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Review text:

This interesting paper describes the symplectic structure on the moduli space of gauge equivalence classes of flat connections on a principal G-bundle over a closed surface Σ in terms of the bar complex of the fundamental group and various differential forms on the classifying space of G, when G is a real reductive Lie group. This moduli space corresponds to the space of conjugacy classes of homomorphisms of the fundamental group $\pi = \pi_1(\Sigma)$ into G. For simplicity consider the open set U of representations whose centralizer equals the center of G; this is a finite-dimensional smooth manifold upon which Ad G acts freely and properly. The quotient manifold U/G enjoys a symplectic structure.

Previous approaches to the symplectic structure involved a mix of infinite-dimensional methods (gauge theory) and finite-dimensional methods (group cohomology). A prominent role is always played by the fundamental cycle of Σ and an invariant inner product on the Lie algebra g of G. For example in "The Yang-Mills Equations over Riemann surfaces," (Phil. Trans. R. Soc. London, **308** (1982), 523–615), the symplectic structure is defined by integrating over M (the fundamental cycle in de Rham cohomology) the inner product of two 1-forms representing tangent vectors to the moduli space of connections. Although it is easy to construct expressions in group cohomology for an exterior 2-form defining the symplectic structure, the first proofs these forms are closed involved gauge theory. Yael Karshon ("An algebraic proof for symplectic structure of moduli space," Proc. A.M.S. **116** (1992), 591–605) found a proof of the closedness of the exterior 2-form defining the symplectic structure in a spirit closer to group cohomology, using a simplicial complex related to G. The present paper refines Karshon's construction and relates it to the simplicial de Rham bicomplex studied by Bott, Shulman and

Stasheff (Adv. Math. **20** (1976), 43–56).

Here is a brief motivation and summary of some of the paper's ideas. The homological model of G is the standard simplicial Lie group $G^\star$ based on Milnor's infinite join construction of the universal G-bundle ("Construction of universal bundles I, II," Ann. Math. **63** (1963), 272–284, 430–436). That is, the p-simplices correspond to $p+1$-tuples $(g_0, \ldots, g_p) \in G^{p+1}$ with the usual face and degeneracy mappings. Its geometrical realization is a model for the total space of the universal G-bundle EG and is acyclic. Its quotient by the natural G-action by left-multiplication is a simplicial manifold (whose p-simplices corresponding to p-tuples in G^p) which realizes the classifying space BG of G. The associated chain complex is the inhomogeneous bar resolution of G. Combining this with the de Rham complex gives a double complex $C^{p,q}(G)$ consisting of exterior differential q-forms on the Cartesian power G^p. Exterior derivative defines a differential $d : C^{p,q}(G) \longrightarrow C^{p,q+1}(G)$. The other differential $\delta : C^{p,q}(G) \longrightarrow C^{p+1,q}(G)$ is the standard Eilenberg-MacLane coboundary of the inhomogeneous bar resolution.

Classical Maurer-Cartan calculus developed in §3 of the paper produces a 3-form $\lambda \in C^{1,3}(G)$ on G with $d\lambda = 0$ and a 2-form $\Omega \in C^{2,2}(G)$ on $G \times G$ with $\delta\Omega = 0$ and $d\Omega = \delta\lambda$. It follows (Corollary 3.5) that the inhomogeneous cochain $\Omega - \lambda$ is a cocycle of degree 4 in the total complex associated to $C^{*,*}(G)$. In these constructions the inner product on g plays a prominent role.

The next ingredient is a simplicial manifold associated to π. Applying Milnor's infinite join construction to π gives a simplicial set $\pi^\star$ modelling the universal π-bundle $E(\pi)$ (equivalently the universal covering space $\tilde{\Sigma}$ of Σ). Its quotient by left-multiplication by π is a simplicial set whose p-simplices correspond to p-tuples in π^p just as above. The product $\pi^\star \times U$ is a simplicial manifold, for which one can define a de Rham bicomplex $C^{p,q}(\pi, U) = \Omega^q(\pi^p \times U)$ consisting of exterior forms on $\pi^p \times U$ with Eilenberg-MacLane coboundary δ and exterior derivative d.

Evaluation of homomorphisms $\phi \in U$ on p-tuples in π defines a simplicial smooth map $\xi : \pi^\star \times U \longrightarrow G^\star$, "modelling" the space U of representations ϕ. Since ϕ is a homomorphism, $\tilde{\xi}$ defines a simplicial smooth map of the quotients which induces a map of the bicomplexes $E : C^{p,q}(\pi, U) \longrightarrow C^{p,q}(G)$ which is a homological model of the "universal family of flat principal G-bundles over Σ," which is the principal G-bundle over $\Sigma \times U$ having total space the quotient of $\tilde{\Sigma} \times U \times G$ byS the π-action defined by $\gamma : (\tilde{\sigma}, \phi, g) \mapsto (\gamma\tilde{\sigma}, \phi, \phi(\gamma)g)$. Cap products of the characteristic classes of this G-bundle over the product $\Sigma \times U$ with homology classes in Σ were used by Newstead ("Characteristic classes of stable bundles over an algebraic curve," Trans. A.M.S. **169** (1972), 337–345) to produce generators of the cohomology ring of U/G when $G = SU(2)$. By analogy, the pullback $E^*(\Omega - \lambda)$ is a 4-cocycle in the total complex $C^{p,q}(\pi, U)$.

However, the main interest is not the space U of representations, but instead its quotient U/G which is the moduli space of flat principal G-bundles. The appropriate homological theory for quotients is the equivariant cohomology, whereby one regards the homology of the quotient X/G of a G-space as the homology of the G-invariant forms on the product $X \times EG$ (which is homotopy-equivalent to X since EG is contractible). Equivariant de Rham theory — in which cochains are G-equivariant polynomial maps $g \longrightarrow \Omega^{\star}{}^{\text{`}}(X)$ — is perfectly suited for this paper's constructions. Unfortunately in the equivariant version of the bicomplex $C^{*,*}(G)$ the inhomogeneous cochain $\Omega - \lambda$ is no longer a total cocycle. §4 of the paper repairs this: there is an equivariant form $\theta : g \longrightarrow \Omega^1(G)$ such that the equivariant cochain $Q_4 = \Omega - \lambda + \theta$ of total degree 4 is now an equivariant cocycle.

§5 assembles all of these constructions. Pull back the equivariant cocycle Q_4 by the evaluation map E to obtain an equivariant cocycle on $\Sigma \times U$. Cap product with the fundamental cycle of Σ defines an equivariantly closed exterior 2-form on U, which descends to a closed exterior 2-form on the moduli space U/G. This general construction leads to other exterior forms on moduli space, which are briefly described at the end of §5. The paper ends (§6) with a discussion of open problems and other directions in which these ideas can be extended.

The results of this paper have been extended to compact surfaces with boundary by K. Guruprasad, J. Huebschmann, L. Jeffrey, and A. Weinstein, "Group systems, groupoids, and moduli spaces of parabolic bundles," Duke Math. J. (to appear).