1. Let $k$ be a field. An element $g \in GL_n(k)$ is called semisimple if it is diagonalizable over an algebraically closure of $k$ and is called unipotent if $(g - Id)^n = 0$ for some $n \in \mathbb{N}$.

Let $k = \mathbb{F}_q$ and $g \in GL_n(k)$. Prove that
(a) $g$ is semisimple if and only if the order of $g$ is coprime to $p$.
(b) $g$ is unipotent if and only if the order of $g$ is a power of $p$.

Solution: (a) Each eigenvalue is an element $\lambda$ in $\mathbb{F}_q^\times = \bigcup_i \mathbb{F}_p^i$. Therefore $\lambda^p - 1 = 1$ for some $i$. If $g$ is semisimple, then the order of $g$ is coprime to $p$.

On the other hand, if $g^n = 1$ for some $n$ with $(n,p) = 1$, then the minimal polynomial of $g$ divides $x^n - 1$. Since $x^n - 1$ has distinct roots over $\mathbb{F}_q$, the minimal polynomial of $g$ has distinct roots. Thus $g$ is semisimple.

(b) If $g$ is unipotent, then $(g - 1)^N = 0$ for some $N$. Choose $i$ such that $p^i > N$, then $(g - 1)^{p^i} = g^{p^i} - 1 = 0$. Thus the order of $g$ divides $p^i$.

If $g^{p^i} = 1$, then $(g - 1)^{p^i} = 0$ and $g$ is unipotent.

2. Show that the number of unipotent elements in $SL_2(\mathbb{F}_q)$ is $q^2$.

Solution: There are three unipotent conjugacy classes: 1, the conjugacy class of \[
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}
\] and the conjugacy class of \[
\begin{pmatrix}
1 & a \\
0 & 1
\end{pmatrix},
\] where $a \in \mathbb{F}_q - \mathbb{F}_q^2$. The first one has 1 element, and the others have $q^2 - 1$ elements each (this can be computed, for example, by checking the order of the centralizer).

3. Let $G = SL_2(\mathbb{F}_q)$ and $St$ be the Steinberg character of $G$. Prove that

$St(g) = \begin{cases} 
\pm |Z_G(g)|_p, & \text{if } g \text{ is semisimple}; \\
0, & \text{otherwise}.
\end{cases}$

Here $| - |_p$ denotes the largest power of $p$ which divides $-$. 

Solution: $R(1)(g) = \sharp(G/B)^g$, where $G/B \cong \mathbb{P}^1(\mathbb{F}_q)$. If $g \in Z$ is central, then $\sharp(G/B)^g = q + 1 = |G|_p + 1$. If $g \in T - Z$, then there are exactly two fixed points in $\mathbb{P}^1$: 0 and $\infty$. If $g = \begin{pmatrix} \epsilon & a \\ 0 & \epsilon \end{pmatrix}$ with $\epsilon = \pm 1$ and $a \neq 1$, then there is only one fixed point. If $g \in T' - Z$, then $g$ is not in any rational Borel subgroup and hence there is no fixed point.

4. Prove that $SL_n(k)$ and $GL_n(k)$ are algebraic sets.

Solution: $SL_n(k) = \{ (a_{ij}) \in k^{n^2}; \det = 1 \}$ and $GL_n(k) = \{ (a_{ij}, x) \in k^{n^2+1}; x \det = 1 \}$.

5. Show that $XY^q - YX^q - 1 \in \mathbb{F}_q[X,Y]$ is irreducible. Deduce that the Drinfeld curve is irreducible.

Solution: Change of variable $(z, t) = (x/y, 1/y)$. It remains to show that $t^{q+1} - z^q - z$ is irreducible. This follows from Einstein’s criterion.
6. Show the product topology of $\mathbb{A}^1 \times \mathbb{A}^1$ does not coincide with the Zariski topology of $\mathbb{A}^1 \times \mathbb{A}^1 \cong \mathbb{A}^2$.

Solution: Diagonal is closed in Zariski topology, but not in the product topology.

7. Let $Z_1 = \{(x, y, z, t) \in \mathbb{A}^4; xy^q - yx^q = 1, zt^q - tz^q = 1, xt \neq yz\}$ be the product to Drinfeld curves and $V = \{(u, a, b) \in \mathbb{A}^3; u \neq 0, u^{q+1} - ab = 1\}$. Prove that the morphism $(x, y, z, t) \mapsto (xt - yz, xt^q - yz^q, x^qt - y^qt)$ induces an isomorphism $Z_1/SL_2(\mathbb{F}_q) \cong V$.

See Bonnafé’s book, page 41-43.