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Affine structures on surfaces and the twisted cubic cone

William M. Goldman

Abstract. We identify the deformation space of marked complete affine strctures on the 2-torus $\mathbb{T}^2$ with the cone over a twisted cubic curve in $\mathbb{R}\mathsf{P}^3$.

Dedicated to Ravi S. Kulkarni, with admiration

Contents

Introduction

An affine structure on an n-dimensional manifold is defined by a system of local corrdinates where the coordinate changes are locally defined by affine transformations of $\mathbb{R}^n$. In this way the manifold is locally modeled on an affine space A (corresponding to $\mathbb{R}^n$, but without the algebraic structure of a vector space). Equivalently, an affine structure on a manifold is a flat torsionfree affine connection. In [**12**], Kuiper described the (geodesically) complete affine structures on $\mathbb{T}^2$ and showed they are of two types: either Eucildean structures (*flat Riemannian structures*), or other structures defined by flat but *non-Riemannian* connections. Figures 1 and 2 depict these two types of structures. In this paper we show that they naturally form a *twisted cubic cone* in $\mathbb{R}^4$.

In a more geometric context, a *complete affine structure* on a manifold M is a representation of M as a quotient $\Gamma\backslash\mathsf{A}$, where A is an affine space, and $\Gamma < \mathsf{Aff}(\mathsf{A})$ a discrete group of affine transformations acting properly on A. More generally, Kuiper classifies affine manifolds covered by *convex domains* in A^2. He shows that if a closed surface admits such a structure then $\chi(M) = 0$, and conjectures this is true without assuming convexity. This was later proved by Benzécri [**7**].

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If Σ is a fixed surface, a *marked affine structure* on Σ is a pair (f, M) where M is an affine manifold and f is diffeomorphism $\Sigma \to M$. Marked affine structures (f, M) and (f', M') are *equivalent* if they differ (up to isotopy) by an affine isomorphism $M \xrightarrow{\phi} M'$ of affine manifolds such that f is isotopic to $f' \circ \phi$. The space of equivalence classes of marked structures is called the *deformation space.* It has a natural topology (see [**10**], §6 Classification) which in general can be quite pathological. However, in many cases it is locally homeomorphic to the "finite-dimensional" space $\mathsf{Hom}\big(\pi_1(\Sigma), G\big)/G$ where $G = \mathsf{Aff}(\mathsf{A})$.[1] This space is analogous to the Teichmüller space, consisting of equivalence classes of marked Riemann surfaces. The mapping class group $\mathsf{Mod}(\Sigma)$ acts simply transitively on the markings, and its quotient is the Riemann moduli space of Riemann surfaces homeomorphic to Σ.

The classification of affine structures on surfaces was completed by Nagano-Yagi [**13**] and Arrowsmith-Furness [**1**]. In particular the deformation space of *all* affine structures on $\mathbb{T}^2$ is not Hausdorff. In this paper we discuss the subspace $\mathcal{CA}(\mathbb{T}^2)$ corresponding to *complete* structures. Baues [**2, 3, 4**] showed that $\mathcal{CA}(\mathbb{T}^2)$ is homeomorphic[2] to $\mathbb{R}^2$.

A parametrization of this space by a "period map" is given in [**6**], and raises the question whether this space has a natural *singular* smooth structure. This paper resolves this question.

The deformation space of *marked* complete structures admits a natural action of the *mapping class group* $\mathsf{Mod}(\mathbb{T}^2) \cong \mathsf{GL}(2, \mathbb{Z})$, permuting the markings. Baues observed that this is the natural *linear* action of $\mathsf{GL}(2, \mathbb{Z})$ on $\mathbb{R}^2$. This action is chaotic, and the quotient $\mathsf{GL}(2, \mathbb{Z})\backslash\mathbb{R}^2$ is intractable. Indeed, this dynamical system is orbit-equivalent to the horocycle flow on the unit tangent bundle $\mathsf{SL}(2, \mathbb{Z})\backslash\mathsf{SL}(2, \mathbb{R})$ of the elliptic modular curve, which is known to be uniquely ergodic.

This dramatically contrasts with the action of $\mathsf{Mod}(\mathbb{T}^2)$ on the deformation space of Euclidean structures. That action is *proper,* with quotient $\mathbb{R}_+ \times \mathfrak{M}_0$ where $\mathfrak{M}_0$ the Riemann moduli space of elliptic curves and the $\mathbb{R}_+$ parameter corresponds to the area of a Euclidean structure on a torus. Of course, Euclidean structures on $\mathbb{T}^2$ are *complete affine* structures. However, since all Euclidean structures are affinely equivalent, the subspace

$$\mathcal{CA}(\mathbb{T}^2)_{\mathbf{Euc}} \subset \mathcal{CA}(\mathbb{T}^2)$$

corresponding to Eudclidean structures collapses to a single point in $\mathcal{CA}(\mathbb{T}^2)$. This point is the origin in $\mathbb{R}^2 \approx \mathcal{CA}(\mathbb{T}^2)$, which maps to the unique singular point in the twisted cubic cone.

The main theorem of the paper is the following:

THEOREM. *The deformation space of affine equivalence classes of marked complete affine structures on $\mathbb{T}^2$ naturally identifies with a twisted cubic cone $\mathfrak{C} \subset \mathbb{R}^4$.*

Outline of the paper

§1 describes the cone on the twisted cubic and its symmetries. §2 describes the theory of affine structures and reduces the classification to commutative nilpotent

[1] $\mathsf{Hom}\big(\pi_1(\Sigma), G\big)$ has the natural structure as a real affine algebraic set, and $\mathsf{Hom}\big(\pi_1(\Sigma), G\big)/G$ is given the quotient topology by the action of $\mathsf{Inn}(G)$ by composition.

[2] Indeed, for a while it was believed [**9**] that the subspace comprising *complete* structures is not Hausdorff. Baues [**2**] addresses this error in the literature.

2-dimensional $\mathbb{R}$-algebras, which are studied in §3. Such an algebra $\mathfrak{A}$, together with a basis, completely determines the marked structure. From the general theory, this already gives a proof of our main theorem. One key point is that completeness is equivalent to the connection being *equiaffine* (sometimes called *parallel volume*). This means its holonomy preserves volume, that is, its linear holonomy has determinant ± 1. In this particular case, it is equivalent to the stronger condition of unipotence of the linear holonomy. Compare the discussion of the Markus conjecture in [**10**], §11 and the proof of completeness of closed affine manifolds with unipotent holonomy in [**10**], §8.4.

§4 gives a direct proof of the main theorem, using the Christoffel symbols $\mathbf{\Gamma}_{ij}^{k}$ of the flat torsionfree equiaffine connection,that the deformation space $\mathcal{C}\mathcal{A}(\mathbb{T}^2)$ of complete affine structures identifies with the cone $\mathfrak{C}$ on the twisted cubic. The Christoffel symbols are just the structure constants of the algebra $\mathfrak{A}$. The key point is that Kuiper's classification implies that the identity component of the affine automorphism group acts simply transitively on M. This implies that the Christoffel symbols are constant, enabling the identification of marked structures with nilpotent commutative associative 2-dimensional algebras over $\mathbb{R}$.

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I am also grateful to the anonymous referee for a careful reading and suggesting many improvements.

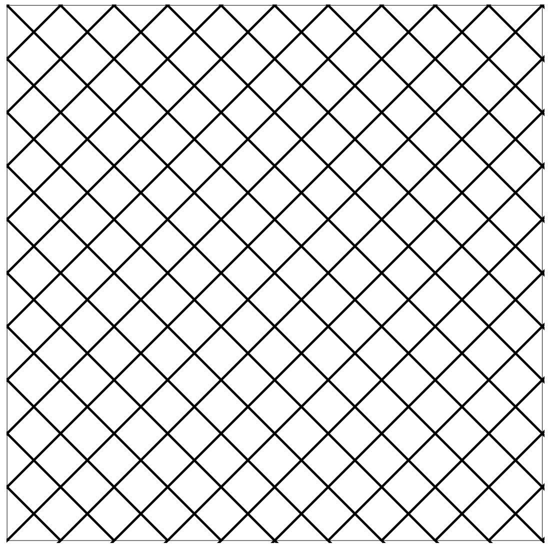

FIGURE 1. One example of a complete affine 2-manifold is a Euclidean flat torus, obtained as the quotient of the Eudlidean plane E^2 by a lattice $\Lambda \in \mathbb{R}^2$ of translations.

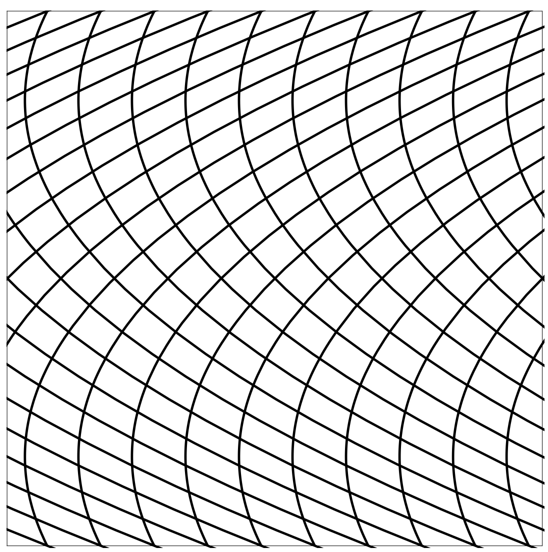

FIGURE 2. Applying a polynomial diffeomorphism of A^2 such as $(x,y) \xmapsto{f} (x+y^2,y)$ conjugates translations to a simply transitive affine action of $\mathbb{R}^2$ which is *not* Euclidean-isometric. The quotient $\mathsf{A}^2/f\Lambda f^{-1}$ is a complete affine 2-torus.

1. The cone on the twisted cubic

The *twisted cubic cone* $\mathfrak{C} \subset \mathbb{R}^4$ is the image of the map

$$\mathbb{R}^2 \hookrightarrow \mathsf{Sym}^3\big(\mathbb{R}^2\big) \cong \mathbb{R}^4 \tag{1.1}$$

$$\begin{bmatrix} A \\ B \end{bmatrix} \longmapsto \begin{bmatrix} A^3 \\ A^2B \\ AB^2 \\ B^3 \end{bmatrix} =: \begin{bmatrix} U \\ X \\ Y \\ Z \end{bmatrix} \in \mathbb{R}^4.$$

which embeds $\mathbb{R}^2$ topologically in $\mathbb{R}^4$. It is singular, and its unique singularity is the origin. It is the cone over the rational normal curve of degree 3 in $\mathbb{R}\mathsf{P}^3$.

Explicitly, $\mathfrak{C}$ is defined by the three homogeneous quadratic equations

$$X^2 = UY$$
$$UZ = XY$$
$$Y^2 = XZ$$

in the coordinates U, X, Y, Z. No two of these equations suffice to define $\mathfrak{C}$, despite $\mathfrak{C}$ having codimension two in $\mathbb{R}^4$.

The map (1.1) is $\mathsf{GL}(2,\mathbb{R})$-equivariant with respect to the standard action on $\mathbb{R}^2$ and the induced action on the symmetric power $\mathsf{Sym}^3\big(\mathbb{R}^2\big)$. The cone $\mathfrak{C} \subset \mathbb{R}^4$ is invariant under this action of $\mathsf{GL}(2,\mathbb{R})$. In particular this action restricts to $\mathsf{GL}(2,\mathbb{Z}) < \mathsf{GL}(2,\mathbb{R})$, and $\mathsf{GL}(2,\mathbb{Z})$ is isomorphic to the mapping class group $\mathsf{Mod}(\mathbb{T}^2)$.

Later in this paper we identify $\mathfrak{C}$ with the deformation space $\mathcal{CA}(\mathbb{T}^2)$ of marked complete affine structures on $\mathbb{T}^2$ and the $\mathsf{GL}(2,\mathbb{Z})$-action on $\mathfrak{C}$ with the $\mathsf{Mod}(\mathbb{T}^2)$-action on $\mathcal{CA}(\mathbb{T}^2)$.

2. Affine structures on the torus

Kuiper [**12**] showed that every complete affine structure on the torus $\mathbb{T}^2$ arises as a flat Euclidean torus $\Lambda\backslash\mathsf{E}^2$ where $\Lambda < \mathbb{R}^2$ is a lattice of translations or a quotient $\Gamma\backslash\mathsf{A}^2$ where $\Gamma < G$ is a lattice where G is the group of affine transformations

$$\begin{bmatrix} x \\ y \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 2b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a + b^2 \\ b \end{bmatrix}$$

of A^2, where $a, b \in \mathbb{R}$. The group G is isomorphic to $\mathbb{R}^2$; indeed is conjugate to the translation subgroup $\mathbb{R}^2 < \mathsf{Aff}(\mathsf{A}^2)$ by the diffeomorphism $(x, y) \longmapsto (x + y^2, y)$.

Since the action of G on A^2 is simply transitive, it induces a left-invariant[3] affine structure on G. Furthermore this passes down to an invariant affine structure on the Lie group $\Gamma\backslash G$, so the complete affine structures on $\mathbb{T}^2$ are all invariant structures on $\mathbb{T}^2$ with respect to its structure as an abelian Lie group.

Left-invariant affine structures on Lie groups form a rich algebraic theory described by (possibly non-associative) algebras, where the algebraic structure is defined by covariant differentiation of left-invariant vector fields. Commutator in this

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[3]Since G is abelian, it is also right-invariant.

algebra is just the usual Lie bracket of left-invariant vector fields (because the connection is torsionfree). Flatness of the connection implies the defining condition that the associator is symmetric in its first two arguments; hence these algebras are called *left-symmetric algebras.* Bi-invariant affine structures correspond to the case when this algebra is associative.[4]

Completeness for these structures is equivalent to several conditions, such as the parallelism of right-invariant volume forms, or unipotence of the affine action by left-multiplication

The theory of affine structures on closed manifolds with nilpotent holonomy is studied in [**8**] (building on Smillie [**14**] for abelian holonomy). In particular the equivalence of geodesic completeness with parallel volume (or *equiaffinity*) and unipotent linear holonomy is shown there, and expounded in §11 of [**10**] in a broader context. For left-invariant affine structures on Lie groups —- that is, for *affine Lie groups,* —- completeness is equivalent to right-invariant volume forms being parallel [**11**]. Thus for *abelian* affine Lie groups, completeness is equivalent to equiaffinity.

As mentioned in the introduction, $\mathbb{T}^2$ is the only closed orientable surface supporting an affine structure. Furthermore, a complete affine structure on $\mathbb{T}^2$ is necessarily invariant Equivalently, every complete affine structure on $\mathbb{T}^2$ is invariant under a structure of $\mathbb{T}^2$ as an *abelian Lie group.* [5] Baues [**3, 4**] surveys the classification of affine structures on surfaces in detail; see also [**10**].

Since the connection is invariant, the covariant derivative of invariant vector fields is invariant, and the *Christoffel symbols,* the coefficients of the covariant derivatives of a basis of invariant vector fields are constant.

3. Commutative nilpotent algebras

THEOREM 3.1. *A marked complete affine structure on $\mathbb{T}^2$ corresponds to a based 2-dimensional vector space V together with a symmetric bilinear form Γ making V into an algebra $\mathfrak{A}$, such that $\mathfrak{A}$ is a commutative $\mathbb{R}$-algebra with $\mathfrak{A}^3 = 0$.*

For an extensive discussion, see [**10**],§8.4. For the reader's convenience we sketch the proof here.

PROOF. Let $\mathbb{T}^2 \longrightarrow M$ be a marked complete affine structure, that is a homeomorphism (defined up to isotopy) onto a complete affine 2-manifold M. Choose a universal covering $\widetilde{M} \to M$ and a developing pair (dev, h) where the developing map $tM \xrightarrow{\mathsf{dev}} \mathsf{A}^2$ is a diffeomorphism and the holonomy homomorphism h is an embedding of $\pi_1(M)$ onto a discrete subgroup $\Gamma < \mathsf{Aff}(\mathsf{A}^2)$ acting properly on A^2. By the general algebraic theory described above (or direct calculation as in Kuiper's original classification), Γ is conjugate to a lattice in a subgroup G_μ of $\mathsf{Aff}(\mathsf{A}^2)$ comprising elements having the form

$$\begin{bmatrix} x \\ y \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 2\mu b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a + \mu b^2/2 \\ b \end{bmatrix}$$

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[4] See §10 of [**10**] for an extensive discussion of this theory.

[5] This was known to Kuiper [**12**], although he didn't state it in this form.

where $a, b \in \mathbb{R}$ and $\mu \in \mathbb{R}$ is a fixed parameter. The Lie algebra of G is generated by affine vector fields:

$$X := \partial/\partial x$$
$$Y_\mu := \mu y \, \partial/\partial x + \partial/\partial y.$$

Under covariant differentiation these vector fields span an algebra with multiplication given by Table 1.

Furthermore every 2-dimensional commutative associative algebra $\mathfrak{A}$ with $\mathfrak{A}^3 = 0$ has a basis as above: If $\mathfrak{A}^2 = 0$, the multiplication is trivial and we may take any basis X, Y (and $\mu = 0$). Otherwise nilpotence implies that $\dim \mathfrak{A}^2 = 1$. Let X base $\mathfrak{A}^2$, and extend $\{X\}$ to a basis $\{X, Y\}$ of $\mathfrak{A}$.

□

In practice V will be the abelian Lie algebra of left-invariant (right-invariant) vector fields, and Γ is the covariant derivative operator

$$\mathsf{V} \times \mathsf{V} \xrightarrow{\Gamma} \mathsf{V}$$
$$(\xi, \eta) \longmapsto \nabla_\xi \eta.$$

The above theory implies that $\mathfrak{A}$ is nilpotent, and indeed $\mathfrak{A}^3 = 0$ (so associativity is obvious).

Such a connection can be described in terms of the twisted cubic cone (1.1) as follows.

The connection (or the algebra structure) relates to the symmetric power $\mathsf{Sym}^3(\mathsf{V})$ as follows. For convenience choose a nonzero element ω in the line $\bigwedge^2(\mathsf{V})$. A vector $v \in \mathsf{V}$ determines a covector $\omega_v \in \mathsf{V}^*$ by

$$u \wedge v \ = \ \omega_v(u)\omega.$$

An endomorphism $\mathcal{E} \in \mathsf{End}(\mathsf{V})$ of rank one is determined by nonzero vectors k (generating the kernel) and j (generating the image) by

$$\mathcal{E} = \omega_k \otimes j.$$

The endomorphism $\mathcal{E}$ is nilpotent if and only if j and k are linearly dependent. In our case, unipotence of the linear action is equivalent to nilpotence of $\mathcal{E}$.

A connection is given by the tensor product $\psi \otimes \mathcal{E}$ where $\psi \in \mathsf{V}^*$ is a covector and $\mathcal{E}$ is an endomorphism. Flatness of the corresponding connection means that in the decomposition $\mathcal{E} = \omega_v \otimes w$, the covectors ω_v and ψ are linearly dependent. Hence a flat connection taking values in nilpotent endomorphisms corresponds to an element of $\mathsf{V}^* \otimes \mathsf{End}(\mathsf{V})$ of the form $\omega_v \otimes \mathcal{E}_v$ where $\mathcal{E}_v := \omega_v \otimes v$, that is in the

	X	Y_μ
X	0	0
Y_μ	0	μX

TABLE 1. A commutative 2-dimensional algebra with $\mathfrak{A}^3 = 0$

image of

$$\begin{aligned} \mathsf{V} &\longrightarrow \mathsf{V}^* \otimes \mathsf{End}(\mathsf{V}) \\ v &\longmapsto \omega_v \otimes \big(\omega_v \otimes v\big) \end{aligned}$$

which is evidently equivalent to (1.1). The final section §4 gives a direct identification using only the assumption of area-preserving holonomy rather than unipotent holonomy.

4. Flat torsionfree equiaffine connections

Choose a coordinate system (x^1, x^2) on A^2 and coordinate vector fields

$$\partial_i := \frac{\partial}{\partial x^i}$$

for $i = 1, 2$. Write ∇_i for the covariant differential operator $\nabla_{\partial i}$ and $\mathbf{\Gamma}_{ij}^{\,k}$ for the *Christoffel symbols* for ∇ with respect to the frame ∂_i:

$$\nabla_i \partial_j \ = \ \mathbf{\Gamma}_{ij}^{\,k} \partial_k$$

(Einstein summation). If ∇ has zero torsion, $\nabla_i \partial_j = \nabla_j \partial_i$, and

$$\mathbf{\Gamma}_{21}^{\,k} \ = \ \mathbf{\Gamma}_{12}^{\,k}. \tag{4.1}$$

for $k = 1, 2$. Since ∇ is *equiaffine* $\nabla\big(dx^1 \wedge dx^2\big) = 0$. It follows that for every vector field ξ, the curvature operator $\eta \longmapsto \nabla_\xi \eta$ has trace zero. Take $\xi = \partial_i$ for $i = 1, 2$ to obtain:

$$\mathbf{\Gamma}_{i1}^{\,1} + \mathbf{\Gamma}_{i2}^{\,2} \ = \ 0 \tag{4.2}$$

for $i = 1, 2$. In the following calculations, (4.2) and (4.1) imply

$$\begin{aligned} \mathbf{\Gamma}_{21}^{\,1} &= \mathbf{\Gamma}_{12}^{\,1} = -\mathbf{\Gamma}_{22}^{\,2} \\ \mathbf{\Gamma}_{21}^{\,2} &= \mathbf{\Gamma}_{12}^{\,2} = -\mathbf{\Gamma}_{11}^{\,1} \end{aligned} \tag{4.3}$$

so we reduce our calculations to the four variables

$$\mathbf{\Gamma}_{11}^{\,1},\ \mathbf{\Gamma}_{11}^{\,2},\ \mathbf{\Gamma}_{22}^{\,1},\ \mathbf{\Gamma}_{22}^{\,2}.$$

Since the covariant derivatives are constant,

$$\begin{aligned} \nabla_1 \nabla_2 \partial_1 &= \nabla_1 \big(\mathbf{\Gamma}_{21}^{\,1}\, \partial_1 + \mathbf{\Gamma}_{21}^{\,2}\, \partial_2\big) \\ &= \big(\mathbf{\Gamma}_{11}^{\,1} \mathbf{\Gamma}_{21}^{\,1} + \mathbf{\Gamma}_{12}^{\,1} \mathbf{\Gamma}_{21}^{\,2}\big)\ \partial_1 + \big(\mathbf{\Gamma}_{11}^{\,2} \mathbf{\Gamma}_{21}^{\,1} + \mathbf{\Gamma}_{12}^{\,2} \mathbf{\Gamma}_{21}^{\,2}\big)\, \partial_2 \end{aligned} \tag{4.4}$$

$$\begin{aligned} \nabla_2 \nabla_1 \partial_1 &= \nabla_2 \big(\mathbf{\Gamma}_{11}^{\,1}\, \partial_1 + \mathbf{\Gamma}_{11}^{\,2}\, \partial_2\big) \\ &= \big(\mathbf{\Gamma}_{21}^{\,1} \mathbf{\Gamma}_{11}^{\,1} + \mathbf{\Gamma}_{22}^{\,1} \mathbf{\Gamma}_{11}^{\,2}\big)\, \partial_1 + \big(\mathbf{\Gamma}_{21}^{\,2} \mathbf{\Gamma}_{11}^{\,1} + \mathbf{\Gamma}_{22}^{\,2} \mathbf{\Gamma}_{11}^{\,2}\big)\, \partial_2 \end{aligned} \tag{4.5}$$

$$\begin{aligned} \nabla_1 \nabla_2 \partial_2 &= \nabla_1 \big(\mathbf{\Gamma}_{22}^{\,1}\, \partial_1 + \mathbf{\Gamma}_{22}^{\,2}\, \partial_2\big) \\ &= \big(\mathbf{\Gamma}_{11}^{\,1} \mathbf{\Gamma}_{22}^{\,1} + \mathbf{\Gamma}_{12}^{\,1} \mathbf{\Gamma}_{22}^{\,2}\big)\, \partial_1 + \big(\mathbf{\Gamma}_{11}^{\,2} \mathbf{\Gamma}_{22}^{\,1} + \mathbf{\Gamma}_{12}^{\,2} \mathbf{\Gamma}_{22}^{\,2}\big)\, \partial_2 \end{aligned} \tag{4.6}$$

$$\begin{aligned} \nabla_2 \nabla_1 \partial_2 &= \nabla_2 \big(\mathbf{\Gamma}_{12}^{\,1}\, \partial_1 + \mathbf{\Gamma}_{12}^{\,2}\, \partial_2\big) \\ &= \big(\mathbf{\Gamma}_{21}^{\,1} \mathbf{\Gamma}_{12}^{\,1} + \mathbf{\Gamma}_{22}^{\,1} \mathbf{\Gamma}_{12}^{\,2}\big)\, \partial_1 + \big(\mathbf{\Gamma}_{21}^{\,2} \mathbf{\Gamma}_{12}^{\,1} + \mathbf{\Gamma}_{22}^{\,2} \mathbf{\Gamma}_{12}^{\,2}\big)\, \partial_2 \end{aligned} \tag{4.7}$$

Flatness of ∇ implies $\nabla_1 \circ \nabla_2 = \nabla_2 \circ \nabla_1$. Subtracting (4.5) from (4.4), $[\nabla_1, \nabla_2]\, \partial_1 = 0$ implies:

$$\mathbf{\Gamma}_{12}^{1}\mathbf{\Gamma}_{21}^{2} = \mathbf{\Gamma}_{22}^{1}\mathbf{\Gamma}_{11}^{2} \tag{4.8}$$

for the ∂_1-component of $[\nabla_1, \nabla_2]\partial_1$

$$\mathbf{\Gamma}_{11}^{2}\mathbf{\Gamma}_{21}^{1} + \mathbf{\Gamma}_{12}^{2}\mathbf{\Gamma}_{21}^{2} = \mathbf{\Gamma}_{21}^{2}\mathbf{\Gamma}_{11}^{1} + \mathbf{\Gamma}_{22}^{2}\mathbf{\Gamma}_{11}^{2} \tag{4.9}$$

for the ∂_2-component of $[\nabla_1, \nabla_2]\partial_1$.

Subtracting (4.7) from (4.6), $[\nabla_1, \nabla_2]\,\partial_2 = 0$ implies:

$$\mathbf{\Gamma}_{11}^{1}\mathbf{\Gamma}_{22}^{1} + \mathbf{\Gamma}_{12}^{1}\mathbf{\Gamma}_{22}^{2} = \mathbf{\Gamma}_{21}^{1}\mathbf{\Gamma}_{12}^{1} + \mathbf{\Gamma}_{22}^{1}\mathbf{\Gamma}_{12}^{2} \tag{4.10}$$

for the ∂_1-component of $[\nabla_1, \nabla_2]\partial_2$.

$$\mathbf{\Gamma}_{11}^{2}\mathbf{\Gamma}_{22}^{1} = \mathbf{\Gamma}_{21}^{2}\mathbf{\Gamma}_{12}^{1} \tag{4.11}$$

for the ∂_2-component of $[\nabla_1, \nabla_2]\partial_2$.

Now apply (4.3) to rewrite (4.8), (4.9), (4.10), and (4.11) in terms of the four variables

$$\mathbf{\Gamma}_{11}^{1},\ \mathbf{\Gamma}_{11}^{2},\ \mathbf{\Gamma}_{22}^{1},\ \mathbf{\Gamma}_{22}^{2}.$$

First, (4.8) becomes

$$\mathbf{\Gamma}_{11}^{1}\mathbf{\Gamma}_{22}^{2} = \mathbf{\Gamma}_{11}^{2}\mathbf{\Gamma}_{22}^{1}, \tag{4.12}$$

Since ∇ is equiaffine, the curvature tensor takes values in traceless endomorphisms. Thus the ∂_1-component of $[\nabla_1, \nabla_2]\,\partial_2$ is the negative of ∂_2-component of $[\nabla_1, \nabla_2]\,\partial_1$. Thus (4.11) is equivalent to (4.8) and (4.11) provides nothing new.

Next, (4.9) and (4.10) respectively become:

$$\left(\mathbf{\Gamma}_{11}^{1}\right)^2 = \mathbf{\Gamma}_{11}^{2}\mathbf{\Gamma}_{22}^{2} \tag{4.13}$$

$$\left(\mathbf{\Gamma}_{22}^{2}\right)^2 = \mathbf{\Gamma}_{11}^{1}\mathbf{\Gamma}_{22}^{1}. \tag{4.14}$$

The three equations (4.12), (4.13) and (4.14) now describe the twisted cubic cone $\mathfrak{C}$ as in §1, by taking:

$$U = \mathbf{\Gamma}_{11}^{2}$$
$$X = \mathbf{\Gamma}_{11}^{1}$$
$$Y = \mathbf{\Gamma}_{22}^{2}$$
$$Z = \mathbf{\Gamma}_{22}^{1},$$

thus identifying the deformation space $\mathcal{CA}(\mathbb{T}^2)$ of marked complete affine structures on $\mathbb{T}^2$ with $\mathfrak{C}$.

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Goldman: Department of Mathematics, University of Maryland, College Park, MD 20742

Email address: `wmg@umd.edu`