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This paper surveys the Poisson structure and integrable systems on the moduli space of flat principal bundles over a compact surface. This paper contains many examples, some of which are quite classical in origin, and puts the integrable systems on moduli spaces in a general context.

Beginning with some generalities on Poisson structures and momentum mappings, the paper describes some of the classical examples of integrable systems in this modern framework. The first examples come from classical mechanics: motion of spinning tops and other rigid bodies. The general theory of Hamiltonian torus actions, which now is quite satisfactorily developed, is described. The main result is the Arnold-Liouville theorem on the existence of invariant tori, the dimension of which are given by the rank of the momentum mapping. The non-periodic Toda lattice is developed in terms of classical r-matrices. Finally examples of Poisson Lie groups are given.

The second chapter develops the Poisson structure on the moduli space of flat connections on a trivial principal bundle over a compact surface S (with possibly nonempty boundary). The space $\mathcal{A}$ of connections is an affine space with underlying vector space $\Omega^1(S;\mathfrak{g})$ and the gauge group $\mathcal{G}$, comprising mappings $S \longrightarrow G$, acts affinely on $\mathcal{A}$. Curvature defines a map $F : \mathcal{A} \longrightarrow \Omega^2(S;\mathfrak{g})$ whose preimage $\mathcal{A}_{fl} = F^{-1}(0)$ determines a quotient $\mathcal{M} = \mathcal{A}_{fl}/\mathcal{G}$ which identifies with the quotient space $\mathrm{Hom}(\pi_1(S), G)/\operatorname{Ad} G$.

In the case of a closed surface and a compact structure group G, Atiyah and Bott ("The Yang-Mills equations over Riemann surfaces," Philos. Trans. Roy. Soc. London Ser. A **308** (1982), 523–615) observed that there is a parallel symplectic form (arising from an $\operatorname{Ad} G$-invariant inner product on $\mathfrak{g}$ and the exterior product on 1-forms) on $\mathcal{A}$ which is $\mathcal{G}$-invariant. Furthermore F defines an equivariant momentum mapping for the $\mathcal{G}$-action on $\mathcal{A}$. In this way, the moduli space $\mathcal{M}$ of equivalence classes of flat connections is a Marsden-Weinstein symplectic quotient.

When S has nonempty boundary, these structures generalize to Poisson structures on $\mathcal{A}$ and $\mathcal{M}$. The symplectic leaves consist of connections (or equivalence classes) whose holonomy around each component of ∂S is constrained to lie in a fixed conjugacy class in G. This theory (when $\partial S \neq \emptyset$) is developed in detail here. In particular, one must pass to a central extension $\hat{\mathbf{g}}$ of the Lie algebra $\mathbf{g} = \Omega^0(M;\mathfrak{g})$ of $\mathcal{G}$ to obtain a Hamiltonian Poisson action on $\mathcal{A}$. Correction terms corresponding to ∂S appear. The three-holed sphere and the one-holed torus furnish examples of homotopy-equivalent non-homeomorphic surfaces whose moduli spaces are identical (each is a copy of $G \times G/\operatorname{Ad} G$). However, the Poisson structures are very different: the symplectic leaves have dimension zero for the sphere and two for the torus, for $G = \mathrm{SU}(2)$. (The symplectic foliation has codimension equal to the number of compoents of ∂S.)

The paper then gives an elegant description of the finite-dimensional discretisation of this theory due to Fock and Roslyi. The compact surface S is considered as a "fat graph," that is, a tubular neighborhood of a graph, its spine. Fock and Roslyi use the fat graph to describe $\mathcal{M}$ and its Poisson structure in purely finite-dimensional terms. The gauge group is replaced by a finite-dimensional "gauge group" $\mathcal{G}^\delta$ and the space of connections by a finite-dimensional "space of connections" $\mathcal{A}^\delta$. Furthermore $\mathcal{G}^\delta$ is a Poisson Lie group, $\mathcal{A}^\delta$ is a Poisson manifold and $\mathcal{G}^\delta \times \mathcal{A}^\delta \longrightarrow \mathcal{A}^\delta$ is a Poisson mapping. (This seems to be closely related to an approach using symplectic groupoids developed by K. Guruprasad, J. Huebschmann, L. Jeffrey, A. Weinstein in "Group systems, groupoids, and moduli spaces of parabolic bundles," Duke Math. J. **89**, No. 2 (1997), 377–412.)

The final chapter describes integrable systems on the moduli space, in terms of functions on $\mathcal{M}$ and their corresponding Hamiltonian flows. In "Invariant functions on Lie groups and Hamiltonian flows of surface group representations," Invent. Math. **85** (1986), 263–302) the reviewer associated to an element of $\gamma \in \pi_1(S)$ and a function $f : G \longrightarrow \mathbb{R}$, invariant under inner automorphisms of G, the function

$$\begin{aligned} f_\gamma : \mathcal{M} &\longrightarrow \mathbb{R} \\ [\phi] &\mapsto f(\phi(\gamma)). \end{aligned}$$

When γ is represented by a simple loop, the corresponding Hamiltonian flow admits an explicit description as a flow on the space of representations.

When $\alpha, \beta \in \pi_1(S)$ are represented by *disjoint* simple curves, then for any invariant functions $f, g : G \longrightarrow \mathbb{R}$, the functions f_α, g_β Poisson commute. (More generally the Poisson bracket of f_α, g_β can be expressed in terms of the intersection points of α and β.) Other functions, due to Alekseev, arise from expanding invariant functions on G upon the defining relation in $\pi_1(S)$. Thus collections of disjoint curves on S and invariant functions on G determines abelian algebras of functions on $\mathcal{M}$.

These constructions provide integrable systems on $\mathcal{M}$. One starts with a maximal collection of homotopically disjoint and nontrivial simple closed curves (a "pants" or "trinion" decomposition) to obtain Hamiltonian $\mathbb{R}^N$-actions for each invariant function. In the case $G = \mathrm{SU}(2)$, the $\mathbb{R}^N$-action has periodic orbits and one obtains a completely integrable system at least on an dense open subset of $\mathcal{M}$. By careful choice of the invariant function, Jeffrey and Weitsman ("Bohr-Sommerfeld orbits in the moduli space of flat connections and the Verlinde dimension formula," Commun. Math. Phys. **150** (1992), 595–630) obtain Hamiltonian torus actions on a dense open set, and the image of the moment map is a convex polytope, which can be related to spherical trigonometry. Many examples of these polytopes and the resulting toric geometry are given. Jeffrey and Weitsman apply this theory to give a simple proof of a theorem of Donaldson's expressing the volume of a symplectic leaf as a Bernoulli polynomial.

The paper concludes with a very interesting example, namely the Ruijsenaars system, according to Fock. This corresponds to the case when S is a once-punctured torus, and generalizes the Calogero system.