THE GEOMETRY OF CROOKED PLANES

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Abstract. Crooked planes are polyhedra used to construct fundamental polyhedra for discrete groups of Lorentz isometries acting properly on Minkowski (2+1)-space. This paper explores intersections of crooked planes. Criteria for the disjointness of crooked planes are developed. These criteria are applied to derive sufficient conditions for affine deformations of a discrete subgroup of $\mathbf{SO}(2,1)$ to act properly on Minkowski space.

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1. Introduction

1.1. **Milnor's Question.** In 1977, Milnor asked ([14]) whether a nonamenable group (for example a free group of rank 2) could act properly by affine transformations. He observed that by applying Tits's alternative (see [15]), this question is equivalent to whether a *nonabelian free* group could act properly by affine transformations. Furthermore he proposed the following construction of such a group: Start with a free discrete subgroup of $\mathbf{SO}^0(2,1)$ (for example a Schottky group acting on the hyperbolic plane) and add translation components to obtain a group of affine transformations which may act freely. However, "it seems difficult to decide whether the resulting group action is properly discontinuous." ([14])

In 1983, Margulis showed that such properly discontinuous groups do indeed exist ([11],[12]), realizing Milnor's suggestion. We study these actions in terms of their quotient spaces, which we call *Margulis space-times.* A Margulis space-time is characterized as a complete flat Lorentz 3-manifold M whose fundamental group is nonamenable (that is, not virtually solvable). If Γ is a discrete group acting properly on $\mathbb{E}$ by Lorentz isometries, then $\mathbb{E}/\Gamma$ is a complete flat Lorentz manifold. Conversely the fundamental group of a complete flat Lorentz manifold is a discrete group acting properly on $\mathbb{E}$ by isometries. (Recall that a group acts properly discontinuously if and only if it acts properly when given the discrete topology.) Since Γ contains a nonabelian free subgroup of finite index, a Margulis space-time M admits a finite covering space whose fundamental group is a nonabelian free group.

1.2. **Fundamental Polyhedra for Schottky groups.** A model for this construction is the *Klein-Maskit combination* procedure applied to *Schottky groups.* Namely, a Schottky group is a discrete group Γ of isometries of hyperbolic space $\mathbf{H}^n$ freely generated by cyclic groups in the following way. If γ is an isometry for which there exists a half-space $H \subset \mathbf{H}^n$ such that $\gamma(H) \subset H$, then $\Delta = H \cap \gamma(H)$ is a fundamental domain for the cyclic group $\langle\gamma\rangle$ generated by γ. Suppose $\gamma_1, \ldots, \gamma_r$ are isometries for which there exist half-spaces H_i such that:

- $\gamma_i H_i \subset H_i$ for each $i = 1, \ldots, r$;
- The half-spaces $H_1, \ldots, H_r, \mathbf{H}^n - H_1, \ldots, \mathbf{H}^n - H_r$ are pairwise disjoint.

Then the cyclic groups $\langle\gamma_i\rangle$ freely generate a discrete group Γ. Furthermore the intersection of their fundamental domains $\Delta_i = H_i \cap \gamma_i H_i$ is a fundamental domain

$$\Delta = \cap_{i=1}^{r} \Delta_i$$

for Γ. Furthermore this construction is *robust* in the sense that for any small perturbation of either the half-spaces H_i or the generators γ_i gives another such fundamental domain.

One would like to imitate this construction for groups of affine transformations. Each generator γ is an affine transformation which leaves invariant a line ℓ, acts by a hyperbolic linear map on the quotient plane $\mathbb{R}^{2,1}/\ell$, and acts by a nontrivial translation along ℓ. Choose a point $p \in \ell$; there is a unique plane P containing p such that $\gamma(P)$ is disjoint (that is, parallel) to P. The region bounded by P and $\gamma(P)$ is a fundamental domain for $\langle\gamma\rangle$, which we call a *slab.*

Slabs unfortunately do not make satisfactory fundamental domains. They fail to be robust as above, and impossible to combine. For example, two slabs never have disjoint complements, a necessary condition for the Klein-Maskit construction. In order to combine properly discontinuous cyclic groups, one needs building blocks which are more adapted to the linear part of the generators, rather than the translational part. Crooked planes serve this purpose, as described by Drumm in his 1990 doctoral dissertation [3].

1.3. **Crooked Planes.** In [3], complete flat Lorentz manifolds with nonabelian free fundamental group were directly constructed using Poincaré's fundamental polyhedron technique ([3],[4]). These examples are diffeomorphic to solid handlebodies. Their fundamental groups are "affine Schottky groups" with fundamental domains derived from fundamental domains of Schottky subgroups of $\mathbf{SO}^0(2,1)$ acting on the hyperbolic plane $\mathbf{H}^2$. These fundamental polyhedra are bounded by certain polyhedral hypersurfaces in $\mathbb{R}^3$, called *crooked planes.* (See Figure 1.) This paper investigates the geometry of crooked planes. In particular we develop criteria for disjointness of crooked planes and, in general, describe intersections of crooked planes.

Conjecture 1.3.1. Every Margulis space-time has a fundamental polyhedron whose faces are crooked planes.

A crooked plane consists of three parts: two half-planes, called *wings* and a pair of opposite planar sectors, called its *stem.* The wings lie in null planes and the stem (whose interior has two connected components) lies in a time-like (indefinite) plane. Crooked planes are sufficiently flexible to build fundamental polyhedra for affine Schottky groups. One starts with a fundamental polygon Δ for the linear part $\mathbb{L}(\Gamma)$ acting on the hyperbolic plane $\mathbf{H}^2$. The sides of Δ correspond to spacelike directions. Using the disjointness criteria developed here, we find crooked planes corresponding to the sides of Δ. Sometimes these crooked planes bound a fundamental domain for the properly discontinuous group Γ.

Here is a basic technical result:

Corollary 4.5.5. Let C_1 and C_2 be two crooked planes. Suppose that the three planes containing the stems and wings of C_1 are each transverse to the three planes containing the stems and wings of C_2. If a wing W of C_1 intersects the stem of C_2, then W intersects one of the wings of C_2.

1.4. **Discrete groups.** These results are applied to groups generated by inversions. If l is a spacelike line, there is a unique *inversion,* ι_l, an isometric involution fixing l. For each point $p \in l$, there is a unique crooked plane C with vertex p invariant under ι_l. Furthermore each of the two connnected components of the complement of C is a fundamental domain for ι_l. Let $l_1, \dots, l_n$ be a family of spacelike lines and $\iota_1, \dots, \iota_n$ be the corresponding inversions. The criterion discussed in [7] for a Schottky group to have properly discontinuous affine deformations implies a criterion for the group Γ generated by $\iota_1, \dots, \iota_n$ to act properly. We examine these criteria in terms of the disjointness criteria for crooked planes corresponding to $l_1, \dots, l_n$.

Although [4] presents a classification of the linear holonomy representations of Margulis space-times, the full moduli space of these structures remains mysterious. Let $\mathbb{R}^{2,1}$ denote a 3-dimensional real vector space with a nondegenerate inner product of signature (2,1). Let $\Gamma \subset \mathbf{SO}(2,1)$ be a subgroup. An affine representation of Γ with the given linear part is determined by its translational part $u : \Gamma \longrightarrow \mathbb{R}^{2,1}$

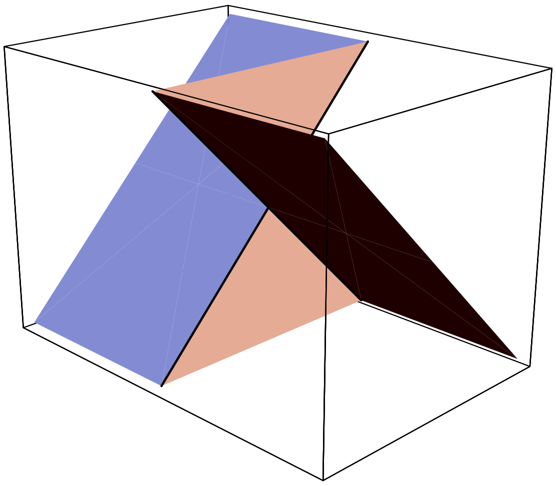

FIGURE 1. A crooked plane

which is a cocycle of Γ with values in the Γ-module $\mathbb{R}^{2,1}$. Thus the space of affine representations of Γ with the given linear part identifies with the vector space $Z^1(\Gamma, \mathbb{R}^{2,1})$ of $\mathbb{R}^{2,1}$-valued cocycles of Γ. Explicitly, if $u \in Z^1(\Gamma, \mathbb{R}^{2,1})$ is such a cocycle, then the affine representation

$$\rho_u : \Gamma \longrightarrow \mathrm{Aff}(\mathbb{E})$$

is given by:

$$\rho_u(\gamma) : x \longmapsto \gamma(x) + u(\gamma)$$

(The cocycle condition on u derives from condition

$$\rho_u(xy) = \rho_u(x)\rho_u(y).$$

Compare §1 of Fried-Goldman-Hirsch [9].) Conjugation by translation by $v_0 \in \mathbb{R}^{2,1}$ alters u by the coboundary

$$\delta v_0 \in Z^1(\Gamma, \mathbb{R}^{2,1})$$

of v_0:

$$\rho_u(\gamma) : x - v_0 \longmapsto \gamma(x - v_0) + u(\gamma) + \delta v_0(\gamma)$$

where

$$\delta v_0 : \gamma \longmapsto v_0 - \gamma(v_0).$$

Thus the space of translational conjugacy classes of affine representations of Γ with the given linear part identifies with the cohomology $H^1(\Gamma, \mathbb{R}^{2,1})$. For given $\Gamma \subset \mathbf{SO}(2,1)$, a subset of $H^1(\Gamma, \mathbb{R}^{2,1})$ corresponds to proper actions. Using fundamental domains for Γ acting on the hyperbolic plane, we have constructed [3],[7] a certain open cone in this subset. Here we interpret the conditions defining this cone in terms of intersections of crooked planes. Understanding how crooked planes intersect is the first part of our program to analyze the moduli space of Margulis space-times.

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2. Background

2.1. **Lorentzian vector spaces.** Define $\mathbb{R}^{2,1}$ to be the 3-dimensional real vector space with the Lorentzian inner product

$$\mathbb{B} : \mathbb{R}^{2,1} \times \mathbb{R}^{2,1} \longrightarrow \mathbb{R}$$

defined by:

$$\mathbb{B}(\mathsf{u},\mathsf{v}) = \mathsf{u}_1\mathsf{v}_1 + \mathsf{u}_2\mathsf{v}_2 - \mathsf{u}_3\mathsf{v}_3,$$

where

$$\mathsf{u} = \begin{bmatrix} \mathsf{u}_1 \\ \mathsf{u}_2 \\ \mathsf{u}_3 \end{bmatrix}$$

and

$$\mathsf{v} = \begin{bmatrix} \mathsf{v}_1 \\ \mathsf{v}_2 \\ \mathsf{v}_3 \end{bmatrix}$$

are written in terms of the usual basis $\{\mathsf{e}_1, \mathsf{e}_2, \mathsf{e}_3\}$. Denote by

$$\mathbf{SO}(2,1) \subset \mathbf{O}(2,1)$$

the group of unimodular linear automorphisms of $\mathbb{R}^{2,1}$, and $\mathbf{SO}^0(2,1)$ its identity component. The *Lorentzian cross-product* $\boxtimes : \mathbb{R}^{2,1} \times \mathbb{R}^{2,1} \longrightarrow \mathbb{R}^{2,1}$ is the skew-symmetric bilinear mapping

$$\mathsf{u} \boxtimes \mathsf{v} = \begin{bmatrix} \mathsf{u}_2\mathsf{v}_3 - \mathsf{u}_3\mathsf{v}_2 \\ \mathsf{u}_3\mathsf{v}_1 - \mathsf{u}_1\mathsf{v}_3 \\ \mathsf{u}_2\mathsf{v}_1 - \mathsf{u}_1\mathsf{v}_2 \end{bmatrix}.$$

which satisfies $\mathbb{B}(\mathsf{x}, \mathsf{x} \boxtimes \mathsf{y}) = 0$. For more details, the reader is referred to [7]. (Minkowski space $\mathbb{R}^{2,1}$ identifies with the Lie algebra $\mathfrak{so}(2,1)$ of $\mathbf{SO}(2,1)$ by:

$$\begin{aligned} \mathbb{R}^{2,1} &\longrightarrow \mathfrak{so}(2,1) \\ \mathsf{v} &\longmapsto (\mathsf{u} \mapsto \mathsf{v} \boxtimes \mathsf{u}). \end{aligned}$$

Under this identification cross-product $\boxtimes$ corresponds to the Lie bracket and the inner product $\mathbb{B}$ corresponds to (-1/2) times the Killing form.)

Let $|\mathsf{v}|$ denote the Euclidean norm of the vector v. Denote by $\langle \mathsf{v}_1, \mathsf{v}_2, \dots, \mathsf{v}_n \rangle$ the subspace spanned by $\mathsf{v}_1, \dots, \mathsf{v}_n$. (For a single vector

v we also reserve the notations $\mathbb{R}\mathsf{v} = \langle \mathsf{v} \rangle$ for the line containing v and $\mathbb{R}_+\mathsf{v}$ for the ray containing v.) For any vector $\mathsf{v} \in \mathbb{R}^{2,1}$, its *Lorentz-perpendicular plane* is defined to be

$$\mathcal{P}(\mathsf{v}) = \{\mathsf{u} \in \mathbb{R}^{2,1} \mid \mathbb{B}\,(\mathsf{u},\mathsf{v}) = 0\}$$

The *light cone*

$$\mathbb{W} = \{\mathsf{x} \in \mathbb{R}^{2,1} \mid \mathbb{B}\,(\mathsf{x},\mathsf{x}) = 0\}$$

consists of all *null* vectors, and its two components form the *time-orientations* of $\mathbb{R}^{2,1}$. (Each component is $\mathbf{SO}^0(2,1)$-invariant.) We refer to the upper nappe

$$\mathbb{W}^+ = \{\mathsf{x} \in \mathbb{W} \mid \mathsf{x}_3 > 0\}, \tag{2.1.1}$$

as the *positive time-orientation.* Similarly the set of *timelike* vectors

$$\mathbb{U} = \{\mathsf{x} \in \mathbb{R}^{2,1} \mid \mathbb{B}\,(\mathsf{x},\mathsf{x}) < 0\}$$

has a component

$$\mathbb{U}^+ = \{\mathsf{x} \in \mathbb{U} \mid \mathsf{x}_3 > 0\}, \tag{2.1.2}$$

also defining the positive time-orientation.

If $a, b \in \mathbb{W}^+$ are nonzero null vectors with positive time-orientation, then $\mathsf{v} = a \boxtimes b$ is spacelike and $\mathbf{x}^-(\mathsf{v}) = a$ and $\mathbf{x}^+(\mathsf{v}) = b$, correcting a misprint in [7], p.152 (3). If $\mathsf{v}_1, \mathsf{v}_2$ are two vectors, then $\mathsf{v}_1 \boxtimes \mathsf{v}_2$ is a vector parallel to the intersection $\mathcal{P}(\mathsf{v}_1) \cap \mathcal{P}(\mathsf{v}_2)$ (which is typically a line).

Let $\mathsf{v} \in \mathbb{W}^+$ be a nonzero null vector. The plane $\mathcal{P}(\mathsf{v})$ equals the plane tangent to the light cone containing the line $\mathbb{R}\mathsf{v}$ and the null line $\mathbb{R}\mathsf{v}$ divides it into two components. The mapping $\mathsf{x} \mapsto \mathsf{x} \boxtimes \mathsf{v}$ defines a diffeomorphism of $\mathbb{W}^+ - \mathbb{R}_+\mathsf{v}$ onto one component. We denote the closure of this component by

$$\mathcal{P}^+(\mathsf{v}) = \mathrm{Cl}\left((\mathbb{W}^+ - \mathbb{R}_+\mathsf{v}) \boxtimes \mathsf{v}\right)$$

The other component equals

$$\mathcal{P}^-(\mathsf{v}) = \mathrm{Cl}\left(\mathsf{v} \boxtimes (\mathbb{W}^+ - \mathbb{R}_+\mathsf{v})\right)$$

A nonzero vector $\mathsf{v} \in \mathbb{R}^{2,1}$ is called

- *spacelike* if $\mathbb{B}\,(\mathsf{v},\mathsf{v}) > 0$;
- *null* (or *lightlike*) if $\mathbb{B}\,(\mathsf{v},\mathsf{v}) = 0$;
- *timelike* if $\mathbb{B}\,(\mathsf{v},\mathsf{v}) < 0$;

A vector v is called *unit-spacelike* if and only if $\mathbb{B}\,(\mathsf{v},\mathsf{v}) = 1$ and *unit-timelike* if and only if $\mathbb{B}\,(\mathsf{v},\mathsf{v}) = -1$. If v is a spacelike vector, $\mathcal{P}(\mathsf{v}) \cap \mathbb{W}$ is the union of 2 null lines. There exists a unique pair

$$\mathbf{x}^-(\mathsf{v}), \mathbf{x}^+(\mathsf{v}) \in \mathcal{P}(\mathsf{v}) \cap \mathbb{W}^+$$

such that

$$|\mathbf{x}^-(\mathsf{v})| = |\mathbf{x}^+(\mathsf{v})| = 1 \tag{2.1.3}$$

and

$$\mathbb{B}\,(\mathsf{v}, \mathbf{x}^-(\mathsf{v}) \boxtimes \mathbf{x}^+(\mathsf{v})) > 0 \tag{2.1.4}$$

Condition (2.1.3) is a Euclidean normalization and condition (2.1.4) means that the ordered triple $(\mathsf{v}, \mathbf{x}^-(\mathsf{v}), \mathbf{x}^+(\mathsf{v}))$ is a positively-oriented basis. If v is spacelike, then

$$\mathbf{x}^-(\mathsf{v}) \boxtimes \mathbf{x}^+(\mathsf{v}) = \frac{2\mathbb{B}\,(\mathsf{v},\mathsf{v})}{\mathbb{B}\,(\mathsf{v},\mathsf{v}) + |\mathsf{v}|^2}\,\mathsf{v}$$

and

$$\mathsf{v} \boxtimes \mathbf{x}^+(\mathsf{v}) = -\mathbb{B}\,(\mathsf{v},\mathsf{v})^{1/2}\mathbf{x}^+(\mathsf{v})$$

Note that $\langle \mathsf{v}, \mathbf{x}^\pm(\mathsf{v})\rangle = \mathcal{P}(\mathbf{x}^\pm(\mathsf{v}))$ is the plane tangent to the null cone at $\mathbf{x}^\pm(\mathsf{v})$. For $\mathsf{x} \in \mathbb{W}^+$ the interior of $\mathcal{P}^+(\mathsf{x})$ is

$$\{\mathsf{v} \in \mathcal{P}(\mathsf{x}) \mid \mathbf{x}^+(\mathsf{v}) = \mathsf{x}/|\mathsf{x}|\}.$$

2.2. **Relation to hyperbolic geometry.** It will be useful to have a concrete model for the hyperbolic plane in this geometry as we will want to pass from fundamental polygons for discrete subgroups of $\mathbf{SO}^0(2,1)$ acting on $\mathbf{H}^2$ to fundamental polyhedra for the corresponding actions on Minkowski space. The hyperboloid defined by $\mathbb{B}\,(\mathsf{v},\mathsf{v}) = -1$ has an induced Riemannian metric of constant curvature -1. Its two components correspond to the two time-orientations; we denote its positively time-oriented component by $\mathbf{H}^2$. Equivalently, we may consider $\mathbf{H}^2$ as the image in the projective space $\mathbb{P}(\mathbb{R}^{2,1})$ of the open cone $\mathbb{U}^+$ of timelike vectors defined in (2.1.2) . Points of $\mathbf{H}^2$ correspond to timelike lines and geodesics in $\mathbf{H}^2$ to spacelike planes (polar to spacelike vectors). If $\mathsf{v}_1, \mathsf{v}_2$ are timelike vectors in $\mathbb{R}^{2,1}$, then the distance ρ between the corresponding points in $\mathbf{H}^2$ is given by:

$$\cosh(\rho) = \frac{|\mathbb{B}\,(\mathsf{v}_1,\mathsf{v}_2)|}{\sqrt{\mathbb{B}\,(\mathsf{v}_1,\mathsf{v}_1)\mathbb{B}\,(\mathsf{v}_2,\mathsf{v}_2)}}$$

Spacelike lines in $\mathbb{R}^{2,1}$ correspond to geodesics in $\mathbf{H}^2$. Suppose that $\mathsf{v} \in \mathbb{R}^{2,1}$ is spacelike. Then the plane $\mathcal{P}(\mathsf{v}) \subset \mathbb{R}^{2,1}$ meets Ω in timelike lines determining a geodesic in $\mathbf{H}^2$.

2.3. **Minkowski 2+1-space.** Consider (2+1)-*dimensional Minkowski space,* the affine space $\mathbb{E}$ corresponding to $\mathbb{R}^{2,1}$. Minkowski space is a simply connected complete flat Lorentzian manifold of dimension 2+1. Conversely any simply connected complete flat Lorentzian manifold of dimension 2+1 is isometric to Minkowski space. For $p, q \in \mathbb{E}$ the vector $\mathsf{v} = q - p$ corresponds to the translation carrying p to q. We also write $q = p + \mathsf{v}$. Denote by $\overline{pq}$ the line segment from p to q. Identify the vector space of translations of $\mathbb{E}$ with the original inner product space $\mathbb{R}^{2,1}$. (In particular the tangent space $T_p\mathbb{E}$ canonically identifies with $\mathbb{R}^{2,1}$.) Denote by

$$\mathbf{H}^0 = \mathbf{SO}^0(2,1) \ltimes \mathbb{R}^{2,1}$$

the identity component of the group of isometries of $\mathbb{E}$ and

$$\mathbf{H} = \mathbf{SO}(2,1) \ltimes \mathbb{R}^{2,1}$$

its (two-component) double cover. Denote by

$$\mathbb{L} : \mathbf{H} \to \mathbf{SO}(2,1)$$

the homomorphism assigning to $h \in \mathbf{H}$ its linear part $\mathbb{L}(h)$.

A geodesic in $\mathbb{E}$ is just a Euclidean straight line, parametrized by constant speed. Geodesics are described in the usual parametric form

$$l = p + \mathbb{R}\mathsf{v} = \{p + t\mathsf{v} \mid t \in \mathbb{R}\} \tag{2.3.1}$$

where p is a point on the line, $t \in \mathbb{R}$ a real parameter, and $\mathsf{v} \in \mathbb{R}^{2,1}$ a vector. l is *spacelike, timelike,* or *null* if and only if the *direction vector* v is spacelike, timelike, or null respectively.

If $l = p + \mathbb{R}\mathsf{v}$ is a line where v is spacelike or timelike, then there is a unique isometry ι_l of order two whose fixed-point set equals l. Explicitly,

$$\iota_l(p + \mathsf{x}) = p - \mathsf{x} + 2\frac{\mathbb{B}(\mathsf{x},\mathsf{v})}{\mathbb{B}(\mathsf{v},\mathsf{v})}\mathsf{v}.$$

We call ι_l the *inversion* in l.

3. Anatomy of a crooked plane

3.1. **Wings and stems.** Let $p \in \mathbb{E}$ be a point and $\mathsf{v} \in \mathbb{R}^{2,1}$ a spacelike vector. Define the *positively oriented crooked plane* $\mathcal{C}(\mathsf{v}, p) \subset \mathbb{E}$ with *vertex* p and *direction vector* v to be the union of two *wings*

$$\mathcal{W}^+(\mathsf{v}, p) = p + \mathcal{P}^+(\mathsf{x}^+(\mathsf{v})), \mathcal{W}^-(\mathsf{v}, p) = p + \mathcal{P}^+(\mathsf{x}^-(\mathsf{v}))$$

and a *stem*

$$\mathcal{S}(\mathsf{v}, p) = p + \ \{\mathsf{x} \in \mathbb{R}^{2,1} \mid \mathbb{B}(\mathsf{v},\mathsf{x}) = 0,\ \mathbb{B}(\mathsf{x},\mathsf{x}) \le 0\}.$$

Each wing is a half-plane, and the stem is the union of two quadrants in a spacelike plane. The positively oriented crooked plane itself is a piecewise linear submanifold, which stratifies into four connected open subsets of planes (two wings and the two components of the interior of the stem), four null rays, and a vertex. (See Figure 1.)

We also define the *negatively oriented crooked plane* $\mathcal{K}(\mathsf{v}, p) \subset \mathbb{E}$ with *vertex* p and *direction vector* v to be the union of two *wings*

$$\mathcal{M}^+(\mathsf{v}, p) = p + \mathcal{P}^-(\mathbf{x}^+(\mathsf{v})), \qquad \mathcal{M}^-(\mathsf{v}, p) = p + \mathcal{P}^-(\mathbf{x}^-(\mathsf{v}))$$

and a *stem*

$$\mathcal{S}(\mathsf{v}, p) = p + \ \{\mathsf{x} \in \mathbb{R}^{2,1} \mid \mathbb{B}\ (\mathsf{v}, \mathsf{x}) = 0,\ \mathbb{B}\ (\mathsf{x}, \mathsf{x}) \leq 0\}.$$

This paper contains calculations only for positively oriented half planes. Furthermore, an unspecified orientation for a crooked plane will assumed to be the positive one.

3.2. **Inversions and half-spaces.** A crooked plane of either orientation is a *polyhedral ruled surface:* its wings are foliated by parallel null lines and its stem decomposes into timelike lines all passing through the vertex. Furthermore the line l through the vertex orthogonal to the stem is the unique spacelike line contained in the crooked plane. Reminiscent of the structure of bisectors in complex hyperbolic space([10]), we call l the *spine* of the crooked plane and the null and timelike lines ruling the crooked plane its *slices.*

The spine of the positively oriented crooked plane $\mathcal{C}(\mathsf{v}, p)$ is

$$l = p + \mathbb{R}\mathsf{v}.$$

Given a spacelike line ℓ and a point $p \in \ell$, there is a unique positively oriented crooked plane with spine ℓ and vertex p. Inversion ι_l in l preserves $\mathcal{C}(\mathsf{v}, p)$, leaving invariant each wing, but permuting the two components of $\mathcal{W}^+(\mathsf{v}, p) - l$ (respectively, $\mathcal{W}^-(\mathsf{v}, p) - l$). Similarly, ι_l permutes the two components of $\mathcal{S}(\mathsf{v}, p) - \{p\}$.

Furthermore each component of the complement $\mathbb{E} - \mathcal{C}(\mathsf{v}, p)$ is a fundamental domain (a crooked half-space) for the group generated by ι_l. Given $p \in \mathbb{E}$ and a unit-spacelike vector v, we define the *crooked half-space* $\mathcal{H}(\mathsf{v}, p)$ to be the component of $\mathbb{E} - \mathcal{C}(\mathsf{v}, p)$ containing

$$p + \{\mathsf{u} \in \mathbb{W}^+ \mid \mathbb{B}\ (\mathsf{v}, \mathsf{u}) > 0\}.$$

Note that $\mathbb{E}$ is the disjoint union

$$\mathcal{H}(\mathsf{v}, p) \cup \mathcal{C}(\mathsf{v}, p) \cup \mathcal{H}(-\mathsf{v}, p).$$

Explicitly the half-space is defined as follows. Choose a point $p \in \mathbb{R}^{2,1}$ and a (spacelike) vector $\mathsf{v} \in \mathbb{R}^{2,1}$ such that $\langle \mathsf{v}, \mathsf{v} \rangle = 1$. Then there exist null vectors u_+, u_- in the positive light-cone defined by

$$x^2 + y^2 - z^2 = 0 \text{ and } z > 0$$

such that $\langle u_\pm, \mathsf{v}\rangle = 0$ and $\{\mathsf{v}, u_-, u_+\}$ is a positively oriented basis. Then the *crooked half-space* with vertex p and direction vector v is defined by:

$$\begin{aligned}\mathcal{H}(\mathsf{v}, p) = \{p + v \in \mathbb{R}^{2,1} \mid \langle u_-, v\rangle > 0 \quad &\text{if } \langle \mathsf{v}, v\rangle > 0,\\ \langle u_-, v\rangle\langle u_+, v\rangle > 0 \quad &\text{if } \langle \mathsf{v}, v\rangle = 0,\\ \langle u_+, v\rangle > 0 \quad &\text{if } \langle \mathsf{v}, v\rangle < 0\}\end{aligned}$$

and the corresponding *crooked plane* is its boundary $\partial\mathcal{H}(\mathsf{v}, p)$.

We have developed criteria for the intersection of these polyhedra $\mathcal{H}(\mathsf{v}, p)$ in terms of the parameters p, v. In [3], fundamental polyhedra for discontinuous groups Γ acting on $\mathbb{R}^{2,1}$ are derived from fundamental polygons for the action of the linear group $L(\Gamma) \subset \mathrm{SO}(2,1)$ on the hyperbolic plane. (The edges of the fundamental polygon are geodesics polar to the spacelike vectors v.) The spacelike geodesic

$$l = p + \mathbb{R}\mathsf{v}$$

lies on the crooked plane $\partial\mathcal{H}(\mathsf{v}, p)$. The crooked half-space $\mathcal{H}(\mathsf{v}, p)$ is a fundamental polyhedron for Lorentz-inversion in l.

Similarly, for each timelike slice s of $\mathcal{C}(\mathsf{v}, p)$, the inversion ι_s preserves $\mathcal{C}(\mathsf{v}, p)$. This inversion permutes the two wings but leaves invariant each component of the stem. The crooked half-spaces comprising the complement of $\mathcal{C}(\mathsf{v}, p)$ are fundamental domains for the cyclic group generated by the inversion. These properties hold as well for negatively oriented crooked planes and half-spaces.

3.3. **Automorphisms.** Let $C = \mathcal{C}(\mathsf{v}, p)$ be a crooked plane. Let ϕ be an automorphism of $\mathbb{R}^{2,1}$ which preserves C. The vertex p of C is fixed by ϕ. The differentiable structure on C is singular at p and along the two null-lines $p + \mathbb{R}\mathbf{x}^\pm(\mathsf{v})$. However p is the only singular point for which the singularity is nonremovable (it has a nonsingular *intrinsic* differentiable structure). Thus $\phi(p) = p$. The spine l of C is the unique spacelike line contained in C; hence $\phi(l) = l$. Conversely every isometry of $\mathbb{R}^{2,1}$ which preserves l preserves C.

3.4. **A specific example.** Let p be the origin and let

$$\mathsf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Then

$$\mathbf{x}^-(\mathsf{v}) = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\1\end{bmatrix}, \qquad \mathbf{x}^+(\mathsf{v}) = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\-1\\1\end{bmatrix}$$

and

$$\begin{aligned}
\mathcal{W}^-(\mathsf{v},p) &= \left\{ \begin{bmatrix}x\\y\\y\end{bmatrix} \middle| \; x \leq 0 \right\}, \\
\mathcal{S}(\mathsf{v},p) &= \left\{ \begin{bmatrix}0\\y\\z\end{bmatrix} \middle| \; |y| \leq |z| \right\}, \\
\mathcal{W}^+(\mathsf{v},p) &= \left\{ \begin{bmatrix}x\\-y\\y\end{bmatrix} \middle| \; x \geq 0 \right\}.
\end{aligned}$$

Also,

$$\begin{aligned}
\mathcal{M}^-(\mathsf{v},p) &= \left\{ \begin{bmatrix}x\\y\\y\end{bmatrix} \middle| \; x \geq 0 \right\}, \\
\mathcal{M}^+(\mathsf{v},p) &= \left\{ \begin{bmatrix}x\\-y\\y\end{bmatrix} \middle| \; x \leq 0 \right\}.
\end{aligned}$$

The spine of the crooked planes $\mathcal{C}(\mathsf{v},p)$ and $\mathcal{K}(\mathsf{v},p)$ is the line:

$$\sigma = \left\{ \begin{bmatrix}x\\0\\0\end{bmatrix} \middle| \; x \in \mathbb{R} \right\}$$

with inversion:

$$\begin{bmatrix}x\\y\\z\end{bmatrix} \longmapsto \begin{bmatrix}x\\-y\\-z\end{bmatrix}.$$

A timelike line contained in $\mathcal{S}(\mathsf{v},p)$ is given by:

$$\tau_t = \left\{ \begin{bmatrix}0\\s\sinh(t)\\s\cosh(t)\end{bmatrix} \middle| \; s \in \mathbb{R} \right\}$$

and the corresponding inversion given by the matrix:

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cosh(2t) & \sinh(2t) \\ 0 & -\sinh(2t) & \cosh(2t) \end{bmatrix}.$$

The isometry group of C is the image of the embedding

$$\mathbf{O}(1,1) \longrightarrow \mathbf{SO}(2,1) \subset \mathbf{H}$$

$$A \mapsto \begin{bmatrix} A & 0 \\ 0 & \det(A) \end{bmatrix}.$$

4. Intersections of wings

4.1. **Intersection of planes.** This section describes the intersection of two wings, and a wing and a stem. The following section describes intersections of stems. From these results follow necessary and sufficient conditions for the intersection of two crooked planes. Since every stem and wing lies in a plane in $\mathbb{E}$, every intersection of a stem or wing with a stem or wing will lie in the intersection of the planes which contain them.

A plane in $\mathbb{E}$ may be written as $p + \mathcal{P}(\mathsf{v})$, for $p \in \mathbb{E}$ and $\mathsf{v} \in \mathbb{R}^{2,1}$. Suppose that $p_1, p_2 \in \mathbb{E}$ and $\mathsf{v}_1, \mathsf{v}_2 \in \mathbb{R}^{2,1}$ satisfy $\mathbb{B}\,(\mathsf{v}_1, \mathsf{v}_2) \neq 0$. The intersection $\mathfrak{P}_1 \cap \mathfrak{P}_2$ of the two planes $\mathfrak{P}_i = p_i + \mathcal{P}(\mathsf{v}_i)$ is a line which can be parametrized as

$$p + \mathbb{R}(\mathsf{v}_1 \boxtimes \mathsf{v}_2) = \{p + k\mathsf{v}_1 \boxtimes \mathsf{v}_2 \mid k \in \mathbb{R}\}$$

for some $p \in \mathfrak{P}_1 \cap \mathfrak{P}_2$. If $\mathbb{B}\,(\mathsf{v}_1, \mathsf{v}_2) = 0$ and v_1 and v_2 are linearly independent then $\mathfrak{P}_1 \cap \mathfrak{P}_2$ can be parametrized as $p + \mathbb{R}\mathsf{v}_1 \boxtimes \mathsf{v}_2$.

4.2. **Intersections of wings.** We now describe the intersection of two null half-planes that are the wings of crooked planes. We need only consider positively oriented null half-planes. The generic case — non-parallel null half-planes — is treated first. Lemma 4.2.1 asserts that the intersection is either empty, a point, or a spacelike line segment. Conversely, every spacelike line segment arises uniquely as the intersection of positively oriented null half-planes (Theorem 4.3.1). The degenerate case of parallel null half-planes is treated in Lemma 4.4.1, where it is shown that either the intersection is either empty or one half-plane contains the other. §4.5 treats intersections of stems and null half-planes.

Lemma 4.2.1. *Let $\mathsf{x}_1, \mathsf{x}_2 \in \mathbb{W}^+$ be linearly independent null vectors and $p_1, p_2 \in \mathbb{E}$ be points. The corresponding null half-planes $\mathfrak{p}_1^+ = p_1 + \mathcal{P}^+(\mathsf{x}_1)$ and $\mathfrak{p}_2^+ = p_2 + \mathcal{P}^+(\mathsf{x}_2)$ are disjoint if and only if $\mathbb{B}\,(p_2 - p_1, \mathsf{x}_1 \boxtimes \mathsf{x}_2) > 0$. In case that $\mathfrak{p}_1^+ \cap \mathfrak{p}_2^+ \neq \emptyset$:*

- $\mathfrak{p}_1^+ \cap \mathfrak{p}_2^+$ *is a point if and only if* $\mathbb{B}\,(p_2 - p_1, \mathsf{x}_1 \boxtimes \mathsf{x}_2) = 0$.
- $\mathfrak{p}_1^+ \cap \mathfrak{p}_2^+$ *is a spacelike line segment if and only if* $\mathbb{B}\,(p_2 - p_1, \mathsf{x}_1 \boxtimes \mathsf{x}_2) < 0$, *with one endpoint on* $\partial\mathfrak{p}_1^+$ *and the other endpoint on* $\partial\mathfrak{p}_2^+$.

Proof. Let ℓ be the intersection of the planes that contain $\mathfrak{p}_1^+$ and $\mathfrak{p}_2^+$. Then $\mathfrak{p}_1^+ \cap \mathfrak{p}_2^+ \subset \ell$. Let $\mathsf{w} = \mathsf{x}_1 \boxtimes \mathsf{x}_2$ and

$$p = \ell \cap (p_1 + P(\mathsf{w})$$

so that

$$\ell = p + \mathbb{R}\mathsf{w}.$$

The subsets $\ell \cap \mathfrak{p}_1^+$ and $\ell \cap \mathfrak{p}_2^+$ of ℓ are characterized as the set of all $p + k\mathsf{w}$ (where $k \in \mathbb{R}$) satisfying the inequalities

$$\mathbb{B}\,((p + k\mathsf{w}) - p_1, -\mathsf{w}) \geq 0,$$

$$\mathbb{B}\,((p + k\mathsf{w}) - p_2, \mathsf{w}) \geq 0,$$

respectively. p was chosen so that $\mathbb{B}\,(p - p_1, \mathsf{w}) = 0$ and for the given parametrization of ℓ

$$\ell \cap \mathfrak{p}_1^+ \longleftrightarrow (-\infty, 0]$$

$$\ell \cap \mathfrak{p}_2^+ \longleftrightarrow \left[\frac{\mathbb{B}\,(p_2 - p_1, \mathsf{w})}{\mathbb{B}\,(\mathsf{w}, \mathsf{w})}, \infty\right).$$

Then:

- $\mathfrak{p}_1^+ \cap \mathfrak{p}_2^+$ is empty if and only if $\mathbb{B}\,(p_2 - p_1, \mathsf{w}) > 0$;
- $\mathfrak{p}_1^+ \cap \mathfrak{p}_2^+$ is a point if and only if $\mathbb{B}\,(p_2 - p_1, \mathsf{w}) = 0$;
- $\mathfrak{p}_1^+ \cap \mathfrak{p}_2^+$ is a line segment if and only if $\mathbb{B}\,(p_2 - p_1, \mathsf{w}) < 0$.

The proof is complete. □

Here are the corresponding lemmas for pairs of negatively oriented half-planes and pairs of half-planes with opposite orientations:

Lemma 4.2.2. *Let* $\mathsf{x}_1, \mathsf{x}_2 \in \mathbb{W}^+$ *be linearly independent null vectors and* $p_1, p_2 \in \mathbb{E}$ *be points. The half-planes*

$$\mathfrak{p}_1^- = p_1 + \mathcal{P}^-(\mathsf{x}_1), \quad \mathfrak{p}_2^- = p_2 + \mathcal{P}^-(\mathsf{x}_2)$$

are disjoint if and only if $\mathbb{B}\,(p_2 - p_1, \mathsf{x}_1 \boxtimes \mathsf{x}_2) < 0$.

Lemma 4.2.3. *Let* $\mathsf{x}_1, \mathsf{x}_2 \in \mathbb{W}^+$ *be linearly independent null vectors and* $p_1, p_2 \in \mathbb{E}$ *be points. The intersection of the half-planes* $\mathfrak{p}^+ = p_1 + \mathcal{P}^+(\mathsf{x}_1)$ *and* $\mathfrak{p}^- = p_2 + \mathcal{P}^-(\mathsf{x}_2)$ *is never empty.*

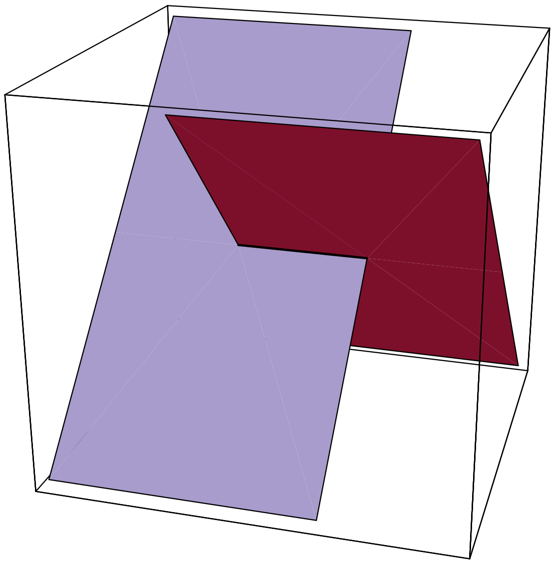

FIGURE 2. An intersection of two null half-planes

Proof. The line ℓ equals the intersection of the planes that contain $\mathfrak{p}^+$ and $\mathfrak{p}^-$. Let $\mathsf{w} = \mathsf{x}_1 \boxtimes \mathsf{x}_2$ and $p = \ell \cap (p_1 + P(\mathsf{w})$ so that $\ell = p + \mathbb{R}\mathsf{w}$. The subsets $\ell \cap \mathfrak{p}^+$ and $\ell \cap \mathfrak{p}^-$ of ℓ consist of all $p + k\mathsf{w}$ (where $k \in \mathbb{R}$) satisfying the inequalities

$$\mathbb{B}\left((p + k\mathsf{w}) - p_1, -\mathsf{w}\right) \geq 0,$$

$$\mathbb{B}\left((p + k\mathsf{w}) - p_2, \mathsf{w}\right) \leq 0,$$

respectively. The point p was chosen so that $\mathbb{B}\left(p - p_1, \mathsf{w}\right) = 0$ and for the given parametrization of ℓ

$$\ell \cap \mathfrak{p}^+ \longleftrightarrow (-\infty, 0]$$

$$\ell \cap \mathfrak{p}^- \longleftrightarrow \left(-\infty, \frac{\mathbb{B}\left(p_2 - p_1, \mathsf{w}\right)}{\mathbb{B}\left(\mathsf{w}, \mathsf{w}\right)}\right].$$

Thus $\mathfrak{p}^+ \cap \mathfrak{p}^-$ is never empty. ☐

4.3. **Representation of spacelike segments.** Every spacelike line segment arises as the intersection of two positively (or negatively) oriented half-planes, and this representation is unique up to the choice of orientation. A null half-plane $\mathcal{P}^+$ or $\mathcal{P}^-$ is uniquely determined by its orientation and boundary $l = \partial\mathcal{P}^+ = \partial\mathcal{P}^-$, which is a null line.

Corollary 4.3.1. *Let σ be a spacelike line segment. Then there exist pairs of disjoint null lines (l_1, l_2) and (k_1, k_2) such that*

$$\sigma = \mathfrak{p}_1^+ \cap \mathfrak{p}_2^+ = \mathfrak{p}_1^- \cap \mathfrak{p}_2^-$$

where $\mathfrak{p}_i^+$ is the unique positively oriented null half-plane such that $l_i = \partial\mathcal{P}^+$ and $\mathfrak{p}_i^-$ is the unique negatively oriented null half-plane such that $k_i = \partial\mathcal{P}^-$. Furthermore (l_1, l_2) and (k_1, k_2) are uniquely determined by σ.

Proof. We shall only consider positively oriented half-planes. Let the endpoints of σ be p_1 and p_2 and let $\mathsf{v} = p_2 - p_1$. Take $l_1 = p_1 + \mathbb{R}\mathbf{x}^+(\mathsf{v})$ and $l_2 = p_2 + \mathbb{R}\mathbf{x}^-(\mathsf{v})$. Since $\mathsf{v} \in \mathcal{P}^+(\mathbf{x}^+(\mathsf{v}))$ and $-\mathsf{v} \in \mathcal{P}^+(\mathbf{x}^-(\mathsf{v}))$,

$$\{p_1 + k\mathsf{v} | 0 \leq k \leq 1\}$$

is the line segment σ.

Conversely, if $\mathfrak{p}_1^+ \cap \mathfrak{p}_2^+ = \sigma$, then $\partial\mathfrak{p}_1^+$ is parallel to $\mathbf{x}^+(\mathsf{v})$ and $\partial\mathfrak{p}_2^+$ is parallel to $\mathbf{x}^-(\mathsf{v})$. By Lemma 4.2.1, the endpoints of σ lie on $\partial\mathfrak{p}_1^+$ and $\partial\mathfrak{p}_2^+$ respectively. This completes the proof of Corollary 4.3.1. ☐

Given a pair l_1, l_2 of null lines, the segment of intersection of the corresponding null half-planes has the following geometric interpretation. The function

$$\begin{aligned} B_{l_1,l_2} : l_1 \times l_2 &\longrightarrow \mathbb{R} \\ (q_1, q_2) &\longmapsto \mathbb{B}\,(q_1 - q_2, q_1 - q_2) \end{aligned}$$

is a quadratic function whose leading term (its Hessian) is indefinite. Let $(p_1, p_2) \in l_1 \times l_2$ be its unique critical point. Then p_1 and p_2 are the endpoints of $\sigma = \mathfrak{p}_1^+ \cap \mathfrak{p}_2^+$.

Here is a formula for these endpoints. Let $q_i \in l_i$ be points and v_i be null vectors parallel to l_i. (Thus $l_i = q_i + \mathbb{R}\mathsf{v}_i$.) The endpoints p_1, p_2 of σ are given by:

$$\begin{aligned} p_1 &= q_1 + \frac{\mathbb{B}\,(q_2 - q_1, \mathsf{v}_2)}{\mathbb{B}\,(\mathsf{v}_1, \mathsf{v}_2)}\, \mathsf{v}_1 \\ p_2 &= q_2 + \frac{\mathbb{B}\,(q_1 - q_2, \mathsf{v}_1)}{\mathbb{B}\,(\mathsf{v}_2, \mathsf{v}_1)}\, \mathsf{v}_2. \end{aligned}$$

4.4. **Parallel null half-planes.** Now consider the intersection of null half-planes for the degenerate case when the null half-planes are parallel:

Lemma 4.4.1. *Let $\mathsf{x} \in \mathbb{W}^+$ be a null vector and $p_1, p_2 \in \mathbb{E}$ be points. Let $\mathfrak{p}_i^+$ denote the half-plane $p_i + \mathcal{P}^+(\mathsf{x})$ and $\mathfrak{p}_i^-$ denote the half-plane $p_i + \mathcal{P}^-(\mathsf{x})$, respectively, for $i = 1, 2$. Then:*

- $\mathfrak{p}_1^+$ *and* $\mathfrak{p}_2^+$ *are disjoint if and only if* $\mathbb{B}\,(p_2 - p_1, \mathsf{x}) \neq 0$*;*

- $\mathfrak{p}_1^-$ *and* $\mathfrak{p}_2^-$ *are disjoint if and only if* $\mathbb{B}\,(p_2 - p_1, \mathsf{x}) \neq 0$*;*

Otherwise, one of the half-planes contains the similarly oriented half-plane and their intersection is a null half-plane.

Proof. We shall only consider positively oriented null half-planes. The intersection $\mathfrak{p}_1^+ \cap \mathfrak{p}_2^+$ lies in

$$(\mathcal{P}(\mathsf{x}) + p_1) \cap (\mathcal{P}(\mathsf{x}) + p_2),$$

which is empty if and only if $\mathbb{B}\,(p_2 - p_1, \mathsf{x}) \neq 0$.

Suppose that $\mathbb{B}\,(p_2 - p_1, \mathsf{x}) = 0$, that is, $p_2 - p_1$ lies in $\mathcal{P}^+(\mathsf{x})$. Choose any spacelike $\mathsf{v} \in \mathcal{P}(\mathsf{x})$. If $\mathbb{B}\,(p_2 - p_1, \mathsf{v}) \geq 0$ then $\mathfrak{p}_1^+ \supset \mathfrak{p}_2^+$. If $\mathbb{B}\,(p_2 - p_1, \mathsf{v}) \leq 0$ then $\mathfrak{p}_1^+ \subset \mathfrak{p}_2^+$. □

4.5. **Intersection of stems and null half-planes.** Next we analyze when a positively oriented null half-plane intersects a stem:

Lemma 4.5.1. *Let* x *be a positively oriented null vector and* v *a unit-spacelike vector such that* $\mathbb{B}\,(\mathsf{x}, \mathsf{v}) < 0$*. Let* $p_1, p_2 \in \mathbb{E}$ *be points. The stem* $S_1 = \mathcal{S}(\mathsf{v}, p_1)$ *and the positively oriented null half-plane* $\mathfrak{p}_2^+ = p_2 + \mathcal{P}^+(\mathsf{x})$ *are disjoint if and only if*

$$\mathbb{B}\,(p_2 - p_1, \mathsf{v} \boxtimes \mathsf{x}) > |\mathbb{B}\,(p_2 - p_1, \mathsf{x})|. \tag{4.5.1}$$

In the case that $S_1 \cap \mathfrak{p}_2^+ \neq \emptyset$ *then:*

- $S_1 \cap \mathfrak{p}_2^+$ *consists of a point (which lies on* ∂S_1 *but not on* $\partial \mathfrak{p}_2^+$*) if and only if:*

$$\mathbb{B}\,(p_2 - p_1, \mathsf{v} \boxtimes \mathsf{x}) < 0 = \mathbb{B}\,(p_2 - p_1, \mathsf{x}). \tag{4.5.2}$$

- $S_1 \cap \mathfrak{p}_2^+$ *consists of a point (which lies on* ∂S_1 *and on* $\partial \mathfrak{p}_2^+$*) if and only if:*

$$\mathbb{B}\,(p_2 - p_1, \mathsf{v} \boxtimes \mathsf{x}) = |\mathbb{B}\,(p_2 - p_1, \mathsf{x})|. \tag{4.5.3}$$

- $S_1 \cap \mathfrak{p}_2^+$ *consists of a spacelike line segment (with at least one vertex on* ∂S_1*) if and only if:*

$$\mathbb{B}\,(p_2 - p_1, \mathsf{v} \boxtimes \mathsf{x}) < |\mathbb{B}\,(p_2 - p_1, \mathsf{x})|, \tag{4.5.4}$$

  where the other endpoint lies on $\partial \mathfrak{p}_2^+$ *if and only if:*

$$\mathbb{B}\,(p_2 - p_1, \mathsf{v} \boxtimes \mathsf{x}) \geq -|\mathbb{B}\,(p_2 - p_1, \mathsf{x})|. \tag{4.5.5}$$

  and on ∂S_1 *if and only if*

$$\mathbb{B}\,(p_2 - p_1, \mathsf{v} \boxtimes \mathsf{x}) \leq -|\mathbb{B}\,(p_2 - p_1, \mathsf{x})|. \tag{4.5.6}$$

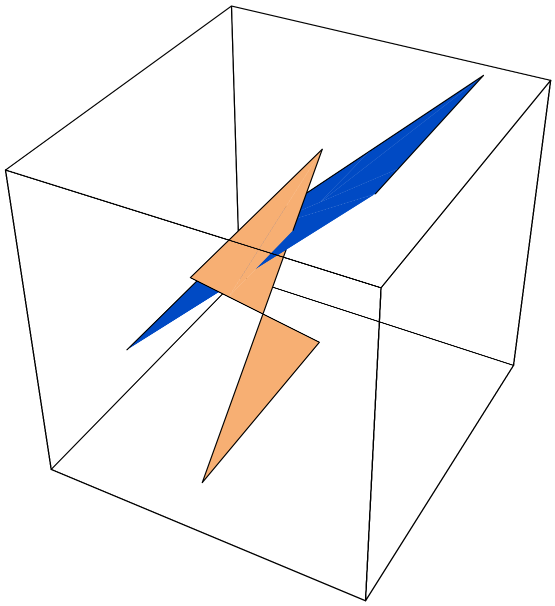

FIGURE 3. Wing intersecting a stem (one endpoint on stem)

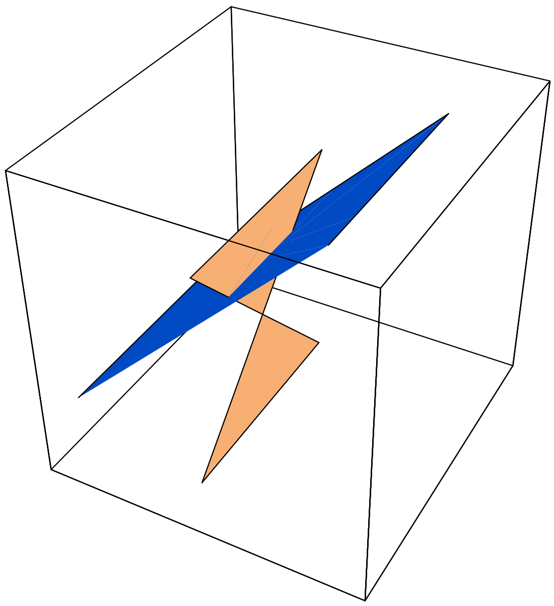

FIGURE 4. Wing intersecting a stem (both endpoints on stem)

Figure 3 shows a null half-plane intersecting a stem of a crooked plane where the segment of intersection has one endpoint on ∂S_1 and the other on $\partial\mathfrak{p}_2^+$, corresponding to (4.5.5) above. Figure 4 shows a null half-plane intersecting a stem of a crooked plane where the segment of intersection has both endpoints on $\partial\mathfrak{p}_2^+$, corresponding to (4.5.6) above.

Proof. Let ℓ be the intersection of the plane containing S_1 and $\mathfrak{p}_2^+$. Let $\mathsf{w} = \mathsf{v} \boxtimes \mathsf{x}$ and

$$p = \ell \cap (p_1 + \mathcal{P}(\mathsf{w})).$$

Parametrize ℓ as $p + \mathbb{R}\mathsf{w}$. The subsets $\ell \cap S_1$ and $\ell \cap \mathfrak{p}_2^+$ of ℓ consist of all $p + k\mathsf{w}$ (where $k \in \mathbb{R}$) satisfying the inequalities

$$\mathbb{B}\left((p + k\mathsf{w}) - p_1, (p + k\mathsf{w}) - p_1\right) \leq 0, \tag{4.5.7}$$

$$\mathbb{B}\left((p + k\mathsf{w}) - p_2, \mathsf{w}\right) \geq 0, \tag{4.5.8}$$

respectively. (4.5.7) is quadratic in k where the coefficient of k^2 is $\mathbb{B}(\mathsf{w},\mathsf{w}) > 0$, so that $\ell \cap S_1$ corresponds to the interval with endpoints

$$\frac{\pm\sqrt{-\mathbb{B}(\mathsf{w},\mathsf{w})\mathbb{B}(p-p_1, p-p_1)}}{\mathbb{B}(\mathsf{w},\mathsf{w})}.$$

Because $\mathbb{B}(p - p_1, \mathsf{w}) = 0$, the vector $p - p_1$ is a linear combination of v and x . Solving $\mathbb{B}(p - p_1, \mathsf{v}) = 0$ and $\mathbb{B}(p - p_2, \mathsf{x}) = 0$ for the coefficients of v and x,

$$p = p_1 + \frac{-\mathbb{B}(p_2 - p_1, \mathsf{x})}{\mathbb{B}(\mathsf{v},\mathsf{x})^2}\mathsf{x} + \frac{\mathbb{B}(p_2 - p_1, \mathsf{x})}{\mathbb{B}(\mathsf{v},\mathsf{x})}\mathsf{v}. \tag{4.5.9}$$

Using (4.5.9) and $\mathbb{B}(\mathsf{w},\mathsf{w}) = \mathbb{B}(\mathsf{v},\mathsf{x})^2$,

$$\ell \cap S_1 \longleftrightarrow \left[\frac{-|\mathbb{B}(p_2 - p_1, \mathsf{x})|}{\mathbb{B}(\mathsf{w},\mathsf{w})}, \frac{|\mathbb{B}(p_2 - p_1, \mathsf{x})|}{\mathbb{B}(\mathsf{w},\mathsf{w})}\right].$$

Solving (4.5.8) for k and using the assumption that $\mathbb{B}(p - p_1, \mathsf{w}) = 0$,

$$\ell \cap \mathfrak{p}_2^+ \longleftrightarrow \left[\frac{\mathbb{B}(p_2 - p_1, \mathsf{w})}{\mathbb{B}(\mathsf{w},\mathsf{w})}, \infty\right).$$

The subset $S_1 \cap \mathfrak{p}_2^+$ of ℓ is empty if and only if

$$\mathbb{B}(p_2 - p_1, \mathsf{w}) > |\mathbb{B}(p_2 - p_1, \mathsf{x})|.$$

Otherwise $S_1 \cap \mathfrak{p}_2^+$ corresponds to the interval

$$\left[\max\left(\frac{\mathbb{B}(p_2 - p_1, \mathsf{w})}{\mathbb{B}(\mathsf{w},\mathsf{w})}, \frac{-|\mathbb{B}(p_2 - p_1, \mathsf{x})|}{\mathbb{B}(\mathsf{w},\mathsf{w})}\right), \frac{|\mathbb{B}(p_2 - p_1, \mathsf{x})|}{\mathbb{B}(\mathsf{w},\mathsf{w})}\right].$$

This concludes the proof of Lemma 4.5.1. $\square$

Corresponding to the intersection of a stem and a negatively oriented wing is:

Lemma 4.5.2. *Let* x *be a positively oriented null vector and* v *a unit-spacelike vector such that* $\mathbb{B}(\mathsf{x},\mathsf{v}) < 0$*. Let* $p_1, p_2 \in \mathbb{E}$ *be points. The stem* $S_1 = \mathcal{S}(\mathsf{v}, p_1)$ *and the negatively oriented null half-plane* $\mathfrak{p}_2^- = p_2 + \mathcal{P}^-(\mathsf{x})$ *are disjoint if and only if*

$$\mathbb{B}(p_2 - p_1, \mathsf{v} \boxtimes \mathsf{x}) < |\mathbb{B}(p_2 - p_1, \mathsf{x})| \tag{4.5.10}$$

The final case occurs when the null half-plane is orthogonal to the stem:

Lemma 4.5.3. *Let* $\mathsf{x} \in \mathbb{W}^+$ *and* v *be a unit-spacelike vector such that* $\mathbb{B}(\mathsf{x},\mathsf{v}) = 0$*. Suppose* $p_1, p_2 \in \mathbb{E}$*. The positively oriented null half-plane* $\mathfrak{p}_1^+ = p_1 + \mathcal{P}^+(\mathsf{x})$ *and the stem* $S_2 = \mathcal{S}(\mathsf{v}, p_2)$ *are disjoint if and only if:*

- $\mathbb{B}(p_2 - p_1, \mathsf{v}) > 0$ *when* x *is a multiple of* $\mathbf{x}^-(\mathsf{v})$
- $\mathbb{B}(p_2 - p_1, \mathsf{v}) < 0$ *when* x *is a multiple of* $\mathbf{x}^+(\mathsf{v})$*.*

Otherwise $S_1 \cap \mathfrak{p}_2^+$ *is a ray with endpoint on* $\partial\mathfrak{p}_2^+$ *(except if* $\mathbb{B}(p_2 - p_1, \mathsf{x}) = 0$*, in which case it is an entire line).*

Proof. We only discuss the case when x is a multiple of $\mathbf{x}^+(\mathsf{v})$. Let ℓ be the intersection of the planes containing $\mathfrak{p}_1^+$ and S_2. Let

$$p = \ell \cap (p_1 + \mathcal{P}(\mathbf{x}^-(\mathsf{v}))).$$

Then $\ell = p + \mathbb{R}\mathsf{x}$. The subsets $\ell \cap \mathfrak{p}_1^+$ and $\ell \cap S_2$ of ℓ consist of all $p + k\mathsf{x}$ (where $k \in \mathbb{R}$) satisfying the inequalities

$$\mathbb{B}((p + k\mathsf{x}) - p_1, \mathsf{v}) \geq 0, \tag{4.5.11}$$

$$\mathbb{B}((p + k\mathsf{x}) - p_2, (p + k\mathsf{x}) - p_2) \leq 0, \tag{4.5.12}$$

respectively. (4.5.11) and (4.5.12) simplify to

$$\mathbb{B}(p - p_1, \mathsf{v}) \geq 0,$$

$$2k\mathbb{B}(p - p_2, \mathsf{x}) + \mathbb{B}(p - p_2, p - p_2) \leq 0.$$

Write $p - p_1$ and $-p - p_2$ as linear combinations of $\mathbf{x}^-(\mathsf{v})$, $\mathbf{x}^+(\mathsf{v})$, and v. Solving

$$\begin{cases} \mathbb{B}(p - p_1, \mathbf{x}^-(\mathsf{v})) = 0 \\ \mathbb{B}(p - p_1, \mathsf{x}) = 0 \\ \mathbb{B}(p - p_2, \mathsf{v}) = 0 \end{cases}$$

for the coefficients of $\mathbf{x}^-(\mathsf{v})$, $\mathbf{x}^+(\mathsf{v})$, and v yields

$$p = p_1 + \mathbb{B}(p_2 - p_1, \mathsf{v})\mathsf{v}, \tag{4.5.13}$$

$$p = p_2 + \frac{\mathbb{B}(p_1 - p_2, \mathbf{x}^-(\mathsf{v}))}{\mathbb{B}(\mathsf{x}, \mathbf{x}^-(\mathsf{v}))}\mathsf{x} + \frac{\mathbb{B}(p_1 - p_2, \mathsf{x})}{\mathbb{B}(\mathsf{x}, \mathbf{x}^-(\mathsf{v}))}\mathbf{x}^-(\mathsf{v}). \tag{4.5.14}$$

Using (4.5.13) and (4.5.14), inequalities (4.5.11) and (4.5.12) further simplify to

$$\mathbb{B}\,(p_2 - p_1, \mathsf{v}) \geq 0,$$

$$2k\mathbb{B}\,(p_1 - p_2, \mathbf{x}^-(\mathsf{v})) \leq \frac{-2\mathbb{B}\,(p_1 - p_2, \mathsf{x})\mathbb{B}\,(p_1 - p_2, \mathbf{x}^-(\mathsf{v})}{\mathbb{B}\,(\mathsf{x}, \mathbf{x}^-(\mathsf{v}))},$$

respectively.

$$\ell \cap \mathfrak{p}_1^+ \longleftrightarrow \begin{cases} \mathbb{R} & \text{if } \mathbb{B}\,(p_2 - p_1, \mathsf{v}) \geq 0, \\ \emptyset & \text{if } \mathbb{B}\,(p_2 - p_1, \mathsf{v}) < 0 \end{cases}$$

$$\ell \cap S_2 \longleftrightarrow \begin{cases} \mathbb{R} & \text{if } \mathbb{B}\,(p_1 - p_2, \mathbf{x}^-(\mathsf{v})) = 0, \\ \left[-2\frac{\mathbb{B}\,(p_1 - p_2, \mathsf{x})}{\mathbb{B}\,(\mathsf{x}, \mathbf{x}^-(\mathsf{v}))},\ \infty\right) & \text{if } \mathbb{B}\,(p_1 - p_2, \mathbf{x}^-(\mathsf{v})) < 0, \\ \left(-\infty, -2\frac{\mathbb{B}\,(p_1 - p_2, \mathsf{x})}{\mathbb{B}\,(\mathsf{x}, \mathbf{x}^-(\mathsf{v}))}\right] & \text{if } \mathbb{B}\,(p_1 - p_2, \mathbf{x}^-(\mathsf{v})) > 0. \end{cases}$$

$S_2 \cap \ell$ is always nonempty, so that $\mathfrak{p}_1^+ \cap S_2$ is empty if and only if $\ell \cap \mathfrak{p}_1^+$ is empty. If $\mathfrak{p}_1^+ \cap S_2$ is nonempty, then $\mathfrak{p}_1^+ \cap S_2 = S_2 \cap \ell$. □

The corresponding lemma for negatively oriented null half-planes is:

Lemma 4.5.4. *Let $\mathsf{x} \in \mathbb{W}^+$ and v be a unit-spacelike vector such that $\mathbb{B}\,(\mathsf{x}, \mathsf{v}) = 0$. Suppose $p_1, p_2 \in \mathbb{E}$. The negatively oriented null half-plane $\mathfrak{p}_1^- = p_1 + \mathcal{P}^-(\mathsf{x})$ and the stem $S_2 = \mathcal{S}(\mathsf{v}, p_2)$ are disjoint if and only if:*

- $\mathbb{B}\,(p_2 - p_1, \mathsf{v}) < 0$ *when x is a multiple of $\mathbf{x}^-(\mathsf{v})$;*
- $\mathbb{B}\,(p_2 - p_1, \mathsf{v}) > 0$ *when x is a multiple of $\mathbf{x}^+(\mathsf{v})$.*

The preceding lemmas imply the following qualitative result about intersections of crooked planes:

Corollary 4.5.5. *Let C_1 and C_2 be two crooked planes. Suppose that the three planes containing the stems and wings of C_1 are each transverse to the three planes containing the stems and wings of C_2. If a wing W of C_1 intersects the stem of C_2, then W intersects one of the wings of C_2.*

5. Intersection of stems

5.1. **Pairs of spacelike vectors.** Consider crooked planes $\mathcal{C}(\mathsf{v}_1, p_1)$ and $\mathcal{C}(\mathsf{v}_2, p_2)$, where $p_1, p_2 \in \mathbb{E}$ are points and $\mathsf{v}_1, \mathsf{v}_2 \in \mathbb{R}^{2,1}$ are unit-spacelike vectors. If the vectors $\mathsf{v}_1, \mathsf{v}_2$ satisfy one of the following equivalent conditions, they are said to be *ultraparallel:*

- $\mathcal{P}(\mathsf{v}_1)$ and $\mathcal{P}(\mathsf{v}_2)$ correspond to ultraparallel geodesics in $\mathbf{H}^2$;

- $\langle \mathsf{v}_1, \mathsf{v}_2\rangle \subset \mathbb{R}^{2,1}$ is an indefinite plane (a secant plane to $\mathbb{W}$);
- $\mathbb{B}\,(\mathsf{v}_1, \mathsf{v}_2)^2 > \mathbb{B}\,(\mathsf{v}_1, \mathsf{v}_1)\mathbb{B}\,(\mathsf{v}_2, \mathsf{v}_2)$;
- $\mathcal{P}(\mathsf{v}_1) \cap \mathcal{P}(\mathsf{v}_2)$ is a spacelike line;
- There exists a unique spacelike line orthogonal to v_1 and v_2;
- $\mathbb{B}\,(\mathsf{v}_1 \boxtimes \mathsf{v}_2, \mathsf{v}_1 \boxtimes \mathsf{v}_2) > 0$

If vectors $\mathsf{v}_1, \mathsf{v}_2$ satisfy one of the following equivalent conditions, they are said to be *asymptotic:*

- $\mathcal{P}(\mathsf{v}_1)$ and $\mathcal{P}(\mathsf{v}_2)$ correspond to asymptotic geodesics in $\mathbf{H}^2$;
- $\langle \mathsf{v}_1, \mathsf{v}_2\rangle \subset \mathbb{R}^{2,1}$ is a degenerate (null) plane (a plane tangent to $\mathbb{W}$);
- $\mathbb{B}\,(\mathsf{v}_1, \mathsf{v}_2)^2 = \mathbb{B}\,(\mathsf{v}_1, \mathsf{v}_1)\mathbb{B}\,(\mathsf{v}_2, \mathsf{v}_2)$;
- $\mathcal{P}(\mathsf{v}_1) \cap \mathcal{P}(\mathsf{v}_2)$ is a null line;
- $\mathbb{B}\,(\mathsf{v}_1 \boxtimes \mathsf{v}_2, \mathsf{v}_1 \boxtimes \mathsf{v}_2) = 0$

If $\mathsf{v}_1, \mathsf{v}_2$ satisfy one of the following equivalent conditions, we say that $\mathsf{v}_1, \mathsf{v}_2$ *cross:*

- $\mathcal{P}(\mathsf{v}_1)$ and $\mathcal{P}(\mathsf{v}_2)$ correspond to intersecting geodesics in $\mathbf{H}^2$;
- $\langle \mathsf{v}_1, \mathsf{v}_2\rangle \subset \mathbb{R}^{2,1}$ is a definite plane (a plane disjoint from $\mathbb{W}$);
- $\mathbb{B}\,(\mathsf{v}_1, \mathsf{v}_2)^2 < \mathbb{B}\,(\mathsf{v}_1, \mathsf{v}_1)\mathbb{B}\,(\mathsf{v}_2, \mathsf{v}_2)$;
- $\mathcal{P}(\mathsf{v}_1) \cap \mathcal{P}(\mathsf{v}_2)$ is a timelike line;
- There exists a unique timelike line orthogonal to v_1 and v_2;
- $\mathbb{B}\,(\mathsf{v}_1 \boxtimes \mathsf{v}_2, vv_1 \boxtimes \mathsf{v}_2) < 0$.

5.2. Ultraparallel Stems.

Lemma 5.2.1. *Let v_1 and v_2 be ultraparallel unit-spacelike vectors. The stems $S_1 = \mathcal{S}(\mathsf{v}_1, p_1)$ and $S_2 = \mathcal{S}(\mathsf{v}_2, p_2)$ are disjoint if and only if:*

$$|\mathbb{B}\,(p_2 - p_1, \mathsf{v}_1 \boxtimes \mathsf{v}_2)| > |\mathbb{B}\,(p_2 - p_1, \mathsf{v}_2)| + |\mathbb{B}\,(p_2 - p_1, \mathsf{v}_1)|. \tag{5.2.1}$$

Otherwise $S_1 \cap S_2$ is a line segment whose endpoints lie on two distinct lines contained in $\partial S_1 \cup \partial S_2$.

Proof. Let ℓ be the intersection of the planes that contain S_1 and S_2. Let

$$\mathsf{w} = \mathsf{v}_1 \boxtimes \mathsf{v}_2$$

and

$$p = \ell \cap (p_1 + P(\mathsf{w}))$$

and parametrize ℓ as $p + \mathbb{R}\mathsf{w}$. The intersection $\ell \cap S_i$ consists of all $p + k\mathsf{w}$ (where $k \in \mathbb{R}$) satisfying the inequality:

$$\mathbb{B}\,((p + k\mathsf{w}) - p_i, (p + k\mathsf{w}) - p_i) \leq 0. \tag{5.2.2}$$

which is equivalent to the quadratic inequality

$$k^2\mathbb{B}\,(\mathsf{w}, \mathsf{w}) + 2k\mathbb{B}\,(p - p_i, \mathsf{w}) + \mathbb{B}\,(p - p_i, p - p_i) \leq 0.$$

Since w is spacelike, $\ell \cap S_i$ correspond to those k in the interval with endpoints

(5.2.3)
$$\frac{-\mathbb{B}\,(p-p_i,\mathsf{w}) \pm \sqrt{(\mathbb{B}\,(p-p_i,\mathsf{w}))^2 - \mathbb{B}\,(\mathsf{w},\mathsf{w})\mathbb{B}\,(p-p_i,p-p_i)}}{\mathbb{B}\,(\mathsf{w},\mathsf{w})}.$$

The point p may be expressed in terms of p_1 or p_2 by writing $p - p_i$ as a linear combination of v_1, v_2, and w. Solving

$$\begin{cases} \mathbb{B}\,(p-p_1,\mathsf{w}) = 0 \\ \mathbb{B}\,(p-p_1,\mathsf{v}_1) = 0 \\ \mathbb{B}\,(p-p_2,\mathsf{v}_2) = 0 \end{cases}$$

for the coefficients for v_1, v_2, and w,

$$p = p_1 + \frac{\mathbb{B}\,(p_2-p_1,\mathsf{v}_2)\mathbb{B}\,(\mathsf{v}_1,\mathsf{v}_2)}{(\mathbb{B}\,(\mathsf{v}_1,\mathsf{v}_2))^2-1}\mathsf{v}_1 + \frac{-\mathbb{B}\,(p_2-p_1,\mathsf{v}_2)}{(\mathbb{B}\,(\mathsf{v}_1,\mathsf{v}_2))^2-1}\mathsf{v}_2, \tag{5.2.4}$$

(5.2.5)
$$p = p_2 + \frac{\mathbb{B}\,(p_2-p_1,\mathsf{v}_1)}{(\mathbb{B}\,(\mathsf{v}_1,\mathsf{v}_2))^2-1}\mathsf{v}_1 + \frac{-\mathbb{B}\,(p_2-p_1,\mathsf{v}_1)\mathbb{B}\,(\mathsf{v}_1,\mathsf{v}_2)}{(\mathbb{B}\,(\mathsf{v}_1,\mathsf{v}_2))^2-1}\mathsf{v}_2 + \frac{-\mathbb{B}\,(p_2-p_1,\mathsf{w})}{\mathbb{B}\,(\mathsf{w},\mathsf{w})}\mathsf{w}.$$

Substituting (5.2.4) and (5.2.5) into (5.2.3), and using

$$\mathbb{B}\,(\mathsf{v}_1,\mathsf{v}_2)^2 - 1 = \mathbb{B}\,(\mathsf{w},\mathsf{w}),$$

$$S_1 \cap \ell \longleftrightarrow \left[\frac{-|\mathbb{B}\,(p_2-p_1,\mathsf{v}_2)|}{\mathbb{B}\,(\mathsf{w},\mathsf{w})}, \frac{|\mathbb{B}\,(p_2-p_1,\mathsf{v}_2)|}{\mathbb{B}\,(\mathsf{w},\mathsf{w})}\right],$$

$$S_2 \cap \ell \longleftrightarrow \left[\frac{\mathbb{B}\,(p_2-p_1,\mathsf{w}) - |\mathbb{B}\,(p_2-p_1,\mathsf{v}_1)|}{\mathbb{B}\,(\mathsf{w},\mathsf{w})}, \frac{\mathbb{B}\,(p_2-p_1,\mathsf{w}) + |\mathbb{B}\,(p_2-p_1,\mathsf{v}_1)|}{\mathbb{B}\,(\mathsf{w},\mathsf{w})}\right].$$

$S_1 \cap S_2$ is empty if and only if either

$$\mathbb{B}\,(p_2-p_1,\mathsf{w}) > |\mathbb{B}\,(p_2-p_1,\mathsf{v}_2)| + |\mathbb{B}\,(p_2-p_1,\mathsf{v}_1)|,$$

or

$$\mathbb{B}\,(p_2-p_1,\mathsf{w}) < -|\mathbb{B}\,(p_2-p_1,\mathsf{v}_2)| - |\mathbb{B}\,(p_2-p_1,\mathsf{v}_1)|$$

as desired. $\square$

5.3. Asymptotic Stems.

Lemma 5.3.1. *Let* v_1 *and* v_2 *be asymptotic unit-spacelike vectors such that* $\mathbf{x}^-(\mathsf{v}_1) = \mathbf{x}^+(\mathsf{v}_2)$. *Let* $S_i = \mathcal{S}(\mathsf{v}_i, p_i)$ *for* $i = 1, 2$. S_1 *and* S_2 *are disjoint if and only if*

$$\mathbb{B}\left(p_2 - p_1, \mathbf{x}^+(\mathsf{v}_1) \boxtimes \mathbf{x}^-(\mathsf{v}_2)\right)$$

has a different sign from $\mathbb{B}(p_2 - p_1, \mathsf{v}_1)$ *and* $\mathbb{B}(p_2 - p_1, \mathsf{v}_2)$. *Otherwise,*

- $S_1 \cap S_2$ *is a point if and only if* $\mathbb{B}\left(p_2 - p_1, \mathbf{x}^+(\mathsf{v}_1) \boxtimes \mathbf{x}^-(\mathsf{v}_2)\right) = 0$ *and* $\mathbb{B}(p_2 - p_1, \mathsf{v}_1)$ *and* $\mathbb{B}(p_2 - p_1, \mathsf{v}_2)$ *have the same signs;*
- $S_1 \cap S_2$ *is a line segment if and only if* $\mathbb{B}\left(p_2 - p_1, \mathbf{x}^+(\mathsf{v}_1) \boxtimes \mathbf{x}^-(\mathsf{v}_2)\right)$, $\mathbb{B}(p_2 - p_1, \mathsf{v}_1)$, *and* $\mathbb{B}(p_2 - p_1, \mathsf{v}_2)$ *have the same signs;*
- $S_1 \cap S_2$ *is a ray if and only if either* $\mathbb{B}(p_2 - p_1, \mathsf{v}_i) = 0$ *for only one* i *or* $\mathbb{B}(p_2 - p_1, \mathsf{v}_1)$ *and* $\mathbb{B}(p_2 - p_1, \mathsf{v}_2)$ *have opposite signs;*
- $S_1 \cap S_2$ *is a line if and only if* $\mathbb{B}(p_2 - p_1, \mathsf{v}_i) = 0$ *for both* i.

Proof. Let ℓ be the intersection of the planes that contain S_1 and S_2. Let $\mathsf{w} = \mathbf{x}^+(\mathsf{v}_1) \boxtimes \mathbf{x}^-(\mathsf{v}_2)$ and $p = \ell \cap (p_1 + \mathcal{P}(\mathsf{w}))$ and parametrize ℓ as $p + \mathbb{R}\mathsf{x}$ where

$$\mathsf{x} = \mathbf{x}^-(\mathsf{v}_1) = \mathbf{x}^+(\mathsf{v}_2).$$

The intersection $\ell \cap S_i$ consists of all $p + k\mathsf{x} \in \ell$ satisfying:

$$\mathbb{B}\left((p + k\mathsf{x}) - p_i, (p + k\mathsf{x}) - p_i\right) \leq 0,$$

which simplifies to

$$2k\mathbb{B}(\mathsf{x}, p - p_i) + \mathbb{B}(p - p_i, p - p_i) \leq 0. \tag{5.3.1}$$

Now express p in terms of p_1 or p_2 by writing $p - p_i$ as a linear combination of $\mathbf{x}^+(\mathsf{v}_1)$, $\mathbf{x}^-(\mathsf{v}_2)$, and x. As p was chosen so that $\mathbb{B}(p - p_1, \mathsf{w}) = 0$, $\mathbb{B}(p - p_1, \mathsf{v}_1) = 0$, and $\mathbb{B}(p - p_2, \mathsf{v}_2) = 0$, write

$$p = p_1 + \frac{\mathbb{B}(p_2 - p_1, \mathsf{v}_2)}{\mathbb{B}(\mathbf{x}^+(\mathsf{v}_1), \mathsf{v}_2)}\mathbf{x}^+(\mathsf{v}_1), \tag{5.3.2}$$

$$p = p_2 + \frac{-\mathbb{B}(p_2 - p_1, \mathsf{v}_1)}{\mathbb{B}(\mathsf{v}_1, \mathbf{x}^-(\mathsf{v}_2))}\mathbf{x}^-(\mathsf{v}_2) + \frac{-\mathbb{B}(p_2 - p_1, \mathsf{w})}{\mathbb{B}(\mathsf{x}, \mathsf{w})}\mathsf{x}. \tag{5.3.3}$$

Substituting (5.3.2) and (5.3.3) into (5.3.1),

$$k\mathbb{B}(p_2 - p_1, \mathsf{v}_2) \leq 0,$$

$$k\mathbb{B}(p_2 - p_1, \mathsf{v}_1) \geq \frac{\mathbb{B}(p_2 - p_1, \mathsf{v}_1)\mathbb{B}(p_2 - p_1, \mathsf{w})}{\mathbb{B}(\mathsf{x}, \mathsf{w})}.$$

$S_1 \cap \ell$ and $S_2 \cap \ell$ correspond to the following subsets of $\mathbb{R}$:

$$S_1 \cap \ell \longleftrightarrow \begin{cases} \mathbb{R} & \text{if } \mathbb{B}\,(p_2 - p_1, \mathsf{v}_2) = 0, \\ [0, \infty) & \text{if } \mathbb{B}\,(p_2 - p_1, \mathsf{v}_2) < 0, \\ (-\infty, 0] & \text{if } \mathbb{B}\,(p_2 - p_1, \mathsf{v}_2) > 0. \end{cases}$$

$$S_2 \cap \ell \longleftrightarrow \begin{cases} \mathbb{R} & \text{if } \mathbb{B}\,(p_2 - p_1, \mathsf{v}_1) = 0, \\ \left[\frac{\mathbb{B}\,(p_2-p_1,\mathsf{w})}{\mathbb{B}\,(\mathsf{x},\mathsf{w})}, \infty\right) & \text{if } \mathbb{B}\,(p_2 - p_1, \mathsf{v}_1) > 0, \\ \left(-\infty, \frac{\mathbb{B}\,(p_2-p_1,\mathsf{w})}{\mathbb{B}\,(\mathsf{x},\mathsf{w})}\right] & \text{if } \mathbb{B}\,(p_2 - p_1, \mathsf{v}_1) < 0. \end{cases}$$

If $\mathbb{B}\,(p_2 - p_1, \mathsf{v}_i) = 0$ for both i then $S_1 \cap S_2$ is a line. If either

- $\mathbb{B}\,(p_2 - p_1, \mathsf{v}_i) = 0$ for only one $i = 1, 2$, or
- $\mathbb{B}\,(p_2 - p_1, \mathsf{v}_1)$ and $\mathbb{B}\,(p_2 - p_1, \mathsf{v}_1)$ have opposite signs,

then $S_1 \cap S_2$ is a ray.

Suppose that $\mathbb{B}\,(p_2 - p_1, \mathsf{v}_1)$ and $\mathbb{B}\,(p_2 - p_1, \mathsf{v}_2)$ have the same sign. If $\mathbb{B}\,(p_2 - p_1, \mathsf{w}) = 0$ then $S_1 \cap S_2$ is a point. If $\mathbb{B}\,(p_2 - p_1, \mathsf{w})$ has the same sign as $\mathbb{B}\,(p_2 - p_1, \mathsf{v}_1)$ and $\mathbb{B}\,(p_2 - p_1, \mathsf{v}_2)$, then $S_1 \cap S_2$ is an interval. If $\mathbb{B}\,(p_2 - p_1, \mathsf{w})$ has a different sign than $\mathbb{B}\,(p_2 - p_1, \mathsf{v}_1)$ and $\mathbb{B}\,(p_2 - p_1, \mathsf{v}_2)$, then $S_1 \cap S_2$ is empty. ☐

5.4. **Crossing Stems.**

Lemma 5.4.1. *Let v_1 and v_2 be unit-spacelike vectors which cross. The intersection of the stems $S_1 = \mathcal{S}(\mathsf{v}_1, p_1)$ and $S_2 = \mathcal{S}(\mathsf{v}_2, p_2)$ lies on a line and is either a line, the union of two disjoint rays, or two rays and a line segment, where all endpoints lie on the set $\partial S_1 \cup \partial S_2$.*

Proof. Use the same calculations as in Lemma 5.2.1 except that

$$\mathbb{B}\,(\mathsf{w}, \mathsf{w}) < 0.$$

Then $S_i \cap \ell$ corresponds to two infinite intervals. The endpoints of the infinite intervals are given by (5.2.3). The intersection $S_1 \cap \ell$ corresponds to the union

$$\left(-\infty, \frac{|-\mathbb{B}\,(p_2 - p_1, \mathsf{v}_2)|}{\mathbb{B}\,(\mathsf{w}, \mathsf{w})}\right] \cup \left[\frac{|\mathbb{B}\,(p_2 - p_1, \mathsf{v}_2)|}{\mathbb{B}\,(\mathsf{w}, \mathsf{w})}, \infty\right),$$

and $S_2 \cap \ell$ corresponds to the union

$$\left(-\infty, \frac{\mathbb{B}\,(p_2 - p_1, \mathsf{w}) - |\mathbb{B}\,(p_2 - p_1, \mathsf{v}_1)|}{\mathbb{B}\,(\mathsf{w}, \mathsf{w})}\right] \cup \left[\frac{\mathbb{B}\,(p_2 - p_1, \mathsf{w}) + |\mathbb{B}\,(p_2 - p_1, \mathsf{v}_1)|}{\mathbb{B}\,(\mathsf{w}, \mathsf{w})}, \infty\right).$$

as desired. ☐

6. Intersections of crooked planes

6.1. **Consistent orientation.** When considering the intersection of crooked planes, some terminology regarding the orientation of the vectors defining the crooked planes is useful. Say that spacelike vectors $\mathsf{v}_1, \mathsf{v}_2, \dots, \mathsf{v}_n$ are *consistently oriented* if $\mathbb{B}(\mathsf{v}_i, \mathsf{v}_j) \leq 0$ and $\mathbb{B}(\mathsf{v}_i, \mathbf{x}^{\pm}(\mathsf{v}_j)) \leq 0$ for $i \neq j$.

Here is the geometric meaning of consistent orientation. A unit-spacelike v defines a half-plane in the hyperbolic plane $\mathbf{H}^2$ as the projective image $H(\mathsf{v})$ of

$$\{u \in \mathbb{U}^+ \mid \mathbb{B}(u, v) < 0\}. \tag{6.1.1}$$

A pair $\mathsf{v}_1, \mathsf{v}_2 \in \mathbb{R}^{2,1}$ of ultraparallel unit-spacelike vectors is consistently oriented if the half-planes $H(\mathsf{v}_1), H(\mathsf{v}_2)$ intersect and neither one contains the other. In that case $H(\mathsf{v}_1) \cap H(\mathsf{v}_2)$ is a strip with two ideal boundary components and boundary components $\partial H(\mathsf{v}_1)$, $\partial H(\mathsf{v}_2)$.

Lemma 6.1.1. *If v_1 and v_2 are consistently oriented unit-spacelike vectors, then*

- $\mathbf{x}^+(\mathsf{v}_1) \boxtimes \mathbf{x}^+(\mathsf{v}_2)$ *is a positive scalar multiple of* $(\mathsf{v}_1 \boxtimes \mathsf{v}_2) - \mathsf{v}_1 + \mathsf{v}_2$.
- $\mathbf{x}^+(\mathsf{v}_1) \boxtimes \mathbf{x}^-(\mathsf{v}_2)$ *is a positive scalar multiple of* $(\mathsf{v}_1 \boxtimes \mathsf{v}_2) + \mathsf{v}_1 + \mathsf{v}_2$.
- $\mathbf{x}^-(\mathsf{v}_1) \boxtimes \mathbf{x}^+(\mathsf{v}_2)$ *is a positive scalar multiple of* $(\mathsf{v}_1 \boxtimes \mathsf{v}_2) - \mathsf{v}_1 - \mathsf{v}_2$.
- $\mathbf{x}^-(\mathsf{v}_1) \boxtimes \mathbf{x}^-(\mathsf{v}_2)$ *is a positive scalar multiple of* $(\mathsf{v}_1 \boxtimes \mathsf{v}_2) + \mathsf{v}_1 - \mathsf{v}_2$.

Proof. $\mathbf{SO}^0(2,1)$ acts transitively on the set of Lorentz unit vectors so write

$$\mathsf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and

$$\mathbf{x}^{\pm}(\mathsf{v}_1) = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ \mp 1 \\ 1 \end{bmatrix}.$$

Applying a hyperbolic element with v_1 as a fixed eigenvector,

$$\mathsf{v}_2 = \begin{bmatrix} -a \\ 0 \\ c \end{bmatrix},$$

where $|c| = \sqrt{a^2 - 1}$. Furthermore

$$\mathbf{x}^{\pm}(\mathsf{v}_2) = \frac{1}{\sqrt{2}} \begin{bmatrix} -c/a \\ \pm 1/a \\ 1 \end{bmatrix}.$$

Since $\mathsf{v}_1, \mathsf{v}_2$ are consistently oriented, $a, c \geq 0$. Thus $a > 1$ and $c = \sqrt{a^2-1}$. Write $\mathbf{x}^{+1}(\mathsf{v}_i) = \mathbf{x}^{+}(\mathsf{v}_i)$ and $\mathbf{x}^{-1}(\mathsf{v}_i) = \mathbf{x}^{-}(\mathsf{v}_i)$. For $j, k \in \{-1, +1\}$

$$\mathsf{v}_1 \boxtimes \mathsf{v}_2 - k\mathsf{v}_1 + j\mathsf{v}_2 = \begin{bmatrix} -k - ja \\ -c \\ jc \end{bmatrix},$$

$$\mathbf{x}^j(\mathsf{v}_1) \boxtimes \mathbf{x}^k(\mathsf{v}_2) = \begin{bmatrix} -j - k/a \\ -c/a \\ jc/a \end{bmatrix},$$

$\mathbf{x}^j(\mathsf{v}_1) \boxtimes \mathbf{x}^k(\mathsf{v}_2)$ is a multiple of $\mathsf{v}_1 \boxtimes \mathsf{v}_2 - k\mathsf{v}_1 + j\mathsf{v}_2$ since

$$\mathbb{B}\ (\mathbf{x}^j(\mathsf{v}_1), \mathsf{v}_1 \boxtimes \mathsf{v}_2 \pm \mathsf{v}_1 + j\mathsf{v}_2) = 0,$$

$$\mathbb{B}\ (\mathbf{x}^k(\mathsf{v}_2), \mathsf{v}_1 \boxtimes \mathsf{v}_2 - k\mathsf{v}_1 \pm \mathsf{v}_2) = 0.$$

Furthermore, $\mathbf{x}^j(\mathsf{v}_1) \boxtimes \mathbf{x}^k(\mathsf{v}_2)$ is a positive multiple of $\mathsf{v}_1 \boxtimes \mathsf{v}_2 - k\mathsf{v}_1 + j\mathsf{v}_2$ since

$$\mathbb{B}\ (\mathbf{x}^j(\mathsf{v}_1) \boxtimes \mathbf{x}^k(\mathsf{v}_2), \mathsf{v}_1 \boxtimes \mathsf{v}_2 - k\mathsf{v}_1 + j\mathsf{v}_2) = \frac{(a \pm 1)^2}{a} > 0$$

as desired. □

6.2. **Ultraparallel crooked planes.**

Theorem 6.2.1. *Let v_1 and v_2 be consistently oriented spacelike Lorentzian unit vectors and $p_1, p_2 \in \mathbb{E}$. Let $C_i = \mathcal{C}(\mathsf{v}_i, p_i)$ for $i = 1, 2$. The positively oriented crooked planes C_1 and C_2 are disjoint if and only if*

$$\mathbb{B}\ (p_2 - p_1, \mathsf{v}_1 \boxtimes \mathsf{v}_2) > |\mathbb{B}\ (p_2 - p_1, \mathsf{v}_2)| + |\mathbb{B}\ (p_2 - p_1, \mathsf{v}_1)|. \tag{6.2.1}$$

If $C_1 \cap C_2$ is nonempty, then $C_1 \cap C_2$ is either a single point or a polygon with 4, 5, 6, 7 or 8 edges.

Proof. Assume that (6.2.1) is satisfied. (5.2.1) is also satisfied and the stems of C_1 and C_2 are disjoint.

Rewrite (6.2.1) as four separate inequalities:

$$\mathbb{B}\ (p_2 - p_1, \mathsf{v}_1 \boxtimes \mathsf{v}_2 - \mathsf{v}_1 + \mathsf{v}_2) > 0 \tag{6.2.2}$$

$$\mathbb{B}\ (p_2 - p_1, \mathsf{v}_1 \boxtimes \mathsf{v}_2 + \mathsf{v}_1 + \mathsf{v}_2) > 0 \tag{6.2.3}$$

$$\mathbb{B}\ (p_2 - p_1, \mathsf{v}_1 \boxtimes \mathsf{v}_2 - \mathsf{v}_1 - \mathsf{v}_2) > 0 \tag{6.2.4}$$

$$\mathbb{B}\ (p_2 - p_1, \mathsf{v}_1 \boxtimes \mathsf{v}_2 + \mathsf{v}_1 - \mathsf{v}_2) > 0. \tag{6.2.5}$$

By Lemma 6.1.1, the inequalities (6.2.2), (6.2.3), (6.2.4), (6.2.5) are respectively equivalent to:

$$\mathbb{B}\left(p_2 - p_1, \mathbf{x}^+(\mathsf{v}_1) \boxtimes \mathbf{x}^+(\mathsf{v}_2)\right) > 0 \tag{6.2.6}$$

$$\mathbb{B}\left(p_2 - p_1, \mathbf{x}^+(\mathsf{v}_1) \boxtimes \mathbf{x}^-(\mathsf{v}_2)\right) > 0 \tag{6.2.7}$$

$$\mathbb{B}\left(p_2 - p_1, \mathbf{x}^-(\mathsf{v}_1) \boxtimes \mathbf{x}^+(\mathsf{v}_2)\right) > 0 \tag{6.2.8}$$

$$\mathbb{B}\left(p_2 - p_1, \mathbf{x}^-(\mathsf{v}_1) \boxtimes \mathbf{x}^-(\mathsf{v}_2)\right) > 0. \tag{6.2.9}$$

The wings $\mathcal{W}^+(\mathsf{v}_1, p_1)$ and $\mathcal{W}^+(\mathsf{v}_2, p_2)$ of the crooked planes are disjoint by applying Lemma 4.2.1 to (6.2.6). Similar applications of Lemma 4.2.1 to (6.2.7),(6.2.8),(6.2.9) imply that

$$\mathcal{W}^+(\mathsf{v}_1, p_1) \cap \mathcal{W}^-(\mathsf{v}_2, p_2) = \emptyset,$$
$$\mathcal{W}^-(\mathsf{v}_1, p_1) \cap \mathcal{W}^+(\mathsf{v}_2, p_2) = \emptyset,$$
$$\mathcal{W}^-(\mathsf{v}_1, p_1) \cap \mathcal{W}^-(\mathsf{v}_2, p_2) = \emptyset$$

respectively. Since the pairs of wings and pairs of stems are disjoint, Corollary 4.5.5 implies the wings and stem of C_1 are disjoint from the wings and stem of C_2. Thus C_1 and C_2 are disjoint.

Conversely, assume that (6.2.1) does not hold. If

$$|\mathbb{B}\left(p_2 - p_1, \mathsf{v}_1 \boxtimes \mathsf{v}_2\right)| \leq |\mathbb{B}\left(p_2 - p_1, \mathsf{v}_2\right)| + |\mathbb{B}\left(p_2 - p_1, \mathsf{v}_1\right)|,$$

Lemma 5.2.1 implies that the stems intersect. Otherwise

$$\mathbb{B}\left(p_2 - p_1, \mathsf{v}_1 \boxtimes \mathsf{v}_2\right) < -|\mathbb{B}\left(p_2 - p_1, \mathsf{v}_2\right)| - |\mathbb{B}\left(p_2 - p_1, \mathsf{v}_1\right)| \tag{6.2.10}$$

For $\mathcal{C}(\mathsf{v}_1, p_1)$ and $\mathcal{C}(\mathsf{v}_2, p_2)$ to intersect it suffices that one wing of $\mathcal{C}(\mathsf{v}_1, p_1)$ intersect a wing of $\mathcal{C}(\mathsf{v}_2, p_2)$. The inequality (6.2.10) is equivalent to 4 inequalities, one of which is

$$\mathbb{B}\left(p_2 - p_1, \mathsf{v}_1 \boxtimes \mathsf{v}_2 + \mathsf{v}_2 + \mathsf{v}_1\right) < 0,$$

which, by Lemma 5.2.1, is equivalent to:

$$\mathbb{B}\left(p_2 - p_1, \mathbf{x}^+(\mathsf{v}_1) \boxtimes \mathbf{x}^-(\mathsf{v}_2)\right) < 0.$$

Applying Lemma 4.2.1, the wings $\mathcal{W}^+(\mathsf{v}_1, p_1)$ and $\mathcal{W}^-(\mathsf{v}_2, p_2)$ intersect. It follows that $\mathcal{C}(\mathsf{v}_1, p_1) \cap \mathcal{C}(\mathsf{v}_2, p_2) \neq \emptyset$ as desired. □

The corresponding theorem for ultraparallel negatively oriented crooked planes is:

Theorem 6.2.2. *Let v_1 and v_2 be consistently oriented unit-spacelike vectors. The negatively oriented crooked planes $\mathcal{K}(\mathsf{v}_1, p_1)$ and $\mathcal{K}(\mathsf{v}_2, p_2)$ are disjoint if and only if*

$$\mathbb{B}\left(p_2 - p_1, \mathsf{v}_1 \boxtimes \mathsf{v}_2\right) < |\mathbb{B}\left(p_2 - p_1, \mathsf{v}_2\right)| + |\mathbb{B}\left(p_2 - p_1, \mathsf{v}_1\right)|. \tag{6.2.11}$$

6.3. **The polygon of intersection.** Assume that $\mathcal{C}(\mathsf{v}_1, p_1)$ and $\mathcal{C}(\mathsf{v}_2, p_2)$ intersect, that is, (6.2.10) does not hold. If

$$\mathbb{B}\,(p_2 - p_1, \mathsf{v}_1 \boxtimes \mathsf{v}_2) = |\mathbb{B}\,(p_2 - p_1, \mathsf{v}_2)| + |\mathbb{B}\,(p_2 - p_1, \mathsf{v}_1)| \tag{6.3.1}$$

then $\mathcal{C}(\mathsf{v}_1, p_1) \cap \mathcal{C}(\mathsf{v}_2, p_2)$ is a single point.

If

$$\mathbb{B}\,(p_2 - p_1, \mathsf{v}_1 \boxtimes \mathsf{v}_2) < |\mathbb{B}\,(p_2 - p_1, \mathsf{v}_2)| + |\mathbb{B}\,(p_2 - p_1, \mathsf{v}_1)|, \tag{6.3.2}$$

then we claim that $\mathcal{C}(\mathsf{v}_1, p_1) \cap \mathcal{C}(\mathsf{v}_2, p_2)$ is a polygon, that is, a piecewise linear 1-dimensional manifold without boundary. $\mathcal{C}(\mathsf{v}_1, p_1) \cap \mathcal{C}(\mathsf{v}_2, p_2)$ is the union of intersections of wings and stems. There are four possible wing-wing intersections, four possible wing-stem intersections, and one stem-stem intersection, all of which are either empty, a point or a finite spacelike line segment (Lemmas 4.2.1, 4.5.1, 5.2.1).

Let $\sigma \subset \mathcal{C}(\mathsf{v}_1, p_1) \cap \mathcal{C}(\mathsf{v}_2, p_2)$ be a spacelike line segment with one endpoint q. Assume that q lies on a null line which is the boundary of a wing and the stem of $\mathcal{C}(\mathsf{v}_1, p_1)$. If q lies in the interior of a wing or stem of $\mathcal{C}(\mathsf{v}_2, p_2)$ then it is clear that q is also the endpoint of exactly one other line segment of $\mathcal{C}(\mathsf{v}_1, p_1) \cap \mathcal{C}(\mathsf{v}_2, p_2)$. Even if q lies on the boundary of a wing and the stem of $\mathcal{C}(\mathsf{v}_2, p_2)$, it can be shown that q is the endpoint of exactly one other line segment of $\mathcal{C}(\mathsf{v}_1, p_1) \cap \mathcal{C}(\mathsf{v}_2, p_2)$. Thus, $\mathcal{C}(\mathsf{v}_1, p_1) \cap \mathcal{C}(\mathsf{v}_2, p_2)$ is a polygon.

We determine the possible number of sides of the polygon

$$\mathcal{C}(\mathsf{v}_1, p_1) \cap \mathcal{C}(\mathsf{v}_2, p_2).$$

First consider the case where the interiors of the stems $\mathcal{S}(\mathsf{v}_1, p_1)$ and $\mathcal{S}(\mathsf{v}_2, p_2)$ of $\mathcal{C}(\mathsf{v}_1, p_1)$ and $\mathcal{C}(\mathsf{v}_2, p_2)$, respectively, are disjoint. Each wing will generically intersect the wings and stem of the other crooked plane in a line segment. The generic intersection of the crooked planes in this case will be polygon with 8 edges. It is possible that for each crooked plane there is one wing such that the intersection of the wing with a stem will be a single point. In this case, the intersection is a polygon with 6, 7, or 8 edges.

Next consider the case where $\mathcal{S}(\mathsf{v}_1, p_1) \cap \mathcal{S}(\mathsf{v}_2, p_2)$ is a line segment with one endpoint on $\partial\mathcal{S}(\mathsf{v}_1, p_1)$ and one endpoint on $\partial\mathcal{S}(\mathsf{v}_2, p_2)$. Each crooked plane has one wing which does not intersect the other crooked plane at more than a point. Each crooked plane also has one wing which intersects a wing and the stem of the other crooked plane in a line segment. (There is no possible degeneration of the wing-stem intersections in this case.) The intersection of the crooked planes in this case is a polygon with 4 edges.

Finally, consider the case where $\mathcal{S}(\mathsf{v}_1, p_1) \cap \mathcal{S}(\mathsf{v}_2, p_2)$ is a line segment with both endpoints on either $\partial\mathcal{S}(\mathsf{v}_1, p_1)$ or $\partial\mathcal{S}(\mathsf{v}_2, p_2)$. Assume the endpoints of the line segment lie on $\partial\mathcal{S}(\mathsf{v}_1, p_1)$. Both wings of $\mathcal{C}(\mathsf{v}_1, p_1)$ will intersect $\mathcal{S}(\mathsf{v}_2, p_2)$ and one wing of $\mathcal{C}(\mathsf{v}_2, p_2)$ in a line segment. Generically, one wing of $\mathcal{C}(\mathsf{v}_2, p_2)$ will intersect the wings of $\mathcal{C}(\mathsf{v}_1, p_1)$ and $\mathcal{S}(\mathsf{v}_1, p_1)$ in a line segment. It is possible that this intersection of a wing of $\mathcal{C}(\mathsf{v}_2, p_2)$ and $\mathcal{S}(\mathsf{v}_1, p_1)$. In this case, the intersection is a polygon with 5 or 6 edges.

6.4. **Asymptotic crooked planes.**

Theorem 6.4.1. *Let v_1 and v_2 be consistently oriented asymptotic unit-spacelike vectors. Let $C_i = \mathcal{C}(\mathsf{v}_i, p_i)$ for $i = 1, 2$. Suppose that $\mathbf{x}^-(\mathsf{v}_1) = \mathbf{x}^+(\mathsf{v}_2)$. Then C_1 and C_2 are disjoint if and only if*

- $\mathbb{B}\,(p_2 - p_1, \mathsf{v}_1) < 0$,
- $\mathbb{B}\,(p_2 - p_1, \mathsf{v}_2) < 0$
- $\mathbb{B}\,(p_2 - p_1, \mathbf{x}^+(\mathsf{v}_1) \boxtimes \mathbf{x}^-(\mathsf{v}_2)) > 0$.

If $\mathbf{x}^+(\mathsf{v}_1) = \mathbf{x}^-(\mathsf{v}_2)$ then C_1 and C_2 are disjoint if and only if

- $\mathbb{B}\,(p_2 - p_1, \mathsf{v}_1) > 0$,
- $\mathbb{B}\,(p_2 - p_1, \mathsf{v}_2) > 0$
- $\mathbb{B}\,(p_2 - p_1, \mathbf{x}^-(\mathsf{v}_1) \boxtimes \mathbf{x}^+(\mathsf{v}_2)) > 0$.

Proof. We only discuss the case for which $\mathbf{x}^-(\mathsf{v}_1) = \mathbf{x}^+(\mathsf{v}_2)$. Let $\mathsf{x} = \mathbf{x}^-(\mathsf{v}_1) = \mathbf{x}^+(\mathsf{v}_2)$.

Suppose that $\mathbb{B}\,(p_2 - p_1, \mathsf{v}_1)$ and $\mathbb{B}\,(p_2 - p_1, \mathsf{v}_2)$ are negative and $\mathbb{B}\,(p_2 - p_1, \mathbf{x}^+(\mathsf{v}_1) \boxtimes \mathbf{x}^-(\mathsf{v}_2))$ is positive. By Lemma 5.3.1, C_1 and C_2 have disjoint stems.

Since $\mathbb{B}\,(p_2 - p_1, \mathsf{v}_1) < 0$ and $\mathbb{B}\,(p_2 - p_1, \mathsf{v}_2) < 0$, $\mathbb{B}\,(p_2 - p_1, \mathsf{x}) \neq 0$. The parallel wings, $\mathcal{W}^-(\mathsf{v}_1, p_1)$ and $\mathcal{W}^+(\mathsf{v}_2, p_2)$ are disjoint by Lemma 4.4.1.

Lemma 4.2.1 implies the other pairs of wings are disjoint. Since

$$\mathbb{B}\,(p_2 - p_1, \mathbf{x}^+(\mathsf{v}_1) \boxtimes \mathbf{x}^-(\mathsf{v}_2)) > 0$$

then $\mathcal{W}^+(\mathsf{v}_1, p_1)$ and $\mathcal{W}^-(\mathsf{v}_2, p_2)$ are disjoint. Now

$$\mathbf{x}^+(\mathsf{v}_2) \boxtimes \mathbf{x}^+(\mathsf{v}_1) = \mathbf{x}^+(\mathsf{v}_1) \boxtimes \mathbf{x}^-(\mathsf{v}_1)$$

is a negative multiple of v_1, and since

$$\mathbb{B}\,(p_2 - p_1, -\mathsf{v}_1) > 0$$

then $\mathcal{W}^+(\mathsf{v}_2, p_2)$ and $\mathcal{W}^+(\mathsf{v}_1, p_1)$ are disjoint. Similarly, $\mathcal{W}^-(\mathsf{v}_2, p_2)$ and $\mathcal{W}^-(\mathsf{v}_1, p_1)$ are disjoint.

Because $\mathbb{B}\,(p_1 - p_2, \mathsf{v}_1) > 0$, the stem $\mathcal{S}(\mathsf{v}_2, p_2)$ and the wing $\mathcal{W}^-(\mathsf{v}_1, p_1)$ are disjoint and the stem $\mathcal{S}(\mathsf{v}_1, p_1)$ and the wing $\mathcal{W}^+(\mathsf{v}_2, p_2)$ are disjoint

(Lemma 4.5.3). Assume:

$$\mathbb{B}\,(p_2 - p_1, \mathsf{v}_2) > 0$$
$$\mathbb{B}\,(p_2 - p_1, \mathbf{x}^-(\mathsf{v}_1) \boxtimes \mathbf{x}^+(\mathsf{v}_2)) > 0$$
$$\mathcal{S}(\mathsf{v}_1, p_1) \cap \mathcal{W}^-(\mathsf{v}_2, p_2) \neq \emptyset$$

By Lemma 4.5.3, the intersection must include a point on $\partial\mathcal{S}(\mathsf{v}_1, p_1)$.

$$\partial\mathcal{S}(\mathsf{v}_1, p_1) = \partial\mathcal{W}^-(\mathsf{v}_1, p_1) \cap \partial\mathcal{W}^+(\mathsf{v}_1, p_1),$$

but the wings from different crooked planes are disjoint. Therefore, $\mathcal{S}(\mathsf{v}_1, p_1)$ and $\mathcal{W}^-(\mathsf{v}_2, p_2)$ are disjoint. Similarly, $\mathcal{S}(\mathsf{v}_2, p_2)$ and $\mathcal{W}^+(\mathsf{v}_1, p_1)$ are disjoint. If $\mathbb{B}\,(p_2 - p_1, \mathsf{v}_1)$ and $\mathbb{B}\,(p_2 - p_1, \mathsf{v}_2)$ are negative and $\mathbb{B}\,(p_2 - p_1, \mathbf{x}^+(\mathsf{v}_1) \boxtimes \mathbf{x}^-(\mathsf{v}_2))$ is positive then the crooked planes are disjoint.

Conversely, assume one of the following conditions:

$$\mathbb{B}\,(p_2 - p_1, \mathsf{v}_1) < 0$$
$$\mathbb{B}\,(p_2 - p_1, \mathsf{v}_2) < 0$$
$$\mathbb{B}\,(p_2 - p_1, \mathbf{x}^-(\mathsf{v}_1) \boxtimes \mathbf{x}^+(\mathsf{v}_2)) < 0$$

If $\mathbb{B}\,(p_2 - p_1, \mathsf{v}_1) < 0$ then $\mathcal{W}^+(\mathsf{v}_2, p_2)$ intersects $\mathcal{S}(\mathsf{v}_1, p_1)$ and $\mathcal{W}^+(\mathsf{v}_1, p_1)$.
If $\mathbb{B}\,(p_2 - p_1, \mathsf{v}_2) < 0$ then $\mathcal{W}^-(\mathsf{v}_1, p_1)$ intersects $\mathcal{S}(\mathsf{v}_2, p_2)$ and $\mathcal{W}^-(\mathsf{v}_1, p_1)$.
If $\mathbb{B}\,(p_2 - p_1, \mathbf{x}^-(\mathsf{v}_1) \boxtimes \mathbf{x}^+(\mathsf{v}_2)) < 0$ then $\mathcal{W}^+(\mathsf{v}_1, p_1)$ intersects $\mathcal{W}^-(\mathsf{v}_2, p_2)$.
□

The corresponding theorem for pairs of negatively oriented crooked planes is:

Theorem 6.4.2. *Let v_1 and v_2 be consistently oriented asymptotic Lorentzian unit vectors. If $\mathbf{x}^-(\mathsf{v}_1) = \mathbf{x}^+(\mathsf{v}_2)$ then the crooked planes $\mathcal{K}(\mathsf{v}_1, p_1)$ and $\mathcal{K}(\mathsf{v}_2, p_2)$ are disjoint if and only if*

- $\mathbb{B}\,(p_2 - p_1, \mathsf{v}_1) > 0$,
- $\mathbb{B}\,(p_2 - p_1, \mathsf{v}_2) > 0$,
- $\mathbb{B}\,(p_2 - p_1, \mathbf{x}^+(\mathsf{v}_1) \boxtimes \mathbf{x}^-(\mathsf{v}_2)) < 0$.

If $\mathbf{x}^+(\mathsf{v}_1) = \mathbf{x}^-(\mathsf{v}_2)$ then the crooked planes $\mathcal{K}(\mathsf{v}_1, p_1)$ and $\mathcal{K}(\mathsf{v}_2, p_2)$ are disjoint if and only if

- $\mathbb{B}\,(p_2 - p_1, \mathsf{v}_1) < 0$,
- $\mathbb{B}\,(p_2 - p_1, \mathsf{v}_2) < 0$
- $\mathbb{B}\,(p_2 - p_1, \mathbf{x}^-(\mathsf{v}_1) \boxtimes \mathbf{x}^+(\mathsf{v}_2)) < 0$.

6.5. Crossing crooked planes.

Theorem 6.5.1. *Let v_1 and v_2 be unit-spacelike vectors which cross. The intersection of crooked planes of indeterminate orientation with spines parallel to v_1 and v_2 is always nonempty.*

Proof. This is an easy corollary of Lemma 5.4.1. Since the stems intersect, the orientations are irrelevant. □

6.6. **Crooked planes with different orientations.** Now consider the intersection of two crooked planes of different orientations.

Theorem 6.6.1. *Let v_1 and v_2 be unit-spacelike vectors which are not parallel. The intersection of the positively oriented crooked plane $\mathcal{C}(\mathsf{v}_1, p_1)$ and the negatively oriented crooked plane $\mathcal{K}(\mathsf{v}_2, p_2)$ is always nonempty.*

Proof. This theorem is an easy corollary of Lemma 4.2.2. □

7. Groups generated by inversions

We apply the preceding considerations to affine deformations of subgroups of $\mathbf{SO}(2,1)$ generated by reflections in the sides of an ultra-ideal polygon. Let $\Delta \subset \mathbf{H}^2$ be a convex n-gon whose pairs of adjacent sides $S_1, \dots, S_n$ are ultraparallel. Let $\rho_i \in \mathbf{SO}(2,1)$ denote the reflection in S_i and $\Gamma \subset \mathbf{SO}(2,1)$ the group generated by $\rho_1, \dots, \rho_n$. Then Γ acts properly and discretely on $\mathbf{H}^2$ with fundamental domain Δ. The subgroup $\Gamma \cap \mathbf{SO}(2,1)^0$ comprising orientation-preserving isometries of $\mathbf{H}^2$ has index two in Γ and is a Schottky group (it has a fundamental domain consisting of $\Delta \cup \rho_i \Delta$ for any i). There is a family of consistently oriented unit-spacelike vectors $\mathsf{v}_1, \dots, \mathsf{v}_n \in \mathbb{R}^{2,1}$ such that the sides S_i of Δ are polar to v_i and

$$\Delta = H(\mathsf{v}_1) \cap \dots H(\mathsf{v}_n)$$

where $H(\mathsf{v}_i)$ is defined in (6.1.1).

An *affine deformation* of Γ is a collection $\mathbf{l} = (l_1, \dots, l_n)$ of oriented spacelike lines such that each l_i has direction vector v_i. Such an affine deformation determines an affine representation of Γ whose image $\Gamma_{\mathbf{l}}$ is generated by inversions ι_j in l_j. A line $l \subset \mathbb{E}$ with direction vector v is parametrized by a point $p \in l$ which is determined up to translation by $\mathbb{R}\mathsf{v}$. The affine deformations of Γ comprise the product

$$\mathbb{E}/\mathbb{R}\mathsf{v}_1 \times \dots \times \mathbb{E}/\mathbb{R}\mathsf{v}_n$$

of the quotient affine spaces $\mathbb{E}/\mathbb{R}\mathsf{v}_j$.

To construct a fundamental domain for the action of $\Gamma_{\mathbf{l}}$, we must find crooked planes $\mathcal{C}_i = \mathcal{C}(\mathsf{v}_i, q_i)$ with spine l_i. (Recall that l_i has direction vector v_i and contains q_i.) Since a crooked plane with given spine l is determined by its vertex, the family $(C) = (\mathcal{C}_1, \dots, \mathcal{C}_n)$ is determined by the n-tuple $\mathbf{q} = (q_1, \dots, q_n) \in l_1 \times \dots \times l_n$.

Conjecture 7.0.2. Let $\Gamma \subset \mathbf{SO}(2,1)$ be as above. An affine deformation Γ_1 of Γ is properly discontinuous if and only if there exist $q \in l_1 \times \cdots \times l_n$ such that $\mathcal{H}(q_1, v_1), \ldots, \mathcal{H}(q_n, v_n)$ bound a fundamental domain

Sufficient conditions for proper discontinuity of an affine deformation of a Schottky group are derived in Margulis [11, 12] and our previous paper[7].

Assume that there exist mutually disjoint crooked planes

$$\mathcal{C}(q_1, v_1), \ldots, \mathcal{C}(q_n, v_n)$$

which bound a common region. The linear part of the index-two subgroup

$$\kappa = \langle \iota_{l_1}\iota_{l_2}, \iota_{l_1}\iota_{l_3}, \ldots \iota_{l_1}\iota_{l_n} \rangle$$

of Γ lies in $\mathbf{SO}(2,1)$. The region bounded by the disjoint crooked planes

$$\mathcal{C}(q_2, v_2), \ldots \mathcal{C}(q_n, v_n), \iota_{l_1}(\mathcal{C}(q_2, v_2)), \ldots \iota_{l_1}(\mathcal{C}(q_n, v_n))$$

is a fundamental domain for the action of κ which acts freely and properly discontinuosly on $\mathbb{R}^{2,1}$ by [3]. Thus, Γ also acts properly discontinuosly on $\mathbb{R}^{2,1}$ and a fundamental domain for the action of Γ is the region bounded by the crooked planes $(\mathcal{C}(q_1, v_1), \ldots, \mathcal{C}(q_n, v_n))$.

References

1. Auslander,L. and Markus, L., *Holonomy of flat affinely connected manifolds,* American J. Math. **62** (1955), 139–159
2. ______ *Flat Lorentz 3-manifolds,* Memoirs Amer. Math. Soc. **30** (1959)
3. Drumm, T., *Fundamental polyhedra for Margulis space-times,* Topology **31** (4) (1992), 677-683
4. ______, *Linear holonomy of Margulis space-times,* J.Diff.Geo. **38** (1993), 679–691
5. ______, *Examples of nonproper affine actions,* Mich. Math. J. **39** (1992), 435–442
6. ______, *Margulis space-times,* Proc. Symp. Pure Math. **54** (1993), Part 2, 191–195
7. Drumm, T. and Goldman, W., *Complete flat Lorentz 3-manifolds with free fundamental group,* Int. J. Math.**1** (1990), 149–161
8. Fried, D. and Goldman, W., *Three-dimensional affine crystallographic groups,* Adv. Math. **47** (1983), 1–49
9. ______ and Hirsch, M. W. *Affine manifolds with nilpotent holonomy,* Comm. Math. Helv. **56** (1981), 487–523
10. Goldman, W., "Complex hyperbolic geometry," Oxford Mathematical Monographs (to appear)
11. Margulis, G., *Free properly discontinuous groups of affine transformations,* Dokl. Akad. Nauk SSSR **272** (1983), 937–940;

12. ______, *Complete affine locally flat manifolds with a free fundamental group,* J. Soviet Math. **134** (1987), 129–134
13. Mess, G., *Lorentz spacetimes of constant curvature,* (preprint)
14. Milnor, J., *On fundamental groups of complete affinely flat manifolds,* Adv. Math. **25** (1977), 178–187
15. Tits, J., *Free subgroups in linear groups,* J. Algebra **20** (1972), 250–270

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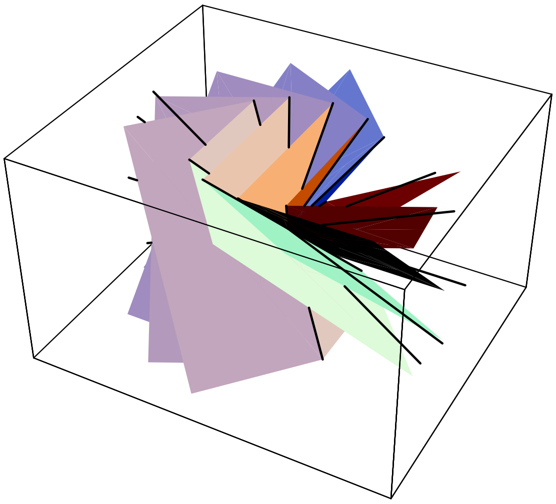

FIGURE 5. Crooked planes through the origin invariant under a cyclic group

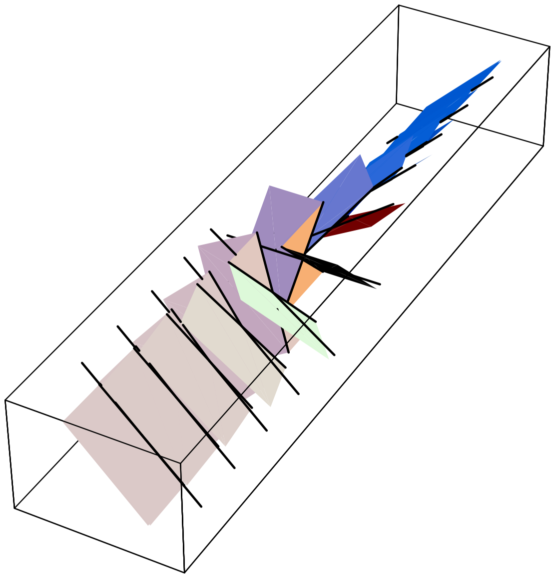

FIGURE 6. Disjoint crooked planes invariant under a cyclic group