THE COMPLEX-SYMPLECTIC GEOMETRY OF $\mathsf{SL}(2,\mathbb{C})$-CHARACTERS OVER SURFACES

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Dedicated to M.S. Raghunathan on his sixtieth birthday

Abstract. The $\mathsf{SL}(2,\mathbb{C})$-character variety X of a closed surface M enjoys a natural complex-symplectic structure invariant under the mapping class group Γ of M. Using the ergodicity of Γ on the $\mathsf{SU}(2)$-character variety, we deduce that every Γ-invariant meromorphic function on X is constant. The trace functions of closed curves on M determine regular functions which generate complex Hamiltonian flows. For simple closed curves, these complex Hamiltonian flows arise from holomorphic flows on the representation variety generalizing the Fenchel-Nielsen twist flows on Teichmüller space and the complex quakebend flows on quasi-Fuchsian space. Closed curves in the complex trajectories of these flows lift to paths in the deformation space $\mathbb{CP}^1(M)$ of complex-projective structures between different $\mathbb{CP}^1$-structures with the same holonomy (grafting). If $\mathcal{P}$ is a pants decomposition, then the trace map $\tau_{\mathcal{P}} : X \longrightarrow \mathbb{C}^{\mathcal{P}}$ defines a holomorphic completely integrable system. Furthermore, if $\Gamma_{\mathcal{P}}$ is the subgroup of Γ preserving $\mathcal{P}$, then every $\Gamma_{\mathcal{P}}$-invariant holomorphic function $X \longrightarrow \mathbb{C}$ factors through $\tau_{\mathcal{P}}$. This holomorphic integrable system is related to the complex Fenchel-Nielsen coordinates on quasi-Fuchsian space $\mathcal{QF}(M)$ developed by Tan and Kourouniotis, and relate to recent formulas of Platis and Series on complex-length functions and complex twist flows on $\mathcal{QF}(M)$.

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Introduction

The $\mathsf{SL}(2,\mathbb{C})$-*character variety* of a closed surface M with fundamental group π is the space of equivalence classes of representations ϕ of the fundamental group π of M into $\mathsf{SL}(2,\mathbb{C})$. The Teichmüller space $\mathfrak{T}(M)$, the moduli space X_U of irreducible flat unitary $\mathsf{SU}(2)$-bundles, the deformation space $\mathbb{CP}^1(M)$ of $\mathbb{CP}^1$-structures on M, and the quasi-Fuchsian space $\mathcal{QF}(M)$ all lie in the smooth stratum X of the $\mathsf{SL}(2,\mathbb{C})$-character variety. These spaces all share several common features: a symplectic geometry derived from the topology of M, and a compatible action of the mapping class group Γ of M. This paper investigates these structures in terms of the Γ-invariant complex-symplectic structure on X.

A *complex-symplectic structure* on a complex manifold is a nondegenerate closed holomorphic exterior 2-form. The $\mathsf{SL}(2,\mathbb{C})$-character variety is an affine variety defined over $\mathbb{C}$ whose set of smooth $\mathbb{C}$-points is a complex-symplectic manifold of complex dimension $-3\chi(M)$ whose complex-symplectic structure is invariant under Γ.

In [46], Fenchel and Nielsen develop a set of coordinates for $\mathfrak{T}(M)$ based on hyperbolic geometry. This set of coordinates is based on a *pants decomposition* $\mathcal{P} = \{\alpha_1, \dots, \alpha_N\}$ on M, that is a set of disjoint simple closed curves α_i cutting M into three-holed spheres ("pants"). In a hyperbolic structure on M, the curves α_i are represented by disjoint simple closed geodesics and their lengths define a function

$$l_{\mathcal{P}} : \mathfrak{T}(M) \longrightarrow (\mathbb{R}_+)^N$$

which consitute half of the Fenchel-Nielsen coordinates on $\mathfrak{T}(M)$. According to Wolpert [48], these functions Poisson-commute and define a completely integrable Hamiltonian system. In the language of classical mechanics, these are the action variables, for which the Fenchel-Nielsen twist vector fields define angle variables. In particular $l_{\mathcal{P}}$ is the moment map for a Hamiltonian $\mathbb{R}^N$-action. Choosing a section σ to $l_{\mathcal{P}}$ determines a symplectomorphim

$$\begin{aligned} (\mathbb{R}_+)^N \times \mathbb{R}^N &\longrightarrow \mathfrak{T}(M) \\ (\lambda; t) &\longmapsto \xi_t \sigma(\lambda) \end{aligned}$$

where ξ_t is the Hamiltonian $\mathbb{R}^N$-action defined by $l_{\mathcal{P}}$. See Wolpert [46, 47, 48] for details.

This picture motivated the study of a symplectic geometry of moduli spaces $\mathsf{Hom}(\pi, G)/G$ developed in [9, 11]. The results there extend directly to the complex-symplectic geometry of deformation spaces $\mathsf{Hom}(\pi, G)/G$ where G is a *complex Lie group* with an Ad-invariant complex-orthogonal structure $\mathbb{B}$ on its Lie algebra. In this paper we consider only the special case $G = \mathsf{SL}(2,\mathbb{C})$ where $\mathbb{B}$ is the trace form. Corresponding to an element $\alpha \in \pi$ is the function $f_\alpha : X \longrightarrow \mathbb{C}$ associating to the equivalence class of ϕ the trace of $\phi(\alpha)$. When α is represented by a *simple* closed curve A, then the complex-Hamiltonian vector field $\mathsf{Ham}(f_\alpha)$ generates a flow which is covered by a *complex twist flow* on $\mathsf{Hom}(\pi, \mathsf{SL}(2,\mathbb{C}))$. We study this flow, computing its periods, and relating it to the action on X of the Dehn twist about A. Following [9, 11], we relate the complex-symplectic geometry and the Hamiltonian actions to the Fenchel-Nielsen twist flows on $\mathfrak{T}(M)$ and the complex earthquakes and bending deformations on $\mathbb{CP}^1(M)$ and $\mathcal{QF}(M)$.

In the presence of a conformal structure on M, the moduli spaces $\mathsf{Hom}(\pi, G)/G$ admit stronger structures (a Kähler structure when G is compact, and hyper-Kähler when G is complex), but these stronger structures fail to be Γ-invariant. However the symplectic (and complex-symplectic) structures are Γ-invariant. The symplectic structure defines a Γ-invariant measure. By [15] and Pickrell-Xia [38], the resulting measure is ergodic under Γ when G is a compact Lie group. This has the following consequence for the holomorphic geometry when G is complex:

Theorem. *There are no nonconstant Γ-invariant meromorphic functions on X.*

The proof uses the inclusion of the set X_U of irreducible unitary characters in X, and the ergodicity of the action of Γ on X_U ([15]). A key idea in the proof is the action of the subgroup $\Gamma_{\mathcal{P}}$ preserving a pants decomposition $\mathcal{P}$ of M. The group $\Gamma_{\mathcal{P}}$ is a free abelian group freely generated by the Dehn twists about the curves in $\mathcal{P}$. This $\mathbb{Z}^N$-action lies in a Hamiltonian $\mathbb{R}^N$-action. The map which associates to $[\phi]$ the collection of traces

$$X_U \longrightarrow \mathbb{R}^N$$

is a moment map for the $\mathbb{R}^N$-action and is also the ergodic decomposition for the $\mathbb{Z}^N$-action. The holomorphic analog is:

Theorem. *Every $\Gamma_{\mathcal{P}}$-invariant meromorphic function on X factors through the map*

$$X \longrightarrow \mathbb{C}^N$$

which associates to a character its values on the curves in $\mathcal{P}$.

The complex twist flows have been extensively studied in the *quasi-Fuchsian space* $\mathcal{QF}(M)$, which is the open subset of the $\mathsf{SL}(2,\mathbb{C})$-character variety comprising equivalence classes of quasi-Fuchsian representations. In this case, the complex twist flows correspond geometrically to quake-bending pleated surfaces in quasi-Fuchsian hyperbolic 3-manifolds, that is, composing Fenchel-Nielsen twist flows (earthquakes) with bending deformations (Epstein-Marden [7]). These quakebends are defined more generally for geodesic laminations, although we only consider deformations supported on simple closed curves here. The complex Fenchel-Nielsen coordinates of Kourouniotis [32, 33, 34]) and Tan [43] are holomorphic Darboux coordinates for the complex-symplectic structure. We recover results of Platis [39] expressing the symplectic duality between the complex twist flows and the complex length functions, and the formula of Series [42] for the derivative of a complex length function under a twist flow in terms of the complex-symplectic geometry of X.

More generally the complex twist flows are defined for $\mathbb{CP}^1$*-structures,* that is, geometric structures with coordinates modelled on $\mathbb{CP}^1$ with coordinate changes in $\mathsf{PSL}(2,\mathbb{C})$. Let $\mathbb{CP}^1(M)$ denote the deformation space of $\mathbb{CP}^1$-structures on M. Using the local biholomorphicity of the holonomy mapping

$$\mathsf{hol} : \mathbb{CP}^1(M) \longrightarrow \mathsf{Hom}(\pi, \mathsf{SL}(2,\mathbb{C}))/\!\!/\mathsf{SL}(2,\mathbb{C}).$$

one obtains complex-symplectic structures, complex length functions, and Hamiltonian complex twist flows on $\mathbb{CP}^1(M)$. The complex twist flows on $\mathbb{CP}^1$-structures can be described geometrically by inserting annuli into a $\mathbb{CP}^1$-manifold split along a simple closed curve which is locally circular. This is a special case of the *grafting* construction considered in Tanigawa [44] and McMullen [36]. In particular closed curves in the complex trajectory of a complex twist flow lift to paths between different $\mathbb{CP}^1$-structures with the same holonomy (Maskit and Hejhal, see Goldman [12]). The holomorphic properties of the grafting construction are discussed in McMullen [36], Tanigawa [44] and Scannell-Wolf [41].

1. Representation varieties and character varieties

Let M be a closed oriented surface with fundamental group π. Let $\mathsf{Hom}(\pi, \mathsf{SL}(2,\mathbb{C}))$ denote the complex affine variety consisting of homomorphisms $\pi \longrightarrow \mathsf{SL}(2,\mathbb{C})$. (It is irreducible, by Goldman [13]; see Benyash-Krivets -Chernousov -Rapinchuk [2] for stronger more general results, and also Li [35].)

The group $\mathsf{SL}(2,\mathbb{C})$ acts by conjugation on the affine algebraic set $\mathsf{Hom}(\pi,\mathsf{SL}(2,\mathbb{C}))$. Let $\mathsf{Hom}(\pi,\mathsf{SL}(2,\mathbb{C}))/\mathsf{SL}(2,\mathbb{C})$ denote the set of $\mathsf{SL}(2,\mathbb{C})$-orbits of $\mathsf{Hom}(\pi,\mathsf{SL}(2,\mathbb{C}))$. Denote the orbit of a representation ϕ by

$$[\phi] \in \mathsf{Hom}(\pi,\mathsf{SL}(2,\mathbb{C})).$$

1.1. **Stable and semistable points.** A homomorphism

$$\phi \in \mathsf{Hom}(\pi,\mathsf{SL}(2,\mathbb{C}))$$

is *irreducible* if it leaves invariant no proper linear subspace of $\mathbb{C}^2$. Equivalently, ϕ is irreducible if the corresponding projective action fixes no point in $\mathbb{CP}^1$. Irreducible homomorphisms are the *stable points* of $\mathsf{Hom}(\pi,\mathsf{SL}(2,\mathbb{C}))$ (with respect to the $\mathsf{SL}(2,\mathbb{C})$-action). The subset of $\mathsf{Hom}(\pi,\mathsf{SL}(2,\mathbb{C}))^s \subset \mathsf{Hom}(\pi,\mathsf{SL}(2,\mathbb{C}))$ consisting of irreducible homomorphisms is Zariski-open and nonsingular. $\mathsf{Hom}(\pi,\mathsf{SL}(2,\mathbb{C}))^s$ is a complex manifold of complex dimension $6g-3$ (see [9]).

More generally, a homomorphism is *reductive* if every invariant subspace possesses an invariant complement. Equivalently, ϕ is reductive if it is either reducible or fixes a pair of distinct points on $\mathbb{CP}^1$ or is *central.* A central homomorphism maps into the center $\{\pm\mathbb{I}\}$ of $\mathsf{SL}(2,\mathbb{C})$ and acts trivially on $\mathbb{CP}^1$. Letting $\mathsf{Fix}(\phi)$ denote the subset of $\mathbb{CP}^1$ fixed by ϕ, a homomorphism ϕ is reductive if and only if $\mathsf{Fix}(\phi)$ equals $\emptyset$, a pair of distinct points, or all of $\mathbb{CP}^1$. Reductive homomorphisms are the *semistable points* of $\mathsf{Hom}(\pi,\mathsf{SL}(2,\mathbb{C}))$ (with respect to the $\mathsf{SL}(2,\mathbb{C})$-action),comprising the subset

$$\mathsf{Hom}(\pi,\mathsf{SL}(2,\mathbb{C}))^{ss} \subset \mathsf{Hom}(\pi,\mathsf{SL}(2,\mathbb{C})).$$

A *semistable* (respectively *stable*) orbit is the orbit of a semistable (respectively stable) point.

Let $\mathsf{Hom}(\pi,\mathsf{SL}(2,\mathbb{C}))/\!\!/\mathsf{SL}(2,\mathbb{C})$ denote the set of $\mathbb{C}$-points of the *categorical quotient* of the $\mathsf{SL}(2,\mathbb{C})$-action on $\mathsf{Hom}(\pi,\mathsf{SL}(2,\mathbb{C}))$, that is, the variety whose coordinate ring is the ring of invariants of $\mathsf{SL}(2,\mathbb{C})$ acting on the coordinate ring $\mathbb{C}[\mathsf{Hom}(\pi,\mathsf{SL}(2,\mathbb{C}))]$ of $\mathsf{Hom}(\pi,\mathsf{SL}(2,\mathbb{C}))$. The natural map

$$\mathsf{Hom}(\pi,\mathsf{SL}(2,\mathbb{C}))/\mathsf{SL}(2,\mathbb{C}) \longrightarrow \mathsf{Hom}(\pi,\mathsf{SL}(2,\mathbb{C}))/\!\!/\mathsf{SL}(2,\mathbb{C})$$

is surjective, but not injective. However $\mathsf{Hom}(\pi,\mathsf{SL}(2,\mathbb{C}))/\!\!/\mathsf{SL}(2,\mathbb{C})$ may be identified with the set of semistable orbits. Thus the inclusion

$$\mathsf{Hom}(\pi,\mathsf{SL}(2,\mathbb{C}))^{ss} \hookrightarrow \mathsf{Hom}(\pi,\mathsf{SL}(2,\mathbb{C}))$$

induces a bijection

$$\mathsf{Hom}(\pi,\mathsf{SL}(2,\mathbb{C}))^{ss}/\mathsf{SL}(2,\mathbb{C}) \longleftrightarrow \mathsf{Hom}(\pi,\mathsf{SL}(2,\mathbb{C}))/\!\!/\mathsf{SL}(2,\mathbb{C}).$$

The group $\mathsf{SL}(2,\mathbb{C})$ acts freely and properly on $\mathsf{Hom}(\pi,\mathsf{SL}(2,\mathbb{C}))^s$. The quotient $X := \mathsf{Hom}(\pi,\mathsf{SL}(2,\mathbb{C}))^s/\mathsf{SL}(2,\mathbb{C})$ is thus a $(6g-6)$-dimensional complex manifold which embeds as a Zariski open subset of the categorical quotient $\mathsf{Hom}(\pi,\mathsf{SL}(2,\mathbb{C}))/\!\!/\mathsf{SL}(2,\mathbb{C})$. Thus the map

$$X = \mathsf{Hom}(\pi,\mathsf{SL}(2,\mathbb{C}))^s/\mathsf{SL}(2,\mathbb{C}) \longrightarrow \mathsf{Hom}(\pi,\mathsf{SL}(2,\mathbb{C}))/\!\!/\mathsf{SL}(2,\mathbb{C})$$

induced by $\mathsf{Hom}(\pi,\mathsf{SL}(2,\mathbb{C}))^s \hookrightarrow \mathsf{Hom}(\pi,\mathsf{SL}(2,\mathbb{C}))$ is an embedding onto an open subset. Thus X is a smooth irreducible complex quasi-affine variety which is dense in the quotient $\mathsf{Hom}(\pi,\mathsf{SL}(2,\mathbb{C}))/\!\!/\mathsf{SL}(2,\mathbb{C})$.

1.2. **Symplectic geometry of deformation spaces.** By the general construction of [9], X has a natural *complex-symplectic structure* Ω, which is Γ-invariant. Furthermore Γ is algebraic in the sense that there exists an algebraic tensor field on $\mathsf{Hom}(\pi,\mathsf{SL}(2,\mathbb{C}))$ inducing Ω. (See [9] for an explicit formula.)

The Zariski tangent space $T_\phi\mathsf{Hom}(\pi,\mathsf{SL}(2,\mathbb{C}))$ is the space

$$Z^1(\pi,\mathfrak{sl}(2,\mathbb{C})_{\mathsf{Ad}\phi})$$

of *1-cocycles* $\pi \longrightarrow \mathfrak{sl}(2,\mathbb{C})_{\mathsf{Ad}\phi}$ and the tangent space $T_\phi(G\cdot\phi)$ of the G-orbit equals the subspace

$$B^1(\pi,\mathfrak{sl}(2,\mathbb{C})_{\mathsf{Ad}\phi}) \subset Z^1(\pi,\mathfrak{sl}(2,\mathbb{C})_{\mathsf{Ad}\phi})$$

of *1-coboundaries.* (Compare Raghunathan [40].) The quotient vector space is the cohomology

$$H^1(\pi,\mathfrak{sl}(2,\mathbb{C})_{\mathsf{Ad}\phi})$$

which, under the de Rham isomorphism is isomorphic to the cohomology $H^1(M;V_\phi)$ where V_ϕ denotes the flat vector bundle over M corresponding to the π-module $\mathfrak{sl}(2,\mathbb{C})_{\mathsf{Ad}\phi}$.

Let $\mathbb{B} : \mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C}) \longrightarrow \mathbb{C}$ be the *trace form* of the standard representation on $\mathbb{C}^2$: if $\alpha,\beta \in \mathfrak{sl}(2,\mathbb{C})$, then the inner product is defined as

$$\mathbb{B}(\alpha,\beta) := \mathsf{tr}(\alpha\beta).$$

Since $\mathbb{B}$ is Ad-invariant, it defines a bilinear pairing of π-modules

$$\mathfrak{sl}(2,\mathbb{C})_{\mathsf{Ad}\phi} \times \mathfrak{sl}(2,\mathbb{C})_{\mathsf{Ad}\phi} \longrightarrow \mathbb{C}$$

or equivalently flat vector bundles

$$V_\phi \times V_\phi \longrightarrow \mathbb{C}.$$

Cup-product defines a bilinear pairing

$$\Omega_\phi : H^1(M;V_\phi) \times H^1(M;V_\phi) \longrightarrow \mathbb{C}$$

with coefficients paired by $\mathbb{B}$. Symmetry of $\mathbb{B}$ implies that Ω_ϕ is skew-symmetric. Since $\mathbb{B}$ is nondegenerate, Ω_ϕ is nondegenerate. It follows from [9] that Ω_ϕ can be expressed as an algebraic tensor on $\mathsf{Hom}(\pi, \mathsf{SL}(2,\mathbb{C}))$, and thus is a holomorphic exterior 2-form. By arguments of [9] or Karshon [29], Weinstein [45], Guruprasad-Huebschmann-Jeffrey-Weinstein [19], Ω is closed. Thus Ω is a nondegenerate closed holomorphic $(2,0)$-form, that is a *complex-symplectic structure.*

1.3. **The mapping class group.** The automorphism group $\mathsf{Aut}(\pi)$ of π acts algebraically on $\mathsf{Hom}(\pi, \mathsf{SL}(2,\mathbb{C}))$ commuting with the action of $\mathsf{SL}(2,\mathbb{C})$. The normal subgroup $\mathsf{Inn}(\pi)$ of inner automorphisms acts trivially on the quotient $\mathsf{Hom}(\pi, \mathsf{SL}(2,\mathbb{C}))/\!\!/\mathsf{SL}(2,\mathbb{C})$. The quotient $\mathsf{Out}(\pi) = \mathsf{Aut}(\pi)/\mathsf{Inn}(\pi)$, acts on $\mathsf{Hom}(\pi, \mathsf{SL}(2,\mathbb{C}))/\!\!/\mathsf{SL}(2,\mathbb{C})$, leaving invariant the subset X. Furthermore the mapping class group $\Gamma := \pi_0(\mathsf{Diff}^+(M))$ of M is isomorphic to $\mathsf{Out}(\pi)$ by Nielsen [37]. The algebraic complex-symplectic structure on X is Γ-invariant.

1.4. **Ergodicity and its holomorphic analog.** The subset X_U of unitary characters is invariant under Γ. Furthermore the complex-symplectic structure restricts to a (real) symplectic structure on X_U which is the Kähler form for a Kähler structure on X_U. In particular the Lebesgue measure class on X_U is invariant under Γ. The main result of [15] is that this action is *ergodic,* that is every Γ-invariant measurable function on X_U is constant almost everywhere.

Ergodicity no longer holds for the $\mathsf{SL}(2,\mathbb{C})$-character variety X (see §4.3). However ergodicity on X_U does imply the following property of holomorphic functions on X:

Theorem 1.4.1. *A Γ-invariant meromorphic function $X \xrightarrow{h} \mathbb{CP}^1$ is constant.*

Proof. The restriction of h to X_U is a Γ-invariant measurable function, and by the main result of Goldman [15], h must be constant almost everywhere. Since h is continuous, it is constant.

Now we argue that in local holomorphic coordinates, X_U is equivalent to $\mathbb{R}^n \subset \mathbb{C}^n$ and a holomorphic function constant on X_U must be globally constant on X. (Compare, for example, Lemma 1 of §2.3 of Platis [39]). For the reader's convenience, we supply a brief proof. Let $\mathcal{U}$ be a nonempty open coordinate neighborhood with local holomorphic coordinates $z = (z^1, \dots, z^n)$. In local holomorphic coordinates, $X_U \cap \mathcal{U}$ is described by

$$z \in \mathbb{R}^n \subset \mathbb{C}^n$$

and h is given by a power series

$$h(z) = \sum_{k=0}^{\infty} a_k z^k$$

which converges in the nonempty open set $z(\mathcal{U}) \in \mathbb{C}^n$. Since the restriction of h to a $z(\mathcal{U}) \cap \mathbb{R}^n$ is constant, $a_k = 0$ for $k > 0$ and thus h is constant on $\mathcal{U}$. Since X is connected, analytic continuation implies that h must be constant. $\square$

We do not know whether $X - X_U$ admits Γ-invariant nonconstant meromorphic functions.

2. The Hamiltonian vector field of a character function

Corresponding to free homotopy classes α of closed curves on M are complex regular functions $f_\alpha : X \longrightarrow \mathbb{C}$. The complex-symplectic structure associates to these functions complex Hamiltonian vector fields $\mathsf{Ham}(f_\alpha)$, which generate holomorphic local flows on X. When α corresponds to a simple closed curve, we define *complex twist flows* on $\mathsf{Hom}(\pi, \mathsf{SL}(2,\mathbb{C}))$ which cover these holomorphic local flows on X. We begin with an preliminary section on the traces in $\mathsf{SL}(2,\mathbb{C})$.

2.1. The variation of the trace function on $\mathsf{SL}(2,\mathbb{C})$.

Let

$$f : \mathsf{SL}(2,\mathbb{C}) \longrightarrow \mathbb{C}$$

be the trace function $f(P) = \mathsf{tr}(P)$ and $\mathbb{B}$ the trace form

$$\begin{aligned} \mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C}) &\longrightarrow \mathbb{C} \\ (X,Y) &\longmapsto \mathsf{tr}(XY). \end{aligned}$$

As in Goldman [11] the differential of f and the orthogonal structure $\mathbb{B}$ determines a variation function

$$F : \mathsf{SL}(2,\mathbb{C}) \longrightarrow \mathfrak{sl}(2,\mathbb{C})$$

characterized by the identity

$$\left.\frac{d}{dt}\right|_{t=0} f\big(P\exp(tX)\big) = \mathbb{B}\big(F(P), X\big).$$

This function is defined by:

$$F(P) = P - \frac{\mathsf{tr}(P)}{2}\mathbb{I}$$

and corresponds to the composition of the inclusion

$$\mathsf{SL}(2,\mathbb{C}) \hookrightarrow \mathfrak{gl}(2,\mathbb{C})$$

with orthogonal projection

$$\mathfrak{gl}(2,\mathbb{C}) \hookrightarrow \mathfrak{sl}(2,\mathbb{C})$$

(orthogonal with respect to the trace form $\mathbb{B}$ on $\mathfrak{sl}(2,\mathbb{C})$). Invariance of the trace

$$f(QPQ^{-1}) = f(P)$$

and Ad-invariance of the orthogonal structure $\mathbb{B}$

$$\mathbb{B}(\mathsf{Ad}(Q)X, \mathsf{Ad}(Q)Y) = \mathbb{B}(X,Y)$$

implies Ad-equivariance of its variation:

$$F(QPQ^{-1}) = \mathsf{Ad}(Q)F(P). \tag{2.1}$$

Taking $Q = P$ shows that $F(P)$ lies in the centralizer of P in $\mathfrak{sl}(2,\mathbb{C})$. In particular the complex one-parameter subgroup of $\mathsf{SL}(2,\mathbb{C})$

$$\zeta_t := \exp\big(tF(P)\big) \tag{2.2}$$

centralizes A in $\mathsf{SL}(2,\mathbb{C})$.

For example, if P is the diagonal matrix

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$$

then $f(P) = \lambda + \lambda^{-1}$ and

$$F(P) = \frac{\lambda - \lambda^{-1}}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

with corresponding complex one-parameter subgroup:

$$\zeta_t = \begin{bmatrix} e^{t(\lambda-\lambda^{-1})/2} & 0 \\ 0 & e^{-t(\lambda-\lambda^{-1})/2} \end{bmatrix}. \tag{2.3}$$

If P is $\pm$ the unipotent matrix

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

then $f(P) = \pm 2$ and

$$F(P) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

with corresponding complex one-parameter subgroup:

$$\zeta_t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \tag{2.4}$$

The following lemma, whose proof is immediate from the above calculations, will be needed in the sequel. Recall that for any subset $S \subset \mathsf{SL}(2,\mathbb{C})$, $\mathsf{Fix}(S)$ denotes the subset of $\mathbb{CP}^1$ fixed by S.

Lemma 2.1.1. *If $P \in \mathsf{SL}(2,\mathbb{C})$, then $\mathsf{Fix}(\zeta_t) = \mathsf{Fix}(P)$ or $\zeta_t = \pm\mathbb{I}$*

2.2. **Character functions.** Let $\alpha \in \pi$. The *character function*

$$\begin{aligned} \mathsf{Hom}(\pi, \mathsf{SL}(2,\mathbb{C})) &\xrightarrow{\tilde{f}_\alpha} \mathbb{C} \\ \phi \quad &\longmapsto f(\phi(\alpha)) \end{aligned}$$

is a regular function. Since f is a class function, $\tilde{f}_\alpha$ is invariant under $\mathsf{SL}(2,\mathbb{C})$ and defines a regular function

$$f_\alpha : X \longrightarrow \mathbb{C}.$$

Let $[\phi] \in X$. Then the tangent space to X equals $H^1(M; V_\phi)$, so the Hamiltonian vector field $\mathsf{Ham}(f_\alpha)$ associates to $[\phi] \in X$ a tangent vector in $H^1(M; V_\phi)$.

Cap product with the fundamental homology class $[M] \in H_2(M;\mathbb{C})$ defines the *Poincaré duality isomorphism:*

$$\cap[M] : H^1(M; V_\phi) \xrightarrow{\cong} H_1(M; V_\phi).$$

Choose a basepoint $x_0 \in M$, an isomorphism of the fiber of V_ϕ over x_0 with $\mathbb{C}^2$, and a representative holonomy homomorphism

$$\phi_0 : \pi_1(M; x_0) \longrightarrow \mathsf{SL}(2,\mathbb{C}).$$

Let $s_0 \in S^1$ be a basepoint. Let $\alpha_0 : (S^1, s_0) \longrightarrow (M, x_0)$ be a based loop in M corresponding to α. Let σ be the parallel section of the flat vector bundle $\alpha_0^* V_\phi$ over S^1 which equals $F(\phi_0(\alpha_0))$ at s_0. Then σ defines a V_ϕ-valued 1-cycle in M, with homology class

$$[\sigma] \in H_1(M; V_\phi).$$

Lemma 2.2.1. *The value of the Hamiltonian vector field* $\mathsf{Ham}(f_\alpha)$ *at a point* $[\phi] \in X$ *is the vector in*

$$T_{[\phi]}X \cong H^1(M; V_\phi)$$

corresponding to the Poincaré dual of the homology class of the V_ϕ*-valued cycle* σ*:*

$$\begin{aligned} \cap[M] : H^1(M; V_\phi) &\longrightarrow H_1(M; V_\phi) \\ \mathsf{Ham}(f_\alpha) &\longmapsto [\sigma]. \end{aligned}$$

Using the above formula and the duality between cup-product and intersections of cycles, we obtain a formula for the Poisson bracket of trace functions ([11]):

Proposition 2.2.2. *Let* α, β *be oriented closed curves meeting transversely in double points* $p_1, \dots, p_k$*. For each* p_i*, choose representatives*

$$\phi_i : \pi_1(M; p_i) \longrightarrow \mathsf{SL}(2,\mathbb{C}).$$

Let α_i and β_i be the elements of $\pi_1(M;p_i)$ representing α,β respectively. Then the Poisson bracket of functions f_α, f_β is:

$$\begin{aligned}\big\{f_\alpha, f_\beta\big\} &= \Omega\big(\mathsf{Ham}(f_\alpha), \mathsf{Ham}(f_\beta)\big)\\ &= \sum_{i=1}^{k} \epsilon(p_i;\alpha,\beta)\ \mathbb{B}\big(F(\phi_i(\alpha_i)), F(\phi_i(\beta_i))\big)\end{aligned}$$

where $\epsilon(p_i;\alpha,\beta)$ denotes oriented intersection number.

Corollary 2.2.3. *If α,β are disjoint, then f_α and f_β Poisson-commute.*

By arguments in [11] (based on a suggestion of S. Wolpert), the converse holds if one of α or β is simple. A purely topological proof of this fact has recently been given by Chas [5].

2.3. **Complex twist flows.** Suppose that α is represented by a simple closed curve A. Then a holomorphic $\mathbb{C}$-action on the representation variety $\mathsf{Hom}(\pi,\mathsf{SL}(2,\mathbb{C}))$ covers the complex Hamiltonian flow on X. The *complex twist flow* τ_α is the holomorphic action of $\mathbb{C}$ of $\mathsf{Hom}(\pi,\mathsf{SL}(2,\mathbb{C}))$ defined as follows.

There are two cases, depending on whether A separates M or not. Denote by $M|A$ the compact surface with boundary whose interior is homeomorphic to the complement $M-A$. Denote the two components of $\partial(M|A)$ by A_+ and A_-. The quotient map $q: M|A \longrightarrow A$ results from identifying these components by a homeomorphism $\eta: A_+ \longrightarrow A_-$.

2.3.1. *The nonseparating case.* If A is nonseparating, then $M|A$ is connected, and choosing a basepoint in A_+ we express π as an HNN-extension

$$\pi_1(M|A)\star_\iota$$

as follows. Choose a basepoint $x_0 \in A$, and lift it to a basepoint $\tilde{x}_0 \in A_+ \subset M|A$. Let α_+ denote the element of $\pi_1(M|A;\tilde{x}_0)$ corresponding to A_+. Let $\tilde{\beta}$ denote a simple arc joining $\tilde{x}_0$ to $\eta(\tilde{x}_0)$ in $M|A$, and α_- the based loop $\tilde{\beta}^{-1}\cdot\alpha_+\cdot\tilde{\beta}$ at the basepoint $\tilde{x}_0$. Then η corresponds to the isomorphism

$$\iota: \langle\alpha_+\rangle \longrightarrow \langle\alpha_-\rangle$$

between the two subgroups of $\pi_1(M|A)$. Let N be the normal closure of the element

$$\beta\alpha_+\beta^{-1}\big(\alpha_-\big)^{-1}$$

of $\pi_1(M|A) \star \langle\beta\rangle$. Then the HNN-extension $\pi_1(M|A)\star_\iota$ is defined as the quotient

$$\pi \cong \Big(\pi_1(M|A) \star \langle\beta\rangle\Big)/N.$$

As in [11], for any $\phi \in \mathsf{Hom}(\pi, \mathsf{SL}(2,\mathbb{C}))$, and $t \in \mathbb{C}$, define

$$\xi_t(\phi) : \gamma \longmapsto \begin{cases} \phi(\gamma) & \text{if } \gamma \in \pi_1(M|A) \\ \phi(\beta)\ \zeta_t & \text{if } \gamma = \beta. \end{cases} \tag{2.5}$$

where

$$\zeta_t = \exp\big(tF(\phi(\alpha_+))\big)$$

is the one-parameter subgroup corresponding to $\phi(\alpha_+)$.

2.3.2. *The separating case.* If A separates M, then let M_+ and M_- denote the two components of $M|A$. Then π is the free product of subgroups corresponding to $\pi_+(M_+)$ and $\pi_1(M_-)$ respectively, amalgamated over the images $\langle\alpha\rangle \longrightarrow \pi_1(M_\pm)$. Then as in [11], for any $\phi \in \mathsf{Hom}(\pi, \mathsf{SL}(2,\mathbb{C}))$, $t \in \mathbb{C}$, define

$$\xi_t^\alpha(\phi) : \gamma \longmapsto \begin{cases} \phi(\gamma) & \text{if } \gamma \in \pi_1(M_+) \\ \zeta_t\ \phi(\gamma)\ \zeta_{-t} & \text{if } \gamma \in \pi_1(M_-) \end{cases}$$

where

$$\zeta_t = \exp\big(tF(\phi(\alpha)\big)$$

is the one-parameter subgroup corresponding to $\phi(\alpha)$.

2.3.3. *Irreducibility.* To show that the Hamiltonian flows act on X, we need to show that the complex twist flows preserve the set of irreducible representations.

Lemma 2.3.1. *Suppose that $\phi \in \mathsf{Hom}(\pi, \mathsf{SL}(2,\mathbb{C}))^s$. Then*

$$\xi_t^\alpha(\phi) \in \mathsf{Hom}(\pi, \mathsf{SL}(2,\mathbb{C}))^s.$$

Proof. Let M' be a component of $M|A$. As above, choose basepoints and identifications to express π as an amalgamated free product or HNN construction with $\pi_1(M') \hookrightarrow \pi_1(M)$. Then

$$\xi_t^\alpha\phi(\pi) \supset \phi(\pi_1(M'))$$

implies that if $\phi_{\pi_1(M')}$ is irreducible, then so is $\xi_t^\alpha\phi$. Thus we may assume that $\phi_{\pi_1(M')}$ is reducible but ϕ is irreducible.

Suppose first that A is nonseparating. We assume $\xi_t^\alpha\phi$ is reducible and derive a contradiction. Let $\Phi = \mathsf{Fix}(\phi(\pi_1(M|A)))$. Since

$$\phi(\alpha_+) \in \phi\big(\pi_1(M|A)\big),$$

the element $\phi(\alpha_+)$ fixes Φ, and Lemma 2.1.1 implies that ζ_t fixes Φ. Since $\xi_t^\alpha\phi(\pi)$ is reducible and is generated by $\phi(\pi_1(M|A))$ and $\phi(\beta)$, it follows that

$$\xi_t^\alpha\phi(\beta) = \phi(\beta)\zeta_t$$

fixes no element of Φ. Since ζ_t fixes Φ, this implies that $\phi(\beta)$ fixes Φ, a contradiction.

Suppose finally that A separates M into two components $M_\pm$. By the remark above, we may assume that $\phi_{\pi_1(M_\pm)}$ is reducible. Let

$$\Phi_\pm = \mathsf{Fix}(\phi(\pi_1(M_\pm))).$$

We may assume that each $\Phi_\pm$ is nonempty but $\Phi_+\cap\Phi_- = \emptyset$. Now $\phi(\alpha)$ fixes each $\Pi_\pm$, and Lemma 2.1.1 implies that ζ_t fixes each $\Pi_\pm$. Since $\phi(\pi_1(M_+))$ and $\zeta_t\phi(\pi_1(M_-))\zeta_t^{-1}$ generate $\xi_t^\alpha\phi(\pi)$,

$$\begin{aligned}
\mathsf{Fix}(\xi_t^\alpha\phi(\pi)) &= \mathsf{Fix}\big(\phi(\pi_1(M_+))\big) \cap \mathsf{Fix}\big(\zeta_t\phi(\pi_1(M_-))\zeta_t^{-1}\big) \\
&= \mathsf{Fix}\big(\phi(\pi_1(M_+))\big) \cap \zeta_t\mathsf{Fix}\big(\phi(\pi_1(M_-))\big)\big) \\
&= \Phi_+ \cap \zeta_t\Phi_- \\
&= \Phi_+ \cap \Phi_- = \emptyset
\end{aligned}$$

as desired. □

2.4. **Gauge-Theoretic Interpretation.** These twist flows admit an interpretation in terms of flat connections. The character variety X identifies with the quotient of the space $\mathcal{F}^{irr}(E)$ of irreducible flat $\mathsf{SL}(2,\mathbb{C})$-connections on a principal $\mathsf{SL}(2,\mathbb{C})$-bundle E over M by the group $\mathcal{G}(E)$ of gauge transformations of E. (Since a principal $\mathsf{SL}(2,\mathbb{C})$-bundle over M which admits a flat connection is necessarily trivial, we may assume E is the product bundle.)

Let $A\subset M$ be a simple closed curve, and choose a basepoint $a_0\in A$. Choose an orientation on A and let α be a based loop on M corresponding to the orientation on A.

Pull E back to a principal $\mathsf{SL}(2,\mathbb{C})$-bundle q^*E over $M|A$ by the quotient map $q: M\longrightarrow M|A$. Choose a point e_0 in the fiber of E over a_0. Let $a_\pm$ be the two elements of $q^{-1}(a_0)$ in $A_\pm$ respectively and $e_\pm$ the elements in the fibers of q^*E over $a_\pm$ corresponding to e_0.

Define the twist flow $\tilde{\xi}_t$ on $\mathcal{F}^{irr}(E)$ as follows. Let $\nabla\in\mathcal{F}^{irr}(E)$ be a flat connection on E. Parallel transport of e_0 along α with respect to α defines a holonomy transformation $\phi_0(\alpha_0)$ as above. Let ζ_t denote the corresponding one-parameter subgroup of $\mathsf{SL}(2,\mathbb{C})$. There is a one-parameter family of gauge-transformations g_t of q^*E supported in a collar neighborhood N of $A_+\subset M|A$ assuming the “value” ζ_t on A_+.

Explicitly, choose a smooth embedding

$$\psi : [0,1] \times S^1 \longrightarrow N \hookrightarrow M|A$$

mapping $\{1\} \times S^1$ diffeomorphically to A_+. Let $r : [0,1] \longrightarrow S^1$ denote the quotient mapping identifying $0, 1 \in [0,1]$. Parallel transport of e_+ defines a trivialization of the pullback of E to $[0,1] \times [0,1]$ so that $\psi^* E$ identifies with the quotient of

$$[0,1] \times [0,1] \times \mathsf{SL}(2,\mathbb{C})$$

by the identification

$$(s, 0, h) \longleftrightarrow (s, 1, \phi_0(\alpha_0)h).$$

Define g_t as the gauge transformation which is the identity map on the complement of N in $M|A$, and equals

$$(s, \theta, h) \longrightarrow (s, \theta, \zeta_{s,t} h)$$

in this trivialization.

Since ζ_t centralizes the holonomy of $q^*\nabla$ along A_+, the identification map $\eta : A_- \longrightarrow A_+$ identifies the restrictions of $(g_t)^*(q^*\nabla)$ to $A_\pm$. Thus a unique flat connection $\tilde{\xi}_t(\nabla)$ exists, satisfying

$$q^*\big(\tilde{\xi}_t(\nabla)\big) = (g_t)^*(q^*\nabla).$$

This is the orbit of the twist flow on flat connections. Clearly the gauge transformation g_t does not arise from a gauge transformation of E, unless the holonomy along A is $\pm\mathbb{I}$. The orbit covers the orbit of the holonomy of ∇ under the twist flow in $X \cong \mathcal{F}^{irr}(E)/\mathcal{G}(E)$.

The flow $\tilde{\xi}$ on $\mathcal{F}^{irr}(E)$ depends only on the choice of the collar neighborhood $N \subset M|A$ and $\psi : [0,1] \times S^1 \longrightarrow N$.

2.5. **Periods.** The subspace $\mathsf{Hom}(\pi, \mathsf{SU}(2))^s$ maps to a (real) symplectic submanifold $X_U \subset X$, and the corresponding real flows define Hamiltonian systems which have been studied in [11], and are closely related to periodic flows studied by Jeffrey-Weitsman [24, 25, 26, 27]. The orbits of these flows are all closed, although the period of the orbit varies with the value of f.

Proposition 2.5.1. *Let $[\phi] \in X_U$ and let $\alpha \in \pi$ be represented by a simple loop A. If $f_\alpha([\phi] \neq \pm 2$, the period of the trajectory of the flow of* $\mathsf{Ham}(f_\alpha)$ *at $[\phi]$ equals*

$$\frac{4\pi}{\sqrt{4 - f_\alpha([\phi])^2}}$$

if A is nonseparating and

$$\frac{2\pi}{\sqrt{4 - f_\alpha([\phi])^2}}$$

if A separates. If $f_\alpha([\phi]) = \pm 2$, then $[\phi]$ is fixed under the flow of $\mathsf{Ham}(f_\alpha)$*.*

Proof. Let $P \in \mathsf{SU}(2)$. Apply an inner automorphism of $\mathsf{SU}(2)$ to assume that P is diagonal:

$$P = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}. \tag{2.6}$$

Then

$$f(P) = 2\cos(\theta), \qquad F(P) = \begin{bmatrix} i\sin(\theta) & 0 \\ 0 & -i\sin(\theta) \end{bmatrix}$$

with corresponding one-parameter subgroup

$$\zeta_t = \exp(tF(P)) = \begin{bmatrix} e^{i\sin(\theta)t} & 0 \\ 0 & e^{-i\sin(\theta)t} \end{bmatrix}$$

Since the $\mathsf{PSL}(2,\mathbb{C})$-action on $\mathsf{Hom}(\pi, \mathsf{SL}(2,\mathbb{C}))^s$ is proper and free, the quotient map is a principal $\mathsf{PSL}(2,\mathbb{C})$-fibration:

$$\mathsf{PSL}(2,\mathbb{C}) \longrightarrow \mathsf{Hom}(\pi, \mathsf{SL}(2,\mathbb{C}))^s \longrightarrow X.$$

Thus the trajectory of the flow ξ_t^α projects diffeomorphically to the trajectory of the flow of $\mathsf{Ham}(f_\alpha)$ on X. The period equals the infimum of all $t > 0$ such that

$$\xi_t^\alpha(\phi) = \phi.$$

If α is nonseparating, then (2.5) implies that T equals the infimum of all $t > 0$ such that

$$\zeta_t \phi(\beta) = \phi(\beta).$$

By (2.2),

$$T = \frac{2\pi}{\sin(\theta)} = \frac{4\pi}{\sqrt{4 - f_\alpha([\phi])^2}}$$

If α separates, then (2.6) implies that T equals the infimum of all $t > 0$ such that

$$\zeta_t \phi(\beta) \zeta_{-t} = \phi(\beta).$$

for all $\beta \in \pi_1(M_-)$. If

$$\phi(\beta) \in \{\pm\mathbb{I}\} = \mathsf{center}(\mathsf{SU}(2))$$

for all $\beta \in \pi_1(M_-)$, then $\phi(\alpha) = 1$. Since $\phi \in \mathsf{Hom}(\pi, \mathsf{SL}(2,\mathbb{C}))^s$, there exists $\beta \in \pi$ such that $\phi(\beta) \neq 1$. The period T is the infimum of all $t > 0$ such that $\zeta_t = \pm\mathbb{I}$. By (2.2),

$$T = \frac{\pi}{\sin(\theta)} = \frac{2\pi}{\sqrt{4 - f_\alpha([\phi])^2}}$$

$\square$

2.6. **Dehn twists.** Closely related to the twist flows are the maps of X_U induced by Dehn twists. Recall that the *Dehn twist* about a simple loop $A \subset M$ is the diffeomorphism supported in a tubular neighborhood of A. If

$$\psi : [0,1] \times S^1 \hookrightarrow M$$
$$\{\frac{1}{2}\} \times S^1 \hookrightarrow A$$

is such a tubular neighborhood, then the Dehn twist τ_A about A is ψ-related to the diffeomorphism of $[0,1] \times S^1$ (restricting to the identity on the boundary) defined by:

$$(s, e^{i\theta}) \longmapsto (s, e^{2\pi i s} e^{i\theta}).$$

The action of τ_A on the fundamental group is given by:

$$(\tau_A)_*\,(\phi) : \gamma \longmapsto \begin{cases} \phi(\gamma) & \text{if } \gamma \in \pi_1(M|A) \\ \phi(\beta)\ \phi(\alpha) & \text{if } \gamma = \beta. \end{cases}$$

if A is nonseparating, and by:

$$(\tau_A)_*\,(\phi) : \gamma \longmapsto \begin{cases} \phi(\gamma) & \text{if } \gamma \in \pi_1(M_+) \\ \phi(\alpha)\phi(\gamma)\phi(\alpha)^{-1} & \text{if } \gamma \in \pi_1(M_-) \end{cases}$$

if A separates.

Let P be as in (2.6). Comparing (2.6) with (2.2), $\zeta_t = P$ for

$$\theta = \sin(\theta) t,$$

that is,

$$t = \frac{\theta}{\sin(\theta)} = \frac{\cos^{-1}\big(f(P)/2\big)}{2\pi\sqrt{4 - f(P)^2}}. \tag{2.7}$$

Combining (2.7), (2.7) with (2.7) implies:

Proposition 2.6.1. *Let $\alpha \in \pi$ correspond to a simple closed curve $A \subset M$. The map $(\tau_A)_*$ on $\mathsf{Hom}(\pi, \mathsf{SL}(2,\mathbb{C}))$ induced by Dehn twist about A equals the time-t map of the flow generated by $\mathsf{Ham}(f_\alpha)$, where*

$$t = \frac{2\cos^{-1}(f_\alpha([\phi])/2)}{\sqrt{4 - f_\alpha([\phi])^2}}.$$

In terms of a parametrization $\mathbb{R}/\mathbb{Z} \longrightarrow X_U$ of this trajectory, $(\tau_A)_*$ acts by translation of

$$\frac{\theta}{2\pi} = \frac{\cos^{-1}\big(f(P)/2\big)}{2\pi}$$

which has infinite order for almost every value of $f(A)$.

By reparametrizing the flow, we obtain Hamiltonian flows whose time-one map is the identity map or the Dehn twist. Jeffrey-Weitsman [25] (§5.1), consider flows of Hamiltonians given by invariant functions

$$\theta(P) = \cos^{-1}\left(\frac{f(P)}{2}\right)$$

which define S^1-actions, but are undefined at $P = \pm\mathbb{I}$. Thus their flows are only defined on the on the dense open subset where $\phi(P)$ is not central. On the other hand their flows are periodic with period 2π if A is nonseparating, and period π if A separates. (Actually they work with the symplectic form which is $1/(4\pi^2)$ of ours, to obtain a 2-form with integral cohomology class. Thus their periods are $1/(2\pi)$ if A is nonseparating and $1/(4\pi)$ if A separates.)

2.6.1. *Orbits of complex twist flows.* On the $\mathsf{SL}(2,\mathbb{C})$-character variety X, the Hamiltonian vector field $\mathsf{Ham}(f_\alpha)$ generates a holomorphic $\mathbb{C}$-action. (2.4) implies that if $f(A) = \pm 2$ and $A \neq \pm\mathbb{I}$ (that is, A is $\pm$-unipotent), then the trajectory defines an injective holomorphic map

$$\begin{aligned} \mathbb{C} &\longrightarrow X \\ t &\longmapsto \xi_t^\alpha([\phi]). \end{aligned} \tag{2.8}$$

If $f(A) \neq \pm 2$, then (2.3) implies that $\zeta_t = 1$ whenever $t \in\ t_0\mathbb{Z}$ where

$$t_0 := \frac{4\pi i}{\lambda - \lambda^{-1}} = \frac{4\pi i}{\big(f(A)^2 - 4\big)^{1/2}} \tag{2.9}$$

(which is well-defined only up to sign). In this case the trajectory of $[\phi]$ is the image of the holomorphic embedding

$$\begin{aligned} \mathbb{C}^* &\longrightarrow X \\ e^z &\longmapsto \xi^{\alpha}_{\left(2z\,/\,(\lambda-\lambda^{-1})\right)}\big([\phi]\big). \end{aligned} \tag{2.10}$$

2.6.2. *Dehn twists.* At a point $[\phi] \in X$ where $f_\alpha(\phi) = \pm 2$, then either

- $\phi(\alpha) = \pm\mathbb{I}$ is a central element and $[\phi]$ is fixed under the entire $\mathbb{C}$-action, or
- $\phi(\alpha)$ is $\pm$ a parabolic element and (2.4) implies $\phi(\alpha) = \pm \exp F(\alpha)$.

In the latter case the embedding (2.8) is equivariant with respect to the action of $\mathbb{Z}$ by translation on $\mathbb{C}$ and the $\mathbb{Z}$-action generated by $(\tau_A)_*$ on X.

Suppose that $f_\alpha([\phi]) \neq \pm 2$. In that case $\zeta_t = P$ precisely when $t \equiv t_1 \big(\bmod t_0\big)$, where

$$t_1 = \frac{\log(\lambda)}{\lambda - \lambda^{-1}} = \frac{\log\big((f(A) \pm (f(A)^2-4)^{1/2})/2\big)}{\big(f(A)^2-2\big)^{1/2}}.$$

(The two choices for $(f(A)^2-4)^{1/2}$ differ by sign but determine equal values for t_1.) In that case the embedding (2.10) is equivariant with respect to the actions generated by multiplication by

$$\lambda = \frac{(f(A) \pm (f(A)^2-4)^{1/2})}{2}$$

on $\mathbb{C}^*$ and by $(\tau_A)_*$ on X.

2.6.3. *Fenchel-Nielsen twist flows for $G = \mathsf{SL}(2,\mathbb{R})$.* Since the complex trace form $\mathbb{B}$ on $\mathfrak{sl}(2,\mathbb{C})$ restricts to the trace form on $\mathfrak{sl}(2,\mathbb{R})$, the complex-symplectic structure Ω on X restricts to the symplectic structure on $\mathfrak{T}(M)$ defined by the trace form of $\mathsf{SL}(2,\mathbb{R})$. By [9] this symplectic structure equals (-2) the Weil-Petersson Kähler form.

Inside the complex twist flows are the *Fenchel-Nielsen twist flows* when the holonomy is hyperbolic. Suppose that $\phi \in \mathsf{Hom}(\pi, \mathsf{SL}(2,\mathbb{R}))$ is a discrete embedding (that is, a *Fuchsian representation*). The subset of X consisting of equivalence classes of Fuchsian representations corresponds bijectively to the space of marked hyperbolic structures on M, that is, the *Teichmüller space* $\mathfrak{T}(M)$. In that case, for every $\mathbb{I} \neq \alpha \in \pi$, the element $\phi(\alpha) \in \mathsf{SL}(2,\mathbb{R})$ is hyperbolic. Geometrically it is a *transvection* along a geodesic γ, and is conjugate to a diagonal matrix

$$P = \pm \begin{bmatrix} e^{l/2} & 0 \\ 0 & e^{-l/2} \end{bmatrix}.$$

l is the distance $\phi(\alpha)$ moves points along γ. In the hyperbolic structure on M, the element α corresponds to a homotopy class of closed loops; γ corresponds to the unique closed geodesic A freely homotopic to a free loop in α and l is the length of this geodesic.

The character function is related to the length function by:

$$f = 2\cosh(l/2).$$

Thus the invariant function l (which is only defined on the subset of $\mathsf{SL}(2,\mathbb{R})$ consisting of hyperbolic elements) defines functions

$$l_\alpha : \mathfrak{T}(M) \longrightarrow \mathbb{R}_+.$$

As in [11], the corresponding variation is

$$L(P) = \pm \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

with corresponding one-parameter subgroup

$$\zeta_t = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix},$$

the one-parameter subgroup of transvections moving at speed 2 along the geodesic.

The inner product of these infinitesimal transvections can be computed as follows. The matrices

$$L_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, L_2 = \begin{bmatrix} -\sin(\psi) & \cos(\psi) \\ \cos(\psi) & \sin(\psi) \end{bmatrix}$$

represent infinitesimal transvections about axes which intersect with angle ψ and

$$\mathbb{B}(L_1, L_2) = \mathsf{tr}(L_1 L_2) = 2\cos(\psi).$$

Thus (as in §3.9 of [11]), Wolpert's cosine formula 2.6.2, follows from Proposition 2.2.2.

If α corresponds to a simple closed curve A, then the corresponding twist flow η_t^α corresponds to the *Fenchel-Nielsen twist flow* on $\mathfrak{T}(M)$ defined geometrically as follows. (Compare Abikoff [1], Buser [4] and Harvey [20].) Represent A by the (necessarily simple) closed geodesic. Split M along A, and identify the boundary components A_+, A_- of $M|A$ by a translation of length t in the positively oriented direction. This is a well-defined path of marked hyperbolic structures in $\mathfrak{T}(M)$; see Wolpert [46] for details. In [46], Wolpert proves that the Fenchel-Nielsen twist flow is Hamiltonian with respect to the Weil-Petersson Kähler form on $\mathfrak{T}(M)$, with Hamiltonian potential the length function l_α [47].

The Fenchel-Nielsen twist flow η^α and the twist flow ξ^α are reparametrizations of one another. Specifically,

$$\xi^\alpha_t = \eta^\alpha_{\sinh(l_\alpha)}.$$

As in [9] (see also [11]), the Weil-Petersson Kähler form equals $-1/2$ the symplectic structure defined by the trace form $\mathbb{B}$ restricted to $\mathsf{SL}(2,\mathbb{R})$, implying the Fenchel-Nielsen flow is Hamiltonian. Since the Fenchel-Nielsen twist flow is Hamiltonian for the length function l_α, the derivative of the length function along the twist vector field equals the Poisson bracket $\{l_\alpha, l_\beta\}$, which can be computed by Proposition 2.2.2. We thus obtain Wolpert's Derivative Formula:

Theorem 2.6.2 ((Wolpert [47, 46]). *Let $\alpha, \beta \in \pi$ where α is represented by a simple closed curve. Then the derivative of the length function l_β with respect to the Fenchel-Nielsen twist flow equals the sum*

$$\sum_{i=1}^{k} \cos(\theta_i)$$

where α and β are represented by closed geodesics, $p_1, \dots, p_k$ are their intersection points, and θ_i is the angle from α to β at p_i.

In particular the action of the Dehn twist τ_A equals the time-t map of the Fenchel-Nielsen twist flow at time $t = l/2$.

3. Abelian Hamiltonian Actions

The classical Fenchel-Nielsen coordinates on $\mathfrak{T}(M)$ can be interpreted as a moment map arising from a *pants decomposition,* that is, a maximal collection of disjoint simple closed curves $\mathcal{P} = \{\alpha_1, \dots, \alpha_N\}$ on M which are each homotopically nontrivial and mutually nonhomotopic. If M has genus g, then $N = 3g - 3$ and the length functions define a map

$$l_{\mathcal{P}} = (l_{\alpha_1}, \dots, l_{\alpha_N}) : \mathfrak{T}(M) \longrightarrow (\mathbb{R}_+)^N$$

which is the moment map for a Hamiltonian $\mathbb{R}^N$-action. This $\mathbb{R}^N$-action is proper and free, and choosing a cross-section (a left-inverse to $l_{\mathcal{P}}$)

$$\sigma : (\mathbb{R}_+)^N \longrightarrow \mathfrak{T}(M)$$

defines a diffeomorphism

$$\begin{aligned} (\mathbb{R}_+)^N \times \mathbb{R}^N &\longrightarrow \mathfrak{T}(M) \\ (\lambda, t) &\longmapsto \eta^{\alpha_1}_{t_1} \dots \eta^{\alpha_N}_{t_N}\big(\sigma(\lambda)\big) \end{aligned} \tag{3.1}$$

The map $l_{\mathcal{P}}$ defines the *action variables* λ while the coordinates $t \in \mathbb{R}^N$ are the *angle variables.* Indeed, Wolpert shows [48] that this completely integrable system defines a symplectomorphism with $(\mathbb{R}_+)^N \times \mathbb{R}^N$: the Fenchel-Nielsen coordinates are Darboux coordinates with respect to the symplectic structure ω_{WP} defined by the Weil-Petersson Kähler form:

$$\omega_{WP} = \sum_{i=1}^{N} d\lambda_i \wedge dt_i$$

This section extends this theory from $\mathsf{SL}(2,\mathbb{R})$ to $\mathsf{SL}(2,\mathbb{C})$.

3.1. **Pants decompositions.** By Corollary 2.2.3, the complex Hamiltonian vector fields of f_α and f_β Poisson commute if α and β are represented by disjoint curves. Thus for a family of disjoint simple closed curves

$$\mathcal{P} = \{A_1, \dots, A_N\}$$

the corresponding complex twist flows of $\mathsf{Ham} f_{\alpha_i}$ generate a $\mathbb{C}^n$-action. Suppose these curves are each homotopically nontrivial and mutually nonhomotopic. The resulting map

$$\tau_{\mathcal{P}} : X \longrightarrow \mathbb{C}^N$$

is a moment map for a complex-Hamiltonian action of $\mathbb{C}^N$ on the complex-symplectic manifold X.

Suppose that $\mathcal{P}$ is *maximal,* that is $N = 3g - 3$, in which case each component

$$P \subset M|(\cup_{i=1}^N A_i)$$

is homeomorphic to a 3-holed sphere. In that case Fricke's theorem [13, 16] implies that the generic inverse image $(\tau_{\mathcal{P}})^{-1}(z)$ is a single $\mathbb{C}^N$-orbit. Specifically, let $(C^N)^s$ denote the subset consisting of $z \in \mathbb{C}^N$ such that, for every (i,j,k) for which $\alpha_i, \alpha_j, \alpha_k$ bound a 3-holed sphere P_{ijk},

$$z_{\alpha_i}^2 + z_{\alpha_j}^2 + z_{\alpha_k}^2 - z_{\alpha_i} z_{\alpha_j} z_{\alpha_k} \neq 4.$$

This expresses the condition that the restriction $\phi|_{\pi_1(P_{ijk})}$ is irreducible; Fricke's theorem asserts the triple

$$\big(f(\phi(\alpha_i)), f(\phi(\alpha_j)), f(\phi(\alpha_k)\big) \in \mathbb{C}^3$$

determines the equivalence class of such an irreducible representation. If $z \in (\mathbb{C}^N)^s$, the fiber $(\tau_{\mathcal{P}})^{-1}(z)$ is a single $\mathbb{C}^N$-orbit.

The subgroup $\Gamma_{\mathcal{P}}$ of Γ preserving the pants decomposition is the free abelian group generated by the Dehn twists τ_{α_i} for $i = 1, \dots, N$.

The resulting $\mathbb{Z}^n$-action lies in the $\mathbb{C}^N$-action by the formulas in §2.6. Namely, at a point $[\phi] \in X$ the orbit is an embedded product

$$\prod_{i=1}^{N} G_i$$

where

$$G_i = \begin{cases} \{1\} & \text{if } \alpha_i([\phi]) = \pm\mathbb{I} \\ \mathbb{C} & \text{if } f_{\alpha_i}([\phi]) = \pm 2 \text{ and } \alpha_i([\phi]) \neq \pm\mathbb{I} \\ \mathbb{C}^* & \text{if } f_{\alpha_i}([\phi]) \neq \pm 2 \end{cases}.$$

For each i with with $f_{\alpha_i}(\phi) \neq 2$, let

$$\lambda_i = \frac{f_{\alpha_i}(\phi) \pm (f_{\alpha_i}(\phi)^2 - 4)^{1/2})}{2}.$$

Then $(k_1, \ldots, k_N) \in \mathbb{Z}^N$ acts by translation by k_i on the i-th factor $G_i \approx \mathbb{C}$ for each i with $f_{\alpha_i}(\phi) = \pm 2$ and by multiplication by λ_i on each factor $G_i \approx \mathbb{C}^*$ with $f_{\alpha_i}(\phi) \neq \pm 2$.

3.2. **The moment map as an ergodic decomposition and its holomorphic extension.** The action of $\Gamma_{\mathcal{P}}$ on X_U is discussed in [15]. In this case the restriction

$$f_{\mathcal{P}} : X_U \longrightarrow [-2, 2]^N$$

is a moment map for a Hamiltonian $\mathbb{R}^n$-action whose orbits are all closed. These orbits are tori, invariant under the $\Gamma_{\mathcal{P}} \cong \mathbb{Z}^N$-action. Almost every fiber has dimension N, and the action is equivalent to an $\mathbb{Z}^N$-action on T^N by translation. For almost all level sets, this action is ergodic. It follows that the moment map $f_{\mathcal{P}}$ for the Hamiltonian $\mathbb{R}^N$-action is also the *ergodic decomposition* for the $\mathbb{Z}^N$-action ([15], Theorem 2.2):

Proposition 3.2.1. *Let $h : X_U \longrightarrow \mathbb{R}$ be a $\Gamma_{\mathcal{P}}$-invariant measurable function. Then there exists a measurable function $\psi : [-2, 2]^N \longrightarrow \mathbb{R}$ such that $h = \psi \circ f_{\mathcal{P}}$ almost everywhere.*

Theorem 3.2.2. *Let $h : X \longrightarrow \mathbb{CP}^1$ be a meromorphic function which is invariant under $\Gamma_{\mathcal{P}}$. Then there exists a meromorphic function $H : \mathbb{C}^{\mathcal{P}} \longrightarrow \mathbb{CP}^1$ such that $h = H \circ \tau_{\mathcal{P}}$.*

Proof. Let G denote the complex-Hamiltonian $\mathbb{C}^{\mathcal{P}}$-action on X and let G_U denote the Hamiltonian $\mathbb{R}^{\mathcal{P}}$-action on X_U. By Proposition 3.2.1, the measurable function h must be almost everywhere constant on the G_U-orbits on X_U. Thus for each $t \in \tau_{\mathcal{P}}(X_U)$, the function h is constant on the preimage $\tau_{\mathcal{P}}^{-1}(t) \cap X_U$ and:

- The preimages of the restriction of $\tau_{\mathcal{P}}$ to X_U are the G_U-orbits in X_U;
- Each G_U-orbit in $\tau_{\mathcal{P}}^{-1}(t)$ is $\mathbb{C}$-Zariski dense in its G-orbit;
- G acts transitively on $\tau_{\mathcal{P}}^{-1}(t)$.

Hence h is constant on each $\tau_{\mathcal{P}}^{-1}(t)$. for $t \in \tau_{\mathcal{P}}(X_U)$.

Now we find a holomorphic section of $\tau_{\mathcal{P}}$. There exists a connected open neighborhood W of X_U, a Zariski-closed subset $Z \subset X$ of positive codimension, and a holomorphic map

$$W - Z \xrightarrow{\sigma} \tau_p p^{-1}(W - Z)$$

so that the composition

$$W - Z \xrightarrow{\sigma} \tau_p p^{-1}(W - Z) \xrightarrow{\tau_{\mathcal{P}}} W - Z$$

equals the identity. Then the meromorphic function

$$h - h \circ \sigma \circ \tau_{\mathcal{P}}$$

on $\tau_p p^{-1}(W - Z)$ vanishes on each fiber $\tau_p p^{-1}(t)$ for $t \in W - Z \cap X_U$. Since $W - Z \cap X_U$ is totally real, it follows that $h - h \circ \sigma \circ \tau_{\mathcal{P}}$ vanishes on all of $W - Z$. Since $W - Z$ is dense in W, this function vanishes on W. Since W is nonempty and open, this function vanishes on all of X. Thus h is constant on the fibers of $\tau_{\mathcal{P}}$ as desired. $\square$

4. Deformation spaces of $\mathbb{CP}^1$-structures

A $\mathbb{CP}^1$-*structure* on M is a geometric structure with local coordinate charts mapping to $\mathbb{CP}^1$ with coordinate changes in the group $\mathsf{PSL}(2,\mathbb{C})$. The holonomy mapping hol maps the deformation space $\mathbb{CP}^1(M)$ locally biholomorphically to the character variety $X - X_U$. Therefore Ω induces a complex-symplectic structure $\mathsf{hol}^*\Omega$ on $\mathbb{CP}^1(M)$. On the other hand, $\mathbb{CP}^1(M)$ is classically the total space of an affine bundle over $\mathfrak{T}(M)$ whose associated vector bundle is the cotangent bundle $T^*\mathfrak{T}(M)$. The canonical exact complex-symplectic structure on $T^*\mathfrak{T}(M)$ defines a complex-symplectic structure on $\mathbb{CP}^1(M)$. Kawai [30] has proved this complex-symplectic structure equals $\mathsf{hol}^*\Omega$:

Theorem. (Kawai) *The complex-symplectic structure on $\mathbb{CP}^1(M)$ induced from the holomorphic cotangent bundle structure on $T^*\mathfrak{T}(M)$ equals the complex-symplectic structure induced from the complex-symplectic structure Ω on X by*

$$\mathbb{CP}^1(M) \xrightarrow{\mathsf{hol}} \mathsf{Hom}(\pi, \mathsf{SL}(2,\mathbb{C}))/\!\!/\mathsf{SL}(2,\mathbb{C}).$$

The complex twist flows on X lift to complex twist flows on $\mathbb{CP}^1(M)$. We relate the complex twist geometry on the open subset $\mathcal{QF}(M) \subset \mathbb{CP}^1(M)$ comprising *quasi-Fuchsian structures* to the quakebends studied by Epstein-Marden [7], Kourouniotis [32, 33, 34], McMullen [36], Platis [39], Series [42], and Tanigawa [44].

4.1. **$\mathbb{CP}^1$-structures.** Let M be a smooth surface. A $\mathbb{CP}^1$-*atlas* on M consists of

- An open covering $\mathcal{U}$ of M by *coordinate patches*;
- For each coordinate patch $U \in \mathcal{U}$, a *coordinate chart*
$$\psi_U : U \longrightarrow \mathbb{CP}^1$$
which is a diffeomorphism of U onto its image $\psi_U(U)$;
- For each connected open subset $C \subset U \cap V$ of the intersection of coordinate patches $U, V \in \mathcal{U}$, a transformation $g_C \in \mathsf{PSL}(2,\mathbb{C})$ such that
$$g_C \circ \psi_U|_{U\cap V} = \psi_V|_{U\cap V}.$$
The *coordinate change* g_C is unique.

Such atlases are partially ordered by inclusion. A $\mathbb{CP}^1$-*structure* on M is a maximal $\mathbb{CP}^1$-atlas. A $\mathbb{CP}^1$-*manifold* is a manifold with a $\mathbb{CP}^1$-structure.

Suppose M and M' are $\mathbb{CP}^1$-manifolds. A mapping $\phi : M \longrightarrow M'$ is a $\mathbb{CP}^1$-*mapping* if and only if for each coordinate chart (U, ψ_U) in M and each coordinate chart $(U', \psi'_{U'})$ in M' the composition
$$(\psi'_{U'}) \circ \phi \circ (\psi_U)^{-1} : \ \psi_U\big(U \cap \phi^{-1}(U')\big) \ \longrightarrow \ \psi'_{U'}\big(\phi(U) \cap U'\big)$$
extends to an element of $\mathsf{PSL}(2,\mathbb{C})$ on each component of $\psi_U(U \cap \phi^{-1}(U')$. A $\mathbb{CP}^1$-mapping is necessarily a local diffeomorphism.

If $f : M \longrightarrow M'$ is a local diffeomorphism, then every $\mathbb{CP}^1$-structure on M' determines a unique $\mathbb{CP}^1$-structure on M such that f is a $\mathbb{CP}^1$-mapping. In particular every covering space of a $\mathbb{CP}^1$-manifold is a $\mathbb{CP}^1$-manifold, and its group of covering transformations is realized by a group of $\mathbb{CP}^1$-automorphisms.

If M is a simply-connected $\mathbb{CP}^1$-manifold, then any chart on M extends to a globally defined *developing map*
$$\mathsf{dev} : M \longrightarrow \mathbb{CP}^1$$
which is a $\mathbb{CP}^1$-mapping. This mapping is unique up to left-composition with transformations in $\mathsf{PSL}(2,\mathbb{C})$.

We denote by Σ the Riemann surface whose underlying manifold is M for which dev is holomorphic.

In general the universal covering space $\tilde{M}$ of a $\mathbb{CP}^1$-manifold admits a developing map into $\mathbb{CP}^1$. The group $\pi_1(M)$ of covering transformations of $\tilde{M} \longrightarrow M$ admits a representation $\rho : \pi_1(M) \longrightarrow \mathsf{PSL}(2,\mathbb{C})$ such that the diagram

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\mathsf{dev}} & \mathbb{CP}^1 \\ {\scriptstyle\gamma}\Big\downarrow & & \Big\downarrow{\scriptstyle\rho(\gamma)} \\ \tilde{M} & \xrightarrow{\mathsf{dev}} & \mathbb{CP}^1 \end{array}$$

commutes for every $\gamma \in \pi_1(M)$. The representation ρ is called the *holonomy representation.* The *developing pair* (dev, ρ) is unique up to the action of $\mathsf{PSL}(2,\mathbb{C})$ (by left-composition with dev and conjugation on ρ). The developing map globalizes the coordinate charts and the holonomy representation globalizes the coordinate changes.

4.1.1. *Deformation spaces.* Let S be a fixed topogical surface with fundamental group $\pi = \pi(S)$. A *marked* $\mathbb{CP}^1$*-structure* on S consists of a $\mathbb{CP}^1$-manifold M and a homeomorphism $f_M : S \longrightarrow M$. Two marked $\mathbb{CP}^1$-structures (M, f_M) and $(M', f_{M'})$ are *equivalent* if and only if there exists a $\mathbb{CP}^1$-isomorphism $\phi : M \longrightarrow M'$ such that

$$\phi \circ f_M \simeq f_{M'}.$$

Let $\mathbb{CP}^1(S)$ denote the set of equivalence classes of marked $\mathbb{CP}^1$-structures on S. Holonomy defines a mapping

$$\mathsf{hol} : \mathbb{CP}^1(S) \longrightarrow \mathsf{Hom}(\pi, \mathsf{SL}(2,\mathbb{C}))/\!\!/\mathsf{SL}(2,\mathbb{C}).$$

As in [14] (see also Earle [6], Hubbard [22] and Kapovich [28]), there is a natural topology on $\mathbb{CP}^1(S)$ such that hol is a local homeomorphism. Furthermore Gallo-Kapovich-Marden [8] proved that the image of hol is $X - X_U - X_D$, where $X_D \subset X - X_U$ is the closed subset consisting of equivalence classes $[\phi]$ of representations leaving invariant a pair of points of $\mathbb{CP}^1$ (these correspond to irreducible representations for which an index-two subgroup acts reducibly).

4.2. **Holonomy and the conformal structure.** Since $\mathsf{PSL}(2,\mathbb{C})$ acts holomorphically on $\mathbb{CP}^1$, a $\mathbb{CP}^1$-atlas on M is a *holomorphic atlas,* that is, an atlas for a complex structure on M. Thus underlying every $\mathbb{CP}^1$-manifold is a Riemann surface. Recording the complex structure underlying a $\mathbb{CP}^1$-structure is a map

$$\mathbb{CP}^1(S) \xrightarrow{\Pi} \mathfrak{T}(M).$$

A *projective structure on a Riemann surface* Σ is a $\mathbb{CP}^1$-structure whose underlying complex structure is Σ.

Lemma 4.2.1. Π *is holomorphic.*

Proof. We recall the classical description of $\mathbb{CP}^1(S)$ as an affine bundle $\mathcal{Q}(M)$ over $\mathfrak{T}(M)$ whose associated vector bundle equals the cotangent bundle $T^*(\mathfrak{T}(M))$. (See for example §2.3 of Earle [6], Hubbard [22] or Gunning [18]). Fix a Riemann surface Σ homeomorphic to M and a projective structure s_0 on Σ. Given any other projective structure on Σ, its developing map is a holomorphic map from the universal covering $\tilde{\Sigma}$ to $\mathbb{CP}^1$. Its Schwarzian derivative (in the local projective coordinates defined by s_0) is a holomorphic quadratic differential which is invariant under π, and therefore defines a holomorphic quadratic differential on Σ. Furthermore this quadratic differential completely determines the developing map up to an element of $\mathsf{PSL}(2,\mathbb{C})$ of $\mathbb{CP}^1$. Thus the fiber $\Pi^{-1}([\Sigma])$ admits a simply transitive action of the complex vector space $H^0(\Sigma;\mathcal{O}(\kappa^2))$ comprising holomorphic quadratic differentials on Σ. The composition law for the Schwarzian derivative implies that changing the origin s_0 effects a translation of $Q(\Sigma)$, so $\Pi^{-1}(\Sigma)$ is an affine space modelled on $H^0(\Sigma;\mathcal{O}(\kappa^2))$, which is the cotangent space $T^*_{[\Sigma]}\mathfrak{T}(M)$ of $\mathfrak{T}(M)$ at $[\Sigma]$.

In particular $\mathbb{CP}^1(S)$ inherits a complex structure, making it a holomorphic affine bundle over $\mathfrak{T}(M)$. In particular the projection Π is holomorphic in this complex structure.

By Lemma 2 of Earle [6] or Hubbard [22], the holonomy mapping hol is a local biholomorphism, and therefore this complex structure on $\mathbb{CP}^1$ induced by hol is the above complex structure. Thus Π is holomorphic with respect to the complex structure induced by hol. ☐

Kawai's theorem implies that Π is a holomorphic Lagrangian fibration. As described in Hubbard [22], the differential $d\Pi$ identifies with the map

$$T_{s_0}\mathbb{CP}^1(S) \cong H^1\big(\Sigma;\mathfrak{sl}(2,\mathbb{C})\big) \longrightarrow H^1\big(\Sigma;\mathcal{O}(\kappa^{-1})\big) \cong T_{[\Sigma]}\mathfrak{T}(M)$$

induced by the short exact sequence of sheaves

$$0 \longrightarrow \mathfrak{sl}(2,\mathbb{C}) \xrightarrow{i} \mathcal{O}(\kappa^{-1}) \xrightarrow{D_3} \mathcal{O}(\kappa^2) \longrightarrow 0. \tag{4.1}$$

Here D_3 is the map of holomorphic sheaves given in local holomorphic coordinates by

$$h(z)\,\frac{\partial}{\partial z} \longmapsto h'''(z)\,dz^2.$$

Its kernel $\mathfrak{sl}(2,\mathbb{C})$ is the locally constant sheaf of locally projective vector fields (whose coefficients in affine coordinates are quadratic). The monomorphism i regards a locally projective vector field as a holomorphic vector field. Then (4.1) induces the exact sequence

$$0 \longrightarrow H^0\big(\Sigma, \mathcal{O}(\kappa^2)\big) \xrightarrow{\delta} H^1\big(\Sigma; \mathfrak{sl}(2,\mathbb{C})\big) \xrightarrow{i_*} H^1\big(\Sigma; \mathcal{O}(\kappa^{-1})\big) \longrightarrow 0.$$

If $\alpha \in H^0\big(\Sigma, \mathcal{O}(\kappa^2)\big)$ is a holomorphic quadratic differential and $\beta \in H^1\big(\Sigma; \mathfrak{sl}(2,\mathbb{C})\big)$, then

$$\langle \delta(\alpha), \beta \rangle = \alpha \cdot i_*(\beta) \tag{4.2}$$

where the first pairing $\langle , \rangle$ is the pairing on $H^1\big(\Sigma; \mathfrak{sl}(2,\mathbb{C})\big)$ induced by the complex Killing form $\mathbb{B}$ on $\mathfrak{sl}(2,\mathbb{C})$ and the second pairing is the Serre duality pairing

$$H^0\big(\Sigma, \mathcal{O}(\kappa^2)\big) \times H^1\big(\Sigma, \mathcal{O}(\kappa^{-1})\big) \longrightarrow H^1\big(\Sigma, \mathcal{O}(\kappa)\big) \cong \mathbb{C}.$$

This discussion implies that Π is a Lagrangian fibration. Suppose

$$\beta, \gamma \in \ker(d\Pi) = \delta(H^0\big(\Sigma, \mathcal{O}(\kappa^2)\big) = \ker(i_*)$$

Then $\gamma = \delta(\alpha)$ for some $\alpha \in H^0\big(\Sigma, \mathcal{O}(\kappa^2)\big)$, and (4.2) implies $\langle \gamma, \beta \rangle = 0$ since $i_*\beta = 0$. Therefore $\ker(d\Pi)$ is isotropic. Since its dimension is half of that of $H^1\big(\Sigma; \mathfrak{sl}(2,\mathbb{C})\big)$, the fibers of Π are Lagrangian. For further details, see Kawai [30].

4.2.1. *Meromorphic functions on $\mathbb{CP}^1(S)$.*

Proposition 4.2.2. *There exist nonconstant Γ-invariant meromorphic functions on $\mathbb{CP}^1(S)$.*

Proof. The *Riemann moduli space of curves* is the quotient $\mathfrak{T}(M)/\Gamma$, which by Knudsen [31], is a quasiprojective variety. Thus $\mathfrak{T}(M)/\Gamma$ admits nonconstant meromorphic functions and $\mathfrak{T}(M)$ admits nonconstant Γ-invariant meromorphic functions.

Let ψ be a nonconstant Γ-invariant meromorphic function on $\mathfrak{T}(M)$. Composing with the projection

$$\Pi : \mathbb{CP}^1(S) \longrightarrow \mathfrak{T}(M)$$

provides a nonconstant Γ-invariant meromorphic function $\psi \circ \Pi$ on $\mathbb{CP}^1(S)$. $\square$

4.3. **Quasi-Fuchsian space.** A $\mathbb{CP}^1$-structure is *Fuchsian* if and only if a developing map embeds $\tilde{M}$ in a hyperbolic plane $\mathsf{H}^1_{\mathbb{C}} \subset \mathbb{CP}^1$. Necessarily the holonomy representation is conjugate to a *Fuchsian representation,* that is, a discrete embedding $\pi \hookrightarrow \mathsf{PU}(1,1)$ (where $\mathsf{PU}(1,1)$ is the stabilizer in $\mathsf{PSL}(2,\mathbb{C})$ of the Poincaré disc $\mathsf{H}^1_{\mathbb{C}}$). However by a construction of Maskit and Hejhal (see also Goldman [12]) there exist $\mathbb{CP}^1$-structures with Fuchsian holonomy with surjective developing maps. A *quasi-Fuchsian* $\mathbb{CP}^1$*-structure* is a $\mathbb{CP}^1$-structure topologically conjugate to a Fuchsian structure, that is, there exists a homeomorphism h of $\mathbb{CP}^1$ such that

$$(h \circ \mathsf{dev}, h \circ \rho \circ h^{-1})$$

is the developing pair for a Fuchsian structure. The limit set Λ of $\rho(\pi)$ is the Jordan curve $h(\partial \mathsf{H}^1_{\mathbb{C}})$. The developing image $\mathsf{dev}(\tilde{M})$ is one component of the complement $\mathbb{CP}^1 - \Lambda$.

Quasi-Fuchsian space $\mathcal{QF}(M)$ is the subset of $\mathbb{CP}^1(S)$ corresponding to quasi-Fuchsian structures. The holonomy representation ρ is a *quasi-Fuchsian representation.* The holonomy mapping

$$\mathsf{hol} : \mathcal{QF}(M) \longrightarrow X$$

embeds $\mathcal{QF}(M)$ as the open subset of X consisting of conjugacy classes of quasi-Fuchsian representations. Quasi-Fuchsian space $\mathcal{QF}(M)$ is both the open subset of $\mathbb{CP}^1(S)$ consisting of quasi-Fuchsian structures and the open subset of $X - X_U$ consisting of characters of quasi-Fuchsian representations.

The conformal structures of the quotients of $\mathbb{CP}^1 - \Lambda$ by $\phi(\pi)$ determine an ordered pair $\mathfrak{T}(M) \times \bar{\mathfrak{T}}(M)$; the first parameter is the marked conformal structure underlying the $\mathbb{CP}^1$-structure. The *simultaneous uniformization theorem* of Bers [3] asserts that the corresponding map

$$\mathcal{QF}(M) \xrightarrow{\approx} \mathfrak{T}(M) \times \bar{\mathfrak{T}}(M)$$

is a biholomorphism. It is evidently Γ-equivariant.

Since $\mathcal{QF}(M) \hookrightarrow \mathbb{CP}^1(S)$, Corollary 4.2.2 implies that nonconstant Γ-invariant meromorphic functions exist on $\mathcal{QF}(M)$.

4.4. **Twist flows of $\mathbb{CP}^1$-structures and grafting.** Let $\alpha \in \pi$. Then the composition

$$\mathbb{CP}^1(S) \xrightarrow{\mathsf{hol}} X \xrightarrow{f_\alpha} \mathbb{C}$$

defines a holomorphic function on $\mathbb{CP}^1(S)$ and its complex-Hamiltonian $\mathsf{Ham}(f_\alpha \circ \mathsf{hol})$ is a holomorphic vector field on $\mathbb{CP}^1(S)$. If α corresponds to a simple closed curve A, then there is a *complex twist flow* $\tilde{\xi}^\alpha_t$ on $\mathbb{CP}^1$ covering the complex twist flow generated by the vector field $\mathsf{Ham}(f_\alpha)$ on X.

This twist flow is defined geometrically on Fuchsian structures as follows (see [12], Kourouniotis [32, 33, 34]). Let s_0 be a Fuchsian $\mathbb{CP}^1$-structure, so a developing map embeds $\tilde{M}$ as a geometric disc $\Delta \subset \mathbb{CP}^1$. Then the simple closed curve $A \subset M$ can be represented by a simple closed geodesic with respect the Poincaré metric induced from that of Δ. Let $\ell(A)$ denote the Poincaré length of this geodesic. A lift $\tilde{A}$ of A to

$\tilde{M}$ develops to a circular arc orthogonal to $\partial\mathsf{dev}(\tilde{M})$. The split surface $M|A$ inherits a $\mathbb{CP}^1$-structure whose boundary components develop to circular arcs. To define $s_t := \tilde{\xi^\alpha}_t(s_0)$, insert an annulus A_θ with $\mathbb{CP}^1$-structure into $M|A$. The angular parameter θ is the imaginary part $\mathsf{Im}(t)$. The annulus is a *θ-annulus* in the sense of §2.12 of [12]. Choose a holomorphic universal covering map

$$E : \mathbb{C} \longrightarrow \mathbb{CP}^1 - \mathsf{Fix}\big(\phi(\alpha)\big)$$

which is periodic with period $2\pi i$. (When $\mathsf{Fix}(\phi(\alpha)) = \{0, \infty\}$, then this is just the exponential map $\exp : \mathbb{C} \longrightarrow \mathbb{C}^*$. For general A, we compose exp with a projective transformation taking $\{0, \infty\}$ to $\mathsf{Fix}(\phi(\alpha))$.)

Define the *θ-strip*

$$S_\theta := \mathbb{R} + i[0, \theta] \subset \mathbb{C}$$

with $\mathbb{CP}^1$-structure induced by E. The developing map E for S_θ is equivariant with respect to the $\mathbb{Z}$-action on S_θ generated by translation by $\ell(A)$ and the $\mathbb{Z}$-action on $\mathbb{CP}^1 - \mathsf{Fix}(\phi(\alpha))$. The *$\theta$-annulus* A_θ is defined as the quotient $\mathbb{CP}^1$-manifold of S_θ.

The *grafted manifold* $M(t)$ is obtained by inserting a θ-annulus into $M|A$. Choose one component of $A_- \subset \partial(M|A)$ and attach A_θ to A_- to obtain a $\mathbb{CP}^1$-manifold homeomorphic to $M|A$ with one boundary component A'_+ corresponding to the component A_+ of $\partial(M|A)$ and another boundary component A' corresponding to the other component of $\partial(A_\theta)$. Identify A'_+ to A' by a transvection of displacement $\mathsf{Re}(t)$.

When $\theta = 0$, this is just the Fenchel-Nielsen twist deformation, obtained from $M|A$ by identifying the two components of $\partial(M|A)$ by a transvection of displacement $\mathsf{Re}(t)$.

Let $P \in \mathsf{SL}(2, \mathbb{C})$. These holomorphic flows on $\mathsf{Hom}(\pi, \mathsf{SL}(2, \mathbb{C}))$ cover holomorphic flows on X which have been extensively studied on quasi-Fuchsian space $\mathcal{QF}(M) \subset X$. On $\mathcal{QF}(M)$ these flows geometrically correpond to the *quakebends* or *complex earthquakes* discussed by Epstein-Marden [7], Kourouniotis [32, 33, 34], McMullen [36], Platis [39], Series [42], and Tanigawa [44]. In particular we obtain the following result of Platis [39]:

Theorem 4.4.1 ((Platis [39], Theorem 7, §2.2). *The complex twist vector field $\tilde{\xi}$ on $\mathcal{QF}(M)$ is Hamiltonian with respect to the complex length function $l^{\mathbb{C}}$.*

Just as the geodesic length function l on the hyperbolic subset of $\mathsf{SL}(2, \mathbb{R})$ and the angle function θ on $\mathsf{SU}(2) - \{\pm\mathbb{I}\}$ are more geometrically natural than the trace functions, we consider the *complex length* "function" on $\mathsf{SL}(2,\mathbb{C})$ defined by:

$$2\cosh\left(\frac{l^{\mathbb{C}}(P)}{2}\right) = f(P)$$

or equivalently

$$l^{\mathbb{C}}(P) = 2\log\left(\frac{\big(f(P) \pm (f(P)^2-4)^{1/2}\big)}{2}\right).$$

(This differs from Tan [43] whose complex length is half of ours. Our definition is consistent with Kourouniotis [32, 33, 34], Platis [39] and Series [42].) This function takes values in $(\mathbb{C}/4\pi i\mathbb{Z})/\{\pm 1\}$, since the logarithm is well-defined only modulo $4\pi i$ and the choice of square root introduces an ambiguity of sign.

Choose the branch $\tilde{l}_{\mathbb{C}}$ of the complex length which is positive on the subset of hyperbolic elements in $\mathsf{SL}(2,\mathbb{R})$; since $\mathcal{QF}(M)$ is simply connected

$$[\phi] \longmapsto \big(\tilde{l}^{\mathbb{C}}(\phi(\alpha_1)), \dots, \tilde{l}^{\mathbb{C}}(\phi(\alpha_N))\big)$$

defines a single-valued function

$$l^{\mathbb{C}}|_{\mathcal{P}} : \mathcal{QF}(M) \longrightarrow \mathbb{C}^N.$$

The restriction of $l^{\mathbb{C}}_{\mathcal{P}}$ to $\mathfrak{T}(M)$ is the length function $l_{\mathcal{P}} : \mathfrak{T}(M) \longrightarrow (\mathbb{R}_+)^N$ defined by (3.1). The fibers of $l^{\mathbb{C}}|_{\mathcal{P}}$ are orbits of the complex-Hamiltonian $\mathbb{C}^N$-action having $l^{\mathbb{C}}|_{\mathcal{P}}$ as moment map. Tan [43] and Kourouniotis [34] show that there exists a section $\sigma : \mathbb{C}^N \longrightarrow \mathcal{QF}(M)$ and a neighborhood of $(\mathbb{R}_+)^N \times \{0\}$ in $\mathbb{C}^N \times \mathbb{C}^N$ which maps biholomorphically to $\mathcal{QF}(M)$.

Korouniotis [33] and Series [42] derive formulas for the derivative of the complex length functions along quakebend flows. We briefly sketch how their formulas can be derived from Proposition 2.2.2 as a Poisson bracket, referring to [33, 42] for details.

Let $P \in \mathsf{SL}(2,\mathbb{C})$ be loxodromic with invariant axis a_P. Then the variation function $L^{\mathbb{C}}$ associated to the invariant function $l^{\mathbb{C}}$ and the complex-orthogonal structure $\mathbb{B}(X,Y) = \mathsf{tr}(XY)$ maps a diagonal matrix to

$$D_0 := \pm \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

(which is not uniquely defined without restrictions to make the function $l^{\mathbb{C}}$ single-valued). By (2.1), the element of $\mathsf{PSL}(2,\mathbb{C})$ corresponding to $L^{\mathbb{C}}(P)$ is the involution fixing a_P. If $P_1, P_2 \in \mathsf{PSL}(2,\mathbb{C})$ are loxodromic, with complex distance z between their invariant axes, then a simple calculation implies that $\mathbb{B}(L^{\mathbb{C}}(P_1), L^{\mathbb{C}}(P_2))$ equals $\cosh(z)$. Applying Proposition 2.2.2 to the complex length functions $l^{\mathbb{C}}_{\beta}$ and $l^{\mathbb{C}}_{\alpha}$, where α

corresponds to a simple closed curve, we obtain the following extension of Theorem 2.6.2:

Theorem 4.4.2 ((Korouniotis [33], Series [42]). *Let $\alpha, \beta \in \pi$ where α is represented by a simple closed curve A. Then the derivative of the complex length function l^C_β on $\mathcal{QF}(M)$ with respect to the quakebend flow with respect to A equals the sum*

$$\sum_{i=1}^{k} \cosh(d_i)$$

where α and β are represented by closed geodesics, $p_1, \ldots, p_k$ are their intersection points, and d_i the complex distance between the axes of $\phi_i(\alpha_i)$ to $\phi_i(\beta_i)$ at p_i, where ϕ_i is a representative homomorphism from $\pi_1(M; p_i)$ and α_i, β_i are elements of $\pi_1(M; p_i)$ representing α and β.

See Korouniotis [33] and Series [42] for details.

Tan [43] and Kourouniotis [32, 33, 34] extend Fenchel-Nielsen coordinates to complex Fenchel-Nielsen coordinates on $\mathcal{QF}(M)$. (See also Series [42] for another exposition.)

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