds. Of course, the relationship with Thurston geometrization is crucial, since the Thurston geometric structures provide a convenient starting point for understanding the more exotic Ehresmann structures. However, the theory is very much in its infancy and certain innocent-sounding questions seem (at least now) to be inaccessible.

The exception is the theory of complete affine structures on 3-manifolds. The classification of such structures on closed 3-manifolds has been understood since the early 1980’s, see Fried-Goldman [61, 54]. The considerably more interesting case of noncompact complete 3-manifolds has only been understood recently. The big breakthrough came in the early 1980’s with Margulis’s resolution [108] of Milnor’s question [115]; see Abels [1] and Goldman [63, 64] for expositions.

For possibly incomplete structures, much less is known. We then describe a few cases where one has definitive information, including the case of closed affine 3-manifolds with nilpotent holonomy and the beautiful classification of Serge Dupont [43] of affine structures on hyperbolic torus bundles.

Finally we discuss a few results concerning geometric structures on closed 3-manifolds, in particular $\mathbb{R}P^3$-structures, flat conformal structures, and spherical CR-structures.

15.1. Complete affine structures on 3-manifolds

Complete affine structures on 3-manifolds were classified by Fried-Goldman [54]. They are finitely covered by complete affine solvmanifolds (see §8.6.2 of Chapter 8) and thus relate to left-invariant affine structures on 3-dimensional Lie groups (see Chapter 10).

15.1.1. Complete affine 3-manifolds. The first step in the classification is the Milnor-Auslander question: Namely, if $M^3 = A^3/\Gamma$, show that $\Gamma$ is solvable. Let $A(\Gamma)$ denote the Zariski closure of $\Gamma$ in $\text{Aff}(A^3)$; clearly it suffices to show that $A(\Gamma)$ is solvable. This is equivalent to showing that the Zariski closure $A(L(\Gamma))$ in $GL(\mathbb{R}^3)$ of the linear
holonomy group $L(\Gamma) \subset GL(\mathbb{R}^3)$ is solvable. The proof is a case-by-case analysis of the possible Levi factors of $A(L(\Gamma))$.

The structures on Euclidean 3-manifolds are classified using 3-dimensional commutative associative algebras; see Chapter 10, Exercise 10.1.4.

15.1.2. Complete affine structures on noncompact 3-manifolds. In his 1977 paper [115], Milnor set the record straight caused by the confusion surrounding Auslander’s flawed proof of Conjecture 8.6.1. Influenced by Tits’s work and amenability, Milnor observed, that for an affine space $A$ of given dimension, the following conditions are all equivalent:

- Every discrete subgroup of $\text{Aff}(A)$ which acts properly on $A$ is amenable.
- Every discrete subgroup of $\text{Aff}(A)$ which acts properly on $A$ is virtually solvable.
- Every discrete subgroup of $\text{Aff}(A)$ which acts properly on $A$ is virtually polycyclic.
- A nonabelian free subgroup of $\text{Aff}(A)$ cannot act properly on $A$.
- The Euler characteristic $\chi(\Gamma \backslash A)$ (when defined) of a complete affine manifold $\Gamma \backslash A$ must vanish (unless $\Gamma = \{1\}$ of course).
- A complete affine manifold $\Gamma \backslash A$ has finitely generated fundamental group $\Gamma$.

(If these conditions were met, one would have a satisfying structure theory, similar to, but somewhat more involved, than the Bieberbach structure theory for flat Riemannian manifolds.)

In for this “conjecture”. For example, the infinitesimal version: Namely, let $G \subset \text{Aff}(A)$ be a connected Lie group which acts properly on $A$. Then $G$ must be an amenable Lie group, which simply means that it is a compact extension of a solvable Lie group. (Equivalently, its Levi subgroup is compact.) Furthermore, he provides a converse: Milnor shows that every virtually polycyclic group admits a proper affine action. (However, Milnor’s actions do not have compact quotient. Benoist found finitely generated nilpotent groups which admit no affine crystallographic action. Benoist’s examples are 11-dimensional.)

However convincing as his “evidence” is, Milnor still proposes a possible way of constructing counterexamples:

“Start with a free discrete subgroup of $O(2,1)$ and add translation components to obtain a group of affine transformations which acts freely. However it seems
difficult to decide whether the resulting group action is properly discontinuous."

This is clearly a geometric problem: As Schottky showed in 1907, free groups act properly by isometries on hyperbolic 3-space, and hence by diffeomorphisms of $A^3$. These actions are not affine.

One might try to construct a proper affine action of a free group by a construction like Schottky’s. Recall that a Schottky group of rank $g$ is defined by a system of $g$ open half-spaces $H_1, \ldots, H_g$ and isometries $A_1, \ldots, A_g$ such that the $2g$ half-spaces

$$H_1, \ldots, H_g, A_1(H_1^c), \ldots, A_g(H_g^c)$$

are all disjoint (where $H^c$ denotes the complement of the closure $\overline{H}$ of $H$). The slab

$$\text{Slab}_i := H_i^c \cap A_i(H_i)$$

is a fundamental domain for the action of the cyclic group $\langle A_i \rangle$. The ping-pong lemma then asserts that the intersection of all the slabs

$$\Delta := \text{Slab}_1 \cap \cdots \cap \text{Slab}_g$$

is a fundamental domain for the group $\Gamma := \langle A_1, \ldots, A_g \rangle$. Furthermore $\Gamma$ is freely generated by $A_1, \ldots, A_g$. The basic idea is the following. Let $B_i^+ := A_i(H_i^c)$ (respectively $B_i^- := H_i$) denote the attracting basin for $A_i$ (respectively $A_i^{-1}$). That is, $A_i$ maps all of $H_i^c$ to $B_i^+$ and $A_i^{-1}$ maps all of $A_i(H_i)$ to $B_i^-$. Let $w(a_1, \ldots, a_g)$ be a reduced word in abstract generators $a_1, \ldots, a_g$, with initial letter $a_i^\pm$. Then

$$w(A_1, \ldots, A_g)(\Delta) \subset B_i^{\pm}.$$

Since all the basins $B_i^{\pm}$ are disjoint, $w(A_1, \ldots, A_g)$ maps $\Delta$ off itself. Therefore $w(A_1, \ldots, A_g) \neq 1$.

Freely acting discrete cyclic groups of affine transformations have fundamental domains which are parallel slabs, that is, regions bounded by two parallel affine hyperplanes. One might try to combine such slabs to form “affine Schottky groups”, but immediately one sees this idea is doomed, if one uses parallel slabs for Schottky’s construction: parallel slabs have disjoint complements only if they are parallel to each other, in which case the group is necessarily cyclic anyway. From this viewpoint, a discrete group of affine transformations seems very unlikely to act properly.

### 15.2. Margulis spacetimes

In the early 1980’s Margulis, while trying to prove that a nonabelian free group can’t act properly by affine transformations, discovered that discrete free groups of affine transformations can indeed act properly!
Around the same time, David Fried and I were also working on these questions, and reduced Milnor’s question in dimension three to what seemed at the time to be one annoying case which we could not handle. Namely, we showed the following: Let $A$ be a three-dimensional affine space and $\Gamma \subset \text{Aff}(A)$. Suppose that $\Gamma$ acts properly on $A$. Then either $\Gamma$ is polycyclic or the restriction of the linear holonomy homomorphism $\Gamma \to \text{GL}(A)$ discretely embeds $\Gamma$ onto a subgroup of $\text{GL}(A)$ conjugate to the orthogonal group $O(2,1)$.

In particular the complete affine manifold $M^3 = \Gamma \backslash A$ is a complete flat Lorentz 3-manifold after one passes to a finite-index torsion-free subgroup of $\Gamma$ to ensure that $\Gamma$ acts freely. In particular the restriction $L|_{\Gamma}$ defines a free properly discrete isometric action of $\Gamma$ on the hyperbolic plane $H^2$ and the quotient $\Sigma^2 := H^2/L(\Gamma)$ is a complete hyperbolic surface with a homotopy equivalence $M^3 := \Gamma \backslash A \simeq H^2/L(\Gamma) =: \Sigma^2$.

Already this excludes the case when $M^3$ is compact, since $\Gamma$ is the fundamental group of a closed aspherical 3-manifold (and has cohomological dimension 3) and the fundamental group of a hyperbolic surface (and has cohomological dimension $\leq 2$). This is a crucial step in the proof of Conjecture 8.6.1 in dimension 3.

That the hyperbolic surface $\Sigma^2$ is noncompact is a much deeper result due to Geoffrey Mess Later proofs and a generalization have been found by Goldman-Margulis (Compare the discussion in §15.2.3.) Since the fundamental group of a noncompact surface is free, $\Gamma$ is a free group. Furthermore $L|_{\Gamma}$ embeds $\Gamma$ as a free discrete group of isometries of hyperbolic space. Thus Milnor’s suggestion is the only way to construct nonsolvable examples in dimension three.

15.2.1. Affine boosts and crooked planes. Since $L$ embeds $\Gamma_0$ as the fundamental group of a hyperbolic surface, $L(\gamma)$ is elliptic only if $\gamma = 1$. Thus, if $\gamma \neq 1$, then $L(\gamma)$ is either hyperbolic or parabolic. Furthermore $L(\gamma)$ is hyperbolic for most $\gamma \in \Gamma_0$.

When $L(\gamma)$ is hyperbolic, $\gamma$ is an affine boost, that is, it has the form

$$\gamma = \begin{bmatrix} e^{\ell(\gamma)} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-\ell(\gamma)} \end{bmatrix} \begin{bmatrix} 0 \\ \alpha(\gamma) \\ 0 \end{bmatrix}$$

(41)

in a suitable coordinate system. (Here the $3 \times 3$ matrix represents the linear part, and the column 3-vector represents the translational part.)
γ leaves invariant a unique (spacelike) line $C_γ$ (the second coordinate line in (41)). Its image in $E^{2,1}/Γ$ is a closed geodesic $C_γ/⟨γ⟩$. Just as for hyperbolic surfaces, most loops in $M^3$ are freely homotopic to such closed geodesics. (For a more direct relationship between the dynamics of the geodesic flows on $Σ^2$ and $M^3$, compare Goldman-Labourie.)

Margulis observed that $C_γ$ inherits a natural orientation and metric, arising from an orientation on $A$, as follows. Choose repelling and attracting eigenvectors $L(γ)^±$ for $L(γ)$ respectively; choose them so they lie in the same component of the nullcone. Then the orientation and metric on $C_γ$ is determined by a choice of nonzero vector $L(γ)^0$ spanning $Fix(L(γ))$. This vector is uniquely specified by requiring that:

- $L(γ)^0 · L(γ)^0 = 1$;
- $(L(γ)^0, L(γ)^-, L(γ)^+)$ is a positively oriented basis.

The restriction of $γ$ to $C_γ$ is a translation by displacement $α(γ)$ with respect to this natural orientation and metric.

Compare this to the more familiar geodesic length function $ℓ(γ)$ associated to a class $γ$ of closed curves on the hyperbolic surface $Σ$. The linear part $L(γ)$ acts by transvection along a geodesic $c_{L(γ)} \subset H^2$. The quantity $ℓ(γ) > 0$ measures how far $L(γ)$ moves points of $c_{L(γ)}$.

This pair of quantities

$$(ℓ(γ), α(γ)) \in \mathbb{R}_+ × \mathbb{R}$$

is a complete invariant of the isometry type of the flat Lorentz cylinder $A/⟨γ⟩$. The absolute value $|α(γ)|$ is the length of the unique primitive closed geodesic in $A/⟨γ⟩$.

A fundamental domain is the parallel slab

$$(Π_{C_γ})^{-1} (p_0 + [0, α(γ)] γ^0)$$

where

$${A} \xrightarrow{Π_{C_γ}} C_γ$$

denotes orthogonal projection onto

$$C_γ = p_0 + \mathbb{R}γ^0.$$  

As noted above, however, parallel slabs can’t be combined to form fundamental domains for Schottky groups, since their complementary half-spaces are rarely disjoint.

In retrospect this is believable, since these fundamental domains are fashioned from the dynamics of the translational part (using the projection $Π_{C_γ}$). While the effect of the translational part is properness, the dynamical behavior affecting most points is influenced by the linear part: While points on $C_γ$ are displaced by $γ$ at a polynomial rate, all other points move at an exponential rate.
Furthermore, parallel slabs are less robust than slabs in $H^2$: while small perturbations of one boundary component extend to fundamental domains, this is no longer true for parallel slabs. Thus one might look for other types of fundamental domains better adapted to the exponential growth dynamics given by the linear holonomy $L(\gamma)$.

Todd Drumm, in his 1990 Maryland thesis defined more flexible polyhedral surfaces, which can be combined to form fundamental domains for Schottky groups of 3-dimensional affine transformations. A crooked plane is a PL surface in $A$, separating $A$ into two crooked half-spaces. The complement of two disjoint crooked halfspace is a crooked slab, which forms a fundamental domain for a cyclic group generated by an affine boost. Drumm proved the remarkable theorem that if $S_1, \ldots, S_g$ are crooked slabs whose complements have disjoint interiors, then given any collection of affine boosts $\gamma_i$ with $S_i$ as fundamental domain, then the intersection $S_1 \cap \cdots \cap S_g$ is a fundamental domain for $\langle \gamma_1, \ldots, \gamma_g \rangle$ acting on all of $A$.

Modeling a crooked fundamental domain for $\Gamma$ acting on $A$ on a fundamental polygon for $\Gamma_0$ acting on $H^2$, Drumm proved the following sharp result:

**Theorem (Drumm).** Every noncocompact torsion-free Fuchsian group $\Gamma_0$ admits a proper affine deformation $\Gamma$ whose quotient is a solid handlebody.

**15.2.2. Marked length spectra.** We now combine the geodesic length function $\ell(\gamma)$ describing the geometry of the hyperbolic surface $\Sigma$ with the Margulis invariant $\alpha(\gamma)$ describing the Lorentzian geometry of the flat affine 3-manifold $M$.

As noted by Margulis, $\alpha(\gamma) = \alpha(\gamma^{-1})$, and more generally

$$\alpha(\gamma^n) = |n| \alpha(\gamma).$$

The invariant $\ell$ satisfies the same homogeneity condition, and therefore

$$\frac{\alpha(\gamma^n)}{\ell(\gamma^n)} = \frac{\alpha(\gamma)}{\ell(\gamma)}$$

is constant along hyperbolic cyclic subgroups. Hyperbolic cyclic subgroups correspond to periodic orbits of the geodesic flow $\phi$ on the unit tangent bundle $U\Sigma$. Periodic orbits, in turn, define $\phi$-invariant probability measures on $U\Sigma$. Goldman-Labourie-Margulis prove that, for any affine deformation, this function extends to a continuous function $Y_\Gamma$ on the space $C(\Sigma)$ of $\phi$-invariant probability measures on $U\Sigma$. Furthermore when $\Gamma_0$ is convex cocompact (that is, contains no parabolic elements), then the affine deformation $\Gamma$ acts properly if and only if $Y_\Gamma$
never vanishes. Since \( C(\Sigma) \) is connected, nonvanishing implies either all \( \Upsilon_\Gamma(\mu) > 0 \) or all \( \Upsilon_\Gamma(\mu) < 0 \). From this follows Margulis’s Opposite Sign Lemma, first proved in to groups with parabolics by Charette and Drumm.

**Theorem** (Margulis). If \( \Gamma \) acts properly, then all of the numbers \( \alpha(\gamma) \) have the same sign.

For an excellent treatment of the original proof of this fact, see the survey article of Abels.

### 15.2.3. Deformations of hyperbolic surfaces.

The Margulis invariant may be interpreted in terms of deformations of hyperbolic structures as follows.

Suppose \( \Gamma_0 \) is a Fuchsian group with quotient hyperbolic surface \( \Sigma_0 = \Gamma_0 \backslash \mathbb{H}^2 \). Let \( \mathfrak{g}_{\text{Ad}} \) be the \( \Gamma_0 \)-module defined by the adjoint representation applied to the embedding \( \Gamma_0 \hookrightarrow \text{O}(2,1) \). The coefficient module \( \mathfrak{g}_{\text{Ad}} \) corresponds to the Lie algebra of right-invariant vector fields on \( \text{O}(2,1) \) with the action of \( \text{O}(2,1) \) by left-multiplication. Geometrically these vector fields correspond to the infinitesimal isometries of \( \mathbb{H}^2 \).

A family of hyperbolic surfaces \( \Sigma_t \) smoothly varying with respect to a parameter \( t \) determines an infinitesimal deformation, which is a cohomology class \( [u] \in H^1(\Gamma_0, \mathfrak{g}_{\text{Ad}}) \). The cohomology group \( H^1(\Gamma_0, \mathfrak{g}_{\text{Ad}}) \) corresponds to infinitesimal deformations of the hyperbolic surface \( \Sigma_0 \). In particular the tangent vector to the path \( \Sigma_t \) of marked hyperbolic structures smoothly varying with respect to a parameter \( t \) defines a cohomology class

\[
[u] \in H^1(\Gamma_0, \mathfrak{g}_{\text{Ad}}).
\]

The same cohomology group parametrizes affine deformations. The translational part \( u \) of a linear representations of \( \Gamma_0 \) is a cocycle of the group \( \Gamma_0 \) taking values in the corresponding \( \Gamma_0 \)-module \( \mathfrak{V} \). Moreover two cocycles define affine deformations which are conjugate by a translation if and only if their translational parts are cohomologous cocycles. Therefore translational conjugacy classes of affine deformations form the cohomology group \( H^1(\Gamma_0, \mathfrak{V}) \). Inside \( H^1(\Gamma_0, \mathfrak{V}) \) is the subset proper corresponding to proper affine deformations.

The adjoint representation \( \text{Ad} \) of \( \text{O}(2,1) \) identifies with the orthogonal representation of \( \text{O}(2,1) \) on \( \mathfrak{V} \). Therefore the cohomology group \( H^1(\Gamma_0, \mathfrak{V}) \) consisting of translational conjugacy classes of affine deformations of \( \Gamma_0 \) can be identified with the cohomology group \( H^1(\Gamma_0, \mathfrak{g}_{\text{Ad}}) \) corresponding to infinitesimal deformations of \( \Sigma_0 \).
THEOREM. Suppose \( u \in Z^1(\Gamma_0, g_{\text{Ad}}) \) defines an infinitesimal deformation tangent to a smooth deformation \( \Sigma_t \) of \( \Sigma \).

- The marked length spectrum \( \ell_t \) of \( \Sigma_t \) varies smoothly with \( t \).
- Margulis’s invariant \( \alpha_u(\gamma) \) represents the derivative
  \[
  \frac{d}{dt} \bigg|_{t=0} \ell_t(\gamma)
  \]

- (Opposite Sign Lemma) If \( [u] \in \text{Proper} \), then all closed geodesics lengthen (or shorten) under the deformation \( \Sigma_t \).

Since closed hyperbolic surfaces do not support deformations in which every closed geodesic shortens, such deformations only exist when \( \Sigma_0 \) is noncompact. This leads to a new proof Mess’s theorem that \( \Sigma_0 \) is not compact. (For another, somewhat similar proof, which generalizes to higher dimensions, see Labourie)

The tangent bundle \( TG \) of any Lie group \( G \) has a natural structure as a Lie group, where the fibration \( TG \xrightarrow{\Pi} G \) is a homomorphism of Lie groups, and the tangent spaces

\[
T_xG = \Pi^{-1}(x) \subset TG
\]

are vector groups. The deformations of a representation \( \Gamma_0 \xrightarrow{\rho_0} G \) correspond to representations \( \Gamma_0 \xrightarrow{\rho} TG \) such that \( \Pi \circ \rho = \rho_0 \). In our case, affine deformations of \( \Gamma_0 \hookrightarrow \text{O}(2, 1) \) correspond to representations in the tangent bundle \( T\text{O}(2, 1) \). When \( G \) is the group \( G(\mathbb{R}) \) of \( \mathbb{R} \)-points of an algebraic group \( G \) defined over \( \mathbb{R} \), then

\[
TG \cong G(\mathbb{R}[\epsilon])
\]

where \( \epsilon \) is an indeterminate with \( \epsilon^2 = 0 \). (Compare This is reminiscent of the classical theory of quasi-Fuchsian deformations of Fuchsian groups, where one deforms a Fuchsian subgroup of \( \text{SL}(2, \mathbb{R}) \) in

\[
\text{SL}(2, \mathbb{C}) = \text{SL}(2, \mathbb{R}[i])
\]

where \( i^2 = -1 \).

15.2.4. Classification. In light of Drumm’s theorem, classifying Margulis spacetimes \( M^3 \) begins with the classification of hyperbolic structures \( \Sigma^2 \). Thus the deformation space of Margulis spacetimes maps to the Fricke space \( \mathfrak{F}(\Sigma) \) of marked hyperbolic structures on the underlying topology of \( \Sigma \).

The main result of is that the positivity (or negativity) of \( \Upsilon_\Gamma \) on on \( \mathfrak{C}(\Sigma) \) is necessary and sufficient for properness of \( \Gamma \). (For simplicity we restrict ourselves to the case when \( L(\Gamma) \) contains no parabolics — that is, when \( \Gamma_0 \) is convex cocompact.) Thus the proper affine deformation
space \( \text{Proper} \) identifies with the open convex cone in \( H^1(\Gamma_0, V) \) defined by the linear functionals \( \Upsilon_\mu \), for \( \mu \) in the compact space \( \mathcal{C}(\Sigma) \). These give uncountably many linear conditions on \( H^1(\Gamma_0, V) \), one for each \( \mu \in \mathcal{C}(\Sigma) \). Since the invariant probability measures arising from periodic orbits are dense in \( \mathcal{C}(\Sigma) \), the cone \( \text{Proper} \) is the interior of half-spaces defined by the countable set of functional \( \Upsilon_\gamma \), where \( \gamma \in \Gamma_0 \).

The zero level sets \( \Upsilon_\gamma^{-1}(0) \) correspond to affine deformations where \( \gamma \) does not act freely. Therefore \( \text{Proper} \) defines a component of the subset of \( H^1(\Gamma_0, V) \) corresponding to affine deformations which are free actions.

Actually, one may go further. An argument inspired by Thurston those measures arising from multi-curves, that is, unions of disjoint simple closed curves. These measures (after scaling) are dense in the Thurston cone \( \text{ML}(\Sigma) \) of measured geodesic laminations on \( \Sigma \). One sees the combinatorial structure of the Thurston cone replicated on the boundary of \( \text{Proper} \subset H^1(\Gamma_0, V) \).

Two particular cases are notable. When \( \Sigma \) is a 3-holed sphere or a 2-holed cross-surface (real projective plane), then the Thurston cone degenerates to a finite-sided polyhedral cone. In particular properness is characterized by 3 Margulis functionals for the 3-holed sphere, and 4 for the 2-holed cross-surface. Thus the deformation space of equivalence classes of proper affine deformations is either a cone on a triangle or a convex quadrilateral, respectively.

When \( \Sigma \) is a 3-holed sphere, these functionals correspond to the three components of \( \partial \Sigma \). The halfspaces defined by the corresponding three Margulis functionals cut off the deformation space (which is a polyhedral cone with 3 faces). The Margulis functionals for the other curves define halfspaces which strictly contain this cone.

When \( \Sigma \) is a 2-holed cross-surface these functionals correspond to the two components of \( \partial \Sigma \) and the two orientation-reversing simple closed curves in the interior of \( \Sigma \). The four Margulis functionals describe a polyhedral cone with 4 faces. All other closed curves on \( \Sigma \) define halfspace strictly containing this cone.

In both cases, an ideal triangulation for \( \Sigma \) models a crooked fundamental domain for \( M \), and \( \Gamma \) is an affine Schottky group, and \( M \) is an open solid handlebody of genus 2 (Charette-Drumm-Goldman depicts these finite-sided deformation spaces.

For the other surfaces where \( \pi_1(\Sigma) \) is free of rank two (equivalently \( \chi(\Sigma) = -1 \)), infinitely many functionals \( \Upsilon_\mu \) are needed to define the deformation space, which necessarily has infinitely many sides. In these cases \( M^3 \) admits crooked fundamental domains corresponding to ideal triangulations of \( \Sigma \), although unlike the preceding cases there is no
single ideal triangulation which works for all proper affine deformations. Once again $M^3$ is a genus two handlebody.

15.3. Nilpotent holonomy

15.4. Dupont’s classification of hyperbolic torus bundles

This builds on Dupont’s classification of affine actions of the two-dimensional group $\text{Aff}(\mathbb{R})$ on $\mathbb{A}^3$, see Dupont [42].

15.5. Examples of projective three-manifolds
APPENDIX A

Transformation groups

Suppose that \( G \) is a group acting on a space \( X \). Denote the action by:

\[
G \times X \xrightarrow{\alpha} X
\]

\((g, x) \mapsto g \cdot x\)

Then the kernel of the action \( \alpha \) consists of all \( g \) such that \( \alpha(g, \cdot) = \mathbb{I} \), that is, \( g \cdot x = x, \forall x \in X \). The action is effective (or faithful if its kernel is trivial).

If \( x \in X \), then its stabilizer is the subgroup

\[
\text{Stab}(x) := \{g \in G \mid g \cdot x = x\}.
\]

The action is free if and only if \( \text{Stab}(x) = 1, \forall x \in X \).

We must relate the actions of \( \text{Aff}(A) \) on \( \mathcal{C}(A) \) and \( G \) on \( \mathcal{C}(P) \). Recall that a topological transformation groupoid consists of a small category \( \mathfrak{G} \) whose objects form a topological space \( X \) upon which a topological group \( G \) acts such that the morphisms \( a \to b \) consist of all \( g \in G \) such that \( g(a) = b \). We write \( \mathfrak{G} = (G, X) \). A homomorphism of topological transformation groupoids is a functor

\[
(X, G) \xrightarrow{(f, F)} (X', G')
\]

arising from a continuous map \( X \xrightarrow{f} X' \) which is equivariant with respect to a continuous homomorphism \( G \xrightarrow{F} G' \).

The space of isomorphism classes of objects in a category \( \mathfrak{G} \) will be denoted \( \text{Iso}(\mathfrak{G}) \). We shall say that \( \mathfrak{G} \) is proper (respectively syndetic) if the corresponding action of \( G \) on \( X \) is proper (respectively syndetic). If \( \mathfrak{G} \) and \( \mathfrak{G}' \) are topological categories, a functor \( \mathfrak{G} \xrightarrow{F} \mathfrak{G}' \) is an equivalence of topological categories if the induced map

\[
\text{Iso}(\mathfrak{G}) \xrightarrow{\text{Iso}(F)} \text{Iso}(\mathfrak{G}')
\]

is a homeomorphism and \( F \) is fully faithful, that is, for each pair of objects \( a, b \) of \( \mathfrak{G} \), the induced map

\[
\text{Hom}(a, b) \xrightarrow{F_*} \text{Hom}(F(a), F(b))
\]
is a homeomorphism. If $F$ is fully faithful it is enough to prove that $\text{Iso}(F)$ is surjective. (Compare Jacobson [78].) We have the following general proposition:

**Lemma A.0.1.** Suppose that

$$(X, G) \xrightarrow{(f,F)} (X', G')$$

is a homomorphism of topological transformation groupoids which is an equivalence of groupoids and such that $f$ is an open map.

- If $(X, G)$ is proper, so is $(X', G')$.
- If $(X, G)$ is syndetic, so is $(X', G')$.

**Proof.** An equivalence of topological groupoids induces a homeomorphism of quotient spaces

$$X/G \rightarrow X'/G'$$

so if $X'/G'$ is compact (respectively Hausdorff) so is $X/G$. Since $(X, G)$ is syndetic if and only if $X/G$ is compact, this proves the assertion about syndeticity. By Koszul [90], p.3, Remark 2, $(X, G)$ is proper if and only if $X/G$ is Hausdorff and the action $(X, G)$ is wandering (or locally proper): each point $x \in X$ has a neighborhood $U$ such that $G(U, U) = \{ g \in G | g(U) \cap U \neq \emptyset \}$ is precompact. Since $(f, F)$ is fully faithful, $F$ maps $G(U, U)$ isomorphically onto $G'(f(U), f(U))$. Suppose that $(X, G)$ is proper. Then $X/G$ is Hausdorff and so is $X'/G'$. We claim that $G'$ acts locally properly on $X'$. Let $x' \in X'$. Then there exists $g' \in G'$ and $x \in X$ such that $g'f(x) = x'$. Since $G$ acts locally properly on $X$, there exists a neighborhood $U$ of $x \in X$ such that $G(U, U)$ is precompact. It follows that $U' = g'f(U)$ is a neighborhood of $x' \in X'$ such that $G'(U', U') \cong G(U, U)$ is precompact, as claimed. Thus $G'$ acts properly on $X'$. \qed
APPENDIX B

Tensor analysis

A vector in $\mathcal{V}$ corresponds to a column vector:

$$v^i \partial_i \longleftrightarrow \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix}$$

and a covector in $\mathcal{V}^*$ corresponds to a row vector:

$$\omega_i dx^i \longleftrightarrow \begin{bmatrix} \omega_1 & \ldots & \omega_n \end{bmatrix}$$

Affine vector fields on $A$ correspond to affine maps $A \to A$:

$$A := (A^i_j x^j + a^i) \partial_i \longleftrightarrow \hat{A} := [A \mid a]$$

where

$$A = \begin{bmatrix} A^1_1 & \ldots & A^1_i & \ldots & A^1_n \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
A^i_1 & \ldots & A^i_j & \ldots & A^i_n \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
A^n_1 & \ldots & A^n_j & \ldots & A^n_n \end{bmatrix}$$

is the linear part and and

$$a = \begin{bmatrix} a^1 \\
\vdots \\
a^i \\
\vdots \\
a^n \end{bmatrix}$$

is the translational part. In this notation,

$$(42) \quad \hat{A} = [A \mid a] = \begin{bmatrix} A^1_1 & \ldots & A^1_i & \ldots & A^1_n & a^1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\ldots & A^i_j & \ldots & a^i & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
A^n_1 & \ldots & A^n_j & \ldots & A^n_n & a^n \end{bmatrix}$$
Under this correspondence, covariant derivative corresponds to composition of affine maps (matrix multiplication):

\[ \nabla_B A \leftrightarrow \hat{A}\hat{B} \]
5. _____, *Flat Lorentz 3-manifolds*, Mem. Amer. Math. Soc. No. 30 (1959), 60. MR 0131842 (24 #A1689)
8. Oliver Baues, *Varieties of discontinuous groups*, in Igodt et al. [77], pp. 147–158. MR 1796130 (2001i:58013)

19. ____, *Sur les variétés localement affines et localement projectives*, Bulletin de la Société Mathématique de France 88 (1960), 229–332. 6.2.3, 6.2.4, 6.2.5, 8.5


42. Serge Dupont, *Variétés projectives à holonomie dans le groupe Aff*[^a](*R*), in Igodt et al. [77], pp. 177–193. MR 1796133 12.3.2.1, 15.4

43. , *Solvariétés projectives de dimension trois*, Geom. Dedicata **96** (2003), 55–89. MR 1956834 12.3.2.1, 15


45. , *Les connexions infinitésimales dans un espace fibré différentiable*, in Thone [139], pp. 29–55. MR 0042768 7.2


49. David Fried, *Radiant flows without cross-sections*, Preprint. 6.3.2


57. ______, *Discontinuous groups and the Euler class*, Ph.D. thesis, University of California, Berkeley, 1980. 5.4, 9.1
82. Shoshichi Kobayashi, *Transformation groups in differential geometry*, Springer-Verlag, New York, 1972, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 70. MR 0355886 (50 #8360)
85. ______, *Projectively invariant distances for affine and projective structures*, Banach Center Publ., vol. 12, PWN, Warsaw, 1984. MR 961077 (89k:53043 (document), 3.3, 3.3.2, 3.3.4, 6.1.4)
91. Variétés localement plates et convexité, Osaka J. Math. 2 (1965), 285–290. MR 0196662 (33 #4849) 8.2.3, 12.3
95. Locally projective spaces of dimension one., The Michigan Mathematical Journal 2 (1953), no. 2, 95–97. 5.4
101. Lectures on representations of surface groups, Zürich Lectures in Advanced Mathematics, European Mathematical Society (EMS), Zürich, 2013. MR 3155540 7.3
BIBLIOGRAPHY 221


113. _______, *Curvatures of left invariant metrics on Lie groups*, Advances in Math. 21 (1976), no. 3, 293–329. MR 0425012 10.2


115. _______, *On fundamental groups of complete affinely flat manifolds*, Advances in Mathematics 25 (1977), no. 2, 178–187. 8.6, 15, 15.1.2


132. *Flat manifolds with non-zero Euler characteristics*, Commentarii Mathematici Helvetici 52 (1977), no. 1, 453–455. 9.2.2

133. *An obstruction to the existence of affine structures*, Inventiones mathematicae 64 (1981), no. 3, 411–415. 11.2.1, 11.2.2


137. *A generalization of Milnor’s inequality concerning affine foliations and affine manifolds*, Commentarii Mathematici Helvetici 51 (1976), no. 1, 183–189. 9.1, 9.1.3


List of Figures

1  Non-Euclidean tesselations by equilateral triangles 35

2  The inside of a properly convex domain admits a projectively invariant distance defined in terms of cross-ratio. This is called the *Hilbert distance*. When the domain is the interior of a conic, then this distance is a Riemannian metric of constant negative curvature. This is the *Klein-Beltrami* projective model of the hyperbolic plane. 47

3  Projective model of a $(3,3,4)$-triangle tesselation of $H^2$ 53

4  Projective deformation of hyperbolic $(3,3,4)$-triangle tesselation 53

5  Tilings corresponding to some complete affine structures on the 2-torus. The first depicts a square Euclidean torus. The second and third pictures depicts non-Riemannian deformations where the holonomy group contains horizontal translations. 121

6  Tilings corresponding to some complete affine structures on the 2-torus. The second picture depicts a complete non-Riemannian deformation where the affine holonomy contains no nontrivial horizontal translation. The corresponding torus contains no closed geodesics. 122

7  Decomposing a genus $g = 2$ surface along $2g$ curves into a $4g$-gon. The single common intersection of the curves is a single point which decomposes into the $4g$ vertices of the polygon. 135

8  Identifying the edges of a $4g$-gon into a closed surface of genus $g$. The sides are paired into $2g$ curves, which meet at the single vertex. 135

9  Some incomplete complex-affine structures on $T^2$ 155

10 Some hyperbolic affine structures on $T^2$ 156

11 Radiant affine structures on $T^2$ developing to a halfplane 157

12 Nonradiant affine structures on $T^2$ developing to a halfplane 158
13 Pathological development for $\mathbb{RP}^2$-torus 186
14 A convex domain tiled by triangles 187
15 Conics tangent to a triangle 191
16 Deforming a conic 191
17 A piecewise conic 192
18 Bulging data 192
19 The deformed conic 192
20 The conic with its deformation 193
List of Tables

<table>
<thead>
<tr>
<th>Table</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Euclidean</td>
<td>159</td>
</tr>
<tr>
<td>2</td>
<td>Complete Non-Riemannian structure</td>
<td>159</td>
</tr>
<tr>
<td>3</td>
<td>Radiant structure on halfplane</td>
<td>159</td>
</tr>
<tr>
<td>4</td>
<td>Nonradiant halfplane</td>
<td>159</td>
</tr>
<tr>
<td>5</td>
<td>Hyperbolic affine structure on $\mathbb{R}^2$</td>
<td>159</td>
</tr>
<tr>
<td>6</td>
<td>Right-invariant vector fields on Aff$(1, \mathbb{R})$</td>
<td>161</td>
</tr>
<tr>
<td>7</td>
<td>Left-invariant vector fields on Aff$(1, \mathbb{R})$</td>
<td>161</td>
</tr>
<tr>
<td>8</td>
<td>Left-invariant vector fields</td>
<td>162</td>
</tr>
<tr>
<td>9</td>
<td>Right-invariant vector fields</td>
<td>162</td>
</tr>
<tr>
<td>10</td>
<td>2-dimensional nonabelian algebra corresponding to complete structure</td>
<td>163</td>
</tr>
<tr>
<td>11</td>
<td>Algebra corresponding to incomplete flat Lorentzian structure</td>
<td>164</td>
</tr>
<tr>
<td>12</td>
<td>Affine structure on parabolic halfplane</td>
<td>165</td>
</tr>
<tr>
<td>13</td>
<td>Algebra corresponding to parabolic 3-dimensional halfspaces</td>
<td>166</td>
</tr>
<tr>
<td>14</td>
<td>A radiant suspension</td>
<td>167</td>
</tr>
<tr>
<td>15</td>
<td>Nonradiant Deformation</td>
<td>168</td>
</tr>
</tbody>
</table>
Index

1-manifolds
  affine, 88
  Euclidean, 87
  projective, 87, 88

action
  free, 15
  proper, 212
  simply transitive, 15
    locally, 148
  syndetic, 212
  transitive, 15
  wandering, 212

affine, 13
  connection, 111
  geometry, 17
  map, 16
  patch, 32
  structure
    bi-invariant, 148
    left-invariant, 147
    on Lie algebra, 150
  subspace, 24
  vector field, 21

affine parameter, 24

algebra
  associative, 148
  commutative associative
    2-dimensional, 152
    3-dimensional, 160
  Koszul-Vinberg, 149
  left-symmetric, 149

algebraicization
  of classical geometries, 6
  of geometric structures, 93

Auslander-Milnor question, 125
  in dimension 3, 201

Benzécri’s theorem
  compactness, 65
  on surfaces, 133

Bieberbach theorems, 126
  boosts
    affine, 204
  Cartesian product, 125
  centroid, 25
  characteristic function, 56
  Chern-Weil theory, 139
  collineation, 29
  complete affine structures, 125
    on solvmanifolds, 127
    on the two-torus, 119
  completeness
    geodesic, 117
  complex affine structure, 99
  cone
    dual, 56
    properly convex, 55
  connections
    affine, 115
    geodesically complete, 116
  convex, 47
    properly, 47
  convex body, 25
  convex cone, 55
  crooked planes, 204
  cross-ratio, 36
  cross-section to flow, 101
  developing map, 79
    tameness of, 113
  ellipsoid, 68
    of inertia, 68

229
embedding of geometries, 96
equivalence of categories, 211
Erlanger Program, 1
Euler class, 141

fibration of geometries, 97
flat tori, 88
moduli of, 108
functor
fully faithful, 211

geodesic completeness, 112
geometric atlas, 76
geometric structure transverse to foliation, 101
geometry
affine, 1, 17
Euclidean, 1
extending, 83
non-Euclidean, 2
projective, 1
similarity, 1
grafting
in dimension one, 91

harmonic
homology, 34
set, 34
Hessian metrics, 183
Hilbert metric, 48
holonomy representation, 79
holonomy sequence, 175, 179
homogeneous coordinates, 28
homothety, 23
Hopf manifold, 88
Hopf manifolds, 98
goedetics on , 124
in dimension one, 88
Hopf-Rinow theorem, 114
hull
crystallographic, 127

invisible, 117
Jacobson product, 149

Kobayashi metric, 50
Kostant-Sullivan theorem, 144

Locally homogeneous Riemannian manifolds, 113
Margulis spacetime, 201
Margulis superrigidity, 174
marking, 107
Markus conjecture, 169
maximal ball, 177
Milnor-Wood inequality, 139
moment of inertia, 68
nonradiant deformation, 166
normality domain, 196
parabolic cylinders, 164
parabolic halfplane, 164
parallel structures, 20
parameter
affine, 24
point of normality, 196
projection, 31
projective reflection, 31
quotient stucture, 113
Riemann moduli space, 107
Riemannian structures
parallel, 20
scale factor homomorphism, 20, 176
Seifert 3-manifold, 172
similarity manifolds, 175
complete, 175
radiant, 95
similarity transformations, 20
singular projective transformation, 38
range, 38
undefined set, 38
Smale-Hirsch immersion theorem, 123
stably parallelizable manifolds, 143
stably trivial vector bundle, 143
suspension
parallel, 101
radiant, 104
syndetic, 65
topological transformation groupoids, 211
tori
  flat, 88
torsor, 16
tube domain, 58
turning number, 136

Unique Extension Property, 29, 77

vector field
  affine, 21
  parallel, 23
  radiant, 23
visible, 117
volume obstruction, 169

Whitney-Graustein theorem, 136