1. Let $k$ be an algebraic closure of $\mathbb{F}_q$. Let $G = \text{sl}_2(k)$ and $\mathbb{B}$ be the subgroup of upper triangle matrices in $SL_2(k)$. Let $\epsilon : \mathbb{B} \to k^\times$ be the character defined by $\epsilon \left( \begin{array}{cc} a & b \\ 0 & a^{-1} \end{array} \right) = a$. For $n \in \mathbb{N}$, set 
\[ \Delta'(n) = \{ f \in k[G]; f(gb) = \epsilon(b)^{-n} f(g) \text{ for all } g \in G, b \in \mathbb{B} \}. \]
Prove that $\Delta'(n) = \text{Sym}^n(V)$ as $G$-modules, where $V$ is the tautological representation of $G$.

2. Let $k$ be an algebraic closure of $\mathbb{F}_q$. Let $G = \text{sl}_3(k)$. Let $Fr : G \to G$ be the standard Frobenius morphism defined by $(a_{ij}) \mapsto (a_{ij}^q)$ and $Fr' : G \to G$ be the Frobenius morphism defined by $(a_{ij}) \mapsto t(a_{ij}^q)^{-1}$. Compute the order of the finite groups $G^{Fr}$ and $G^{Fr'}$.

3. Again let $k$ be an algebraic closure of $\mathbb{F}_q$. Let $G = \text{sl}_n(k)$ and let $Fr, Fr'$ be two different Frobenius morphisms defined as in Problem 2.
   (1) Compute the number of $G^{Fr}$-conjugacy classes of rational maximal tori.
   (2) Compute the number of $G^{Fr'}$-conjugacy classes of rational maximal tori.

4. Let $G$ be the group of type $G_2$ over $k$ and $G^{Fr} = G_2(\mathbb{F}_q)$. The Weyl group of $G_2$ is generated by $s_1$ and $s_2$ with relations $s_1^2 = s_2^2 = (s_1 s_2)^6 = 1$ and the action on $W$ induced by Frobenius is trivial. Compute the number of $G^{Fr}$-conjugacy classes of rational maximal tori.

5. Let $(L, \delta)$ and $(M, \eta)$ be two cuspidal pairs. Prove that 
\[ < R^G_L \delta, R^G_M \eta > = |\{ x \in G^{Fr}; xLx^{-1} = M, \text{Ad}(x)\delta = \eta \}|/|L^{Fr}|. \]

6. Let $k$ be an algebraically closed field and $G = SL_2(k)$. Let $s$ be the nonidentity element in the Weyl group $S_2$. Consider the multiplication map $f : BsB \times_B BsB \to G$. Prove that there exists a decomposition $BsB \times_B BsB = X_1 \sqcup X_2$, where $X_1$ is closed and $X_2$ is open and $f |_{X_1} : X_1 \to B$ is a line bundle and $f |_{X_2} : X_2 \to BsB$ is a line bundle with zero section removed.

7. Compute the number of geometric conjugacy classes of $(T, \theta)$ for $SL_2(\mathbb{F}_q)$. What are the Lusztig’s series associated to each geometric conjugacy class?