ERGODIC THEORY ON MODULI SPACES

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Dedicated to the memory of Peter Greenberg and Russell Mills

Abstract. Let M be a compact surface with $\chi(M) < 0$ and let G be a compact Lie group whose Levi factor is a product of groups locally isomorphic to SU(2) (for example SU(2)). Then the mapping class group Γ_M of M acts on the moduli space $X(M)$ of flat G-bundles over M (possibly twisted by a fixed central element of G). When M is closed, then Γ_M preserves a symplectic structure on $X(M)$ which has finite total volume on M. More generally, the subspace of $X(M)$ corresponding to flat bundles with fixed behavior over ∂M carries a Γ_M-invariant symplectic structure. The main result is that Γ_M acts ergodically on $X(M)$ with respect to the measure induced by the symplectic structure.

Contents

Date: April 25, 1997.
1991 *Mathematics Subject Classification.* 57M05 (Low-dimensional topology), 22D40 (Ergodic theory on groups).
Key words and phrases. Fundamental group of surface, mapping class group, Dehn twist, symplectic structure, character variety, moduli spaces, stable holomorphic vector bundle, Teichmüller space, parabolic structure.
The author gratefully acknowledges support from National Science Foundation grants DMS-8613576, DMS-8902619, DMs-9 504764, the Alfred P. Sloan Foundation, and the University of Maryland Institute for Advanced Computer Studeies.

1. Introduction

1.1. **Statement of results.** Let M be a closed oriented surface of genus $g > 1$ and let π denote its fundamental group. The mapping class group of M is by definition $\pi_0 \operatorname{Diff}(M)$. According to J. Nielsen, the mapping class group is isomorphic to the outer automorphism group $\Gamma_M = \operatorname{Out}(\pi)$. Let G be a Lie group and let $\operatorname{Hom}(\pi, G)$ denote the space of homomorphisms $\pi \to G$. Let $\operatorname{Hom}(\pi, G)/G$ denote the space of G-conjugacy classes of homomorphisms $\pi \to G$. We may geometrically identify $\operatorname{Hom}(\pi, G)/G$ with the moduli space of isomorphism classes of flat principal G-bundles over M. Then Γ_M acts on $\operatorname{Hom}(\pi, G)/G$. When G is a semisimple group then this action preserves a natural symplectic structure and hence a volume form (Goldman [12]). Although $\operatorname{Hom}(\pi, G)/G$ is compact, it has singularities and the symplectic structure is singular along the singular stratum; by Huebschmann [19], the measure of $\operatorname{Hom}(\pi, G)/G$ is nonetheless finite.

For many purposes it is useful to consider more general *twisted moduli spaces,* where the relation in the fundamental group is "twisted" by an element ζ_0 of the center of G. For example for $G = \mathrm{SU}(2)$, the space $\mathrm{Hom}(\pi, G)/G$ identifies with the moduli space of semistable holomorphic rank two vector bundles with fixed determinant of even degree and the $-I$-twisted moduli space $\mathrm{Hom}_{-}(\pi, G)/G$ identifies with the moduli space of semistable holomorphic rank two vector bundles with fixed determinant of odd degree. Furthermore $\mathrm{Hom}_{-}(\pi, G)/G$ is smooth. (See §2.2 for the precise definition).

Theorem 1.1. *Let G be a compact Lie group locally isomorphic to a product of copies of* $\mathrm{SU}(2)$ *and* $\mathrm{U}(1)$. *Then for every $\zeta_0 \in Z(G)$, the mapping class group Γ_M is ergodic on* $\mathrm{Hom}_{\zeta_0}(\pi, G)/G$.

Corollary 1.2. *Let $G = \mathrm{SU}(2)$. Then the actions of Γ_M on* $\mathrm{Hom}(\pi, G)/G$ *and* $\mathrm{Hom}_{-}(\pi, G)/G$ *are ergodic.*

This corollary was the original result of the paper. As we finished writing the paper, it became clear how to extend the techniques to cover the hypotheses above. The modifications necessary to prove Theorem 1.1 for products are given in §9.

Conjecture 1.3. *Γ_M is ergodic on* $\mathrm{Hom}_{\zeta_0}(\pi, G)/G$ *for any compact Lie group and $\zeta_0 \in Z(G)$.*

1.2. **The Narasimhan-Seshadri foliation of the universal moduli space.** Corollary 1.2 has a more geometric interpretation involving the Riemann moduli space $\mathcal{M}_g$ of curves of genus g. Let $\mathcal{T}_g$ denote the Teichmüller space of M; then $\mathcal{M}_g$ equals the quotient $\mathcal{T}_g/\Gamma_M$. The fiber product

$$\mathcal{B}_g = (\mathcal{T}_g \times \mathrm{Hom}(\pi, G)/G)/\Gamma_M$$

is the total space of a bundle over $\mathcal{M}_g$. The fiber over a point of $\mathcal{M}_g$ corresponding to a Riemann surface M consists of the Narasimhan-Seshadri moduli space of rank 2 semistable vector bundles over M with trivial determinant bundle. (See [36].) The foliation of $\mathcal{T}_g \times \mathrm{Hom}(\pi, G)/G$ by leaves $\mathcal{T}_g \times \{[\phi]\}$ is invariant under Γ_M and hence defines a foliation $\mathcal{F}$ of $\mathcal{B}_g$ transverse to the fibration over $\mathcal{M}_g$. Corresponding to the Γ_M-invariant measure on $\mathrm{Hom}(\pi, G)/G$ is an invariant transverse measure to $\mathcal{F}$. Similarly there is a foliated bundle $(\mathcal{B}'_g, \mathcal{F}')$ over $\mathcal{M}_g$ with fiber the moduli space of $(-I)$-twisted representations of $\pi_1(M)$.

Theorem 1.4. *The foliation $\mathcal{F}$ of $\mathcal{B}_g$ (respectively $\mathcal{F}'$ of $\mathcal{B}'_g$) is ergodic.*

1.3. **Moduli spaces over surfaces with boundary.** Theorem 1.1 is proved by inductively gluing together "smaller surfaces," which are necessarily compact surfaces-with-boundary. The inductive process involves the following generalization of Theorem 1.1. The *character variety* of M is defined to be

$$X(M) = \mathrm{Hom}(\pi_1(M), G)/G$$

if M is path-connected and the Cartesian product

$$X(M) = \prod_{M_i \in \pi_0(M)} X(M_i)$$

in general. If M is a surface-with-boundary, then the inclusions of the boundary components $\partial_i M \hookrightarrow M$ induce a restriction map

$$\partial^\# : X(M) \longrightarrow X(\partial M) = \prod X(\partial_i M)$$

In this way we regard $X(M)$ as a family of *relative character varieties* over $X(\partial M)$. Since each $X(\partial_i M)$ identifies with the set $[G]$ of conjugacy classes in G, the base of this family is the product of copies of $[G]$, one for each boundary component. Explicitly, choose $\gamma_1, \dots, \gamma_m \in \pi_1(M)$ corresponding to the boundary components. For each γ_i choose a conjugacy class $\mathcal{C}_i \in G$. Then the relative character variety (defined with respect to

$$\mathcal{C} = (\mathcal{C}_1, \dots, \mathcal{C}_m) \in X(\partial M)$$

equals

$$X_{\mathcal{C}}(M) = \{[\rho] \in \mathrm{Hom}(\pi, G)/G \mid \rho(\gamma_i) \in \mathcal{C}_i, \quad \forall i = 1, \dots, m\}$$

The *mapping class group* Γ_M of M equals the group of relative isotopy classes of diffeomorphisms $M \longrightarrow M$ restricting to the identity on ∂M. The mapping class group identifies with the group of equivalence classes of automorphisms of $\pi_1(M)$ preserving the conjugacy class of each cyclic subgroup $\langle \gamma_i \rangle$ corresponding to a boundary component.

Theorem 1.5. *Let G be a compact Lie group locally isomorphic to a product of copies of* $\mathrm{SU}(2)$ *and* $\mathrm{U}(1)$. *Then the mapping class group Γ_M is ergodic on $X_{\mathcal{C}}(M)$.*

This theorem includes Theorem 1.1 as the special case when M is a once-punctured surface and the boundary holonomy maps to the conjugacy class $\{\zeta_0\}$.

1.4. **Parabolic structures.** Our methods apply equally well to compact surfaces M with finitely many punctures. The moduli spaces of representations of $\pi_1(M)$ in $\mathrm{SU}(2)$ are moduli spaces of *holomorphic vector bundles with parabolic structure.* Fixing the conjugacy class of the holonomy of each component of the boundary amounts to a parabolic structure. For fixed boundary structure, the moduli space has the structure of a projective variety, as in Mehta-Seshadri [33]. For the symplectic geometry in this case, see Alekseev-Malkin [2], Biswas-Guruprasad [4], King-Sengupta [28], Huebschmann [19], and Jeffrey [21],[22]. The theorem we prove for compact surfaces applies to these moduli spaces as well; we simply state the result (which is just 1.5 translated into the language of holomorphic vector bundles with parabolic structure):

Theorem 1.6. *Let M be a compact Riemann surface with m distinguished points $p_1, \dots, p_m$. Fix a rank-two parabolic structure $\mathcal{P}$ on M at $p_1, \dots, p_m$. Let Γ denote the group of isotopy classes of diffeomorphisms of M which fix each p_i. Then Γ is ergodic on the moduli space of semistable rank two vector bundles with parabolic structure $\mathcal{P}$ over M.*

1.5. **Further speculations.** The space $\mathrm{Hom}(\pi, G)/G$ is a "nonabelian" extension of the space $\mathrm{Hom}(\pi, \mathrm{U}(1))$ of 1-dimensional unitary characters of $\pi_1(M)$. (Alternatively this space is the first cohomology group of M with circle coefficients.) This compact $2g$-dimensional torus identifies with the Jacobi variety of holomorphic line bundles of degree 0 when M is provided with a complex structure. The action of Γ_M on $\mathrm{Hom}(\pi, \mathrm{U}(1))$ factors through the action of the integral symplectic group $\mathrm{Sp}(\mathbb{Z}^{2g})$ on the torus $\mathbb{R}^{2g}/\mathbb{Z}^{2g}$. This action is ergodic, although it is quite far from being effective. The action of Γ_M on the $\mathrm{SU}(2)$-moduli space is effective (Theorem 10.1).

Several other actions of the mapping class group are known to be ergodic. According to Masur [31],[32], the action of the mapping class group on the Thurston space of measured foliations $\mathcal{MF}$ is ergodic. Indeed, the action on the product $\mathcal{MF} \times \mathcal{MF}$ is ergodic. These actions are analogous to the actions of a lattice subgroup of a semisimple Lie group on the boundary of the corresponding symmetric space (Compare Ivanov [20]). However, these ergodic actions do not have finite invariant measure. Theorem 1.1 seems to be the first geometrically natural example of an effective ergodic action of the mapping class group with finite invariant measure. (Effectiveness of the action is proved in §10.) The referee has pointed out that every group Γ has a

"Bernoulli type" ergodic action on $\{0,1\}^\Gamma$ which is effective with finite invariant measure.

Related is the $\mathrm{Out}(\pi)$-action on the $\mathrm{Hom}(\pi, G)/G$ when $G = \mathrm{PSL}(2,\mathbb{R})$ (see [14]). One component of this space consists of flat structures on the $\mathrm{PSL}(2,\mathbb{R})$-bundle corresponding to the unit tangent bundle of M. This component consists of embeddings of π onto discrete subgroups of $\mathrm{PSL}(2,\mathbb{R})$ and naturally identifies with the Teichmüller space $\mathcal{T}_g$ of M. The action of $\mathrm{Out}(\pi)$ on this component is properly discontinuous. (Its quotient is the Riemann moduli space $\mathcal{M}_g$ discussed above.) The proper discontinuity of this action markedly contrasts the ergodicity of $\mathrm{Out}(\pi)$ on the space of $\mathrm{SU}(2)$-representations.

Conjecture 1.7. *Let G be a compact Lie group. For each G-invariant symmetric bilinear form $B : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{R}$, let ω_B be the corresponding exterior 2-form on* $\mathrm{Hom}(\pi, G)/G$. *Then the subalgebra of the de Rham algebra* $\Omega^*(\mathrm{Hom}(\pi, G)/G)$ *consisting of* $\mathrm{Out}(\pi)$*-invariant differential forms on* $\mathrm{Hom}(\pi, G)/G$ *is the algebra generated by all ω_B.*

(Although other $\mathrm{Out}(\pi)$-invariant cohomology classes in

$$H^*(\mathrm{Hom}(\pi, G)/G)$$

exist, it seems difficult to represent them by invariant *differential forms.*)

Zimmer [43] has defined a notion of algebraic hull for smooth ergodic actions on smooth manifolds. This is the smallest algebraic subgroup of the general linear group for which an invariant measurable reduction of structure group exists. Since the action is symplectic the algebraic hull of this action lies in the symplectic group $\mathrm{Sp}(\mathbb{R}^{6g-6})$. We conjecture that this algebraic hull actually equals the symplectic group.

Individual elements of Γ_M are ergodic on $\mathrm{Hom}(\pi, \mathrm{U}(1))$. In particular a mapping class $\phi \in \Gamma_M$ is ergodic on the torus $\mathrm{Hom}(\pi, \mathrm{U}(1))$ if and only if the induced homology map $H_1(\phi) \in \mathrm{Sp}(\mathbb{Z}^{2g})$ is hyperbolic. Can a cyclic subgroup of Γ_M be ergodic on $\mathrm{Hom}(\pi, G)/G$ when $G = \mathrm{SU}(2)$? Another interesting question is whether the action of Γ_M on the tangent bundle of $\mathrm{Hom}(\pi, G)/G$ is ergodic. Brown [5, 6] gives examples of hyperbolic mapping classes which do not act ergodically on moduli spaces of the once-punctured torus.

For G abelian, the action of Γ_M factors through the homomorphism

$$\Gamma_M \cong \mathrm{Out}(\pi) \longrightarrow \mathrm{Aut}(H^*(M;\mathbb{Z})) \cong \mathrm{Sp}(\mathbb{Z}^{2g})$$

The *Torelli subgroup* is defined as the kernel of this homomorphism.

Conjecture 1.8. *Let M be a closed surface. Then the action of the Torelli subgroup of Γ_M on X is ergodic.*

The proof of Theorem 1.5 involves a careful analysis of the action of a Dehn twist on X. It is known that the Dehn twists generate Γ. The proof analyzes the functions invariant under certain subgroups generated by Dehn twists. In particular there are noncompact subgroups of Γ whose action on X is not ergodic. Scot Adams has observed that this implies that the action of Γ is not mixing, since mixing is preserved upon passage to noncompact subgroups and mixing implies ergodic.

On the other hand, these actions are *weak-mixing:* An ergodic action with finite invariant measure of Γ on X is weak-mixing if the diagonal action of Γ on $X \times X$ is ergodic. Now

$$\mathrm{Hom}(\pi, G_1 \times G_2) = \mathrm{Hom}(\pi, G_1) \times \mathrm{Hom}(\pi, G_2)$$

and, more generally, if $\zeta_1 \in Z(G_1)$ and $\zeta_2 \in Z(G_2)$, then the twisted moduli spaces satisy

$$\mathrm{Hom}_{(\zeta_1,\zeta_2)}(\pi, G_1 \times G_2) = \mathrm{Hom}_{\zeta_1}(\pi, G_1) \times \mathrm{Hom}_{\zeta_2}(\pi, G_2)$$

The character varieties decompose similarly. Since the class of groups G satisfying the hypotheses of Theorem 1.5 is closed under Cartesian products, Γ_M is ergodic on $X \times X$ and hence Γ_M is weak-mixing on X.

As the referee observed, the conclusion of weak-mixing is interesting given the structure of the proof. The proof reduces the action of Γ_M to irrational rotations on a torus, actions which are not ergodic but not weak-mixing. (Most proofs of weak-mixing involve some hyperbolicity, which is noticeably absent here. In fact, one conclusion [5, 6] is that "hyperbolic" mapping classes need not display hyperbolicity in their action on the moduli spaces.) What seems to be important here is the size and complexity of the group Γ_M.

1.6. **Outline of proof.** Let $\alpha \subset M$ be a simple closed curve corresponding to an element $\alpha \in \pi_1(M)$. Let $G = \mathrm{SU}(2)$. The map

$$\begin{aligned} \mathrm{Hom}(\pi, G) &\longrightarrow [-2, 2] \\ \rho &\longmapsto \mathrm{tr}(\rho(\alpha)) \end{aligned}$$

is invariant under inner automorphisms of G acting on $\mathrm{Hom}(\pi, G)$ and thus defines a function s_α on the quotient

$$X(M) = \mathrm{Hom}(\pi, G)/G$$

which is a symplectic manifold (up to a singular set of measure zero). The corresponding Hamiltonian flow ξ_α has closed orbits on $X(M)$, almost every one of which is a circle. By [13], this flow lifts to a flow on $\mathrm{Hom}(\pi, G)$ which can be described as follows: Split M along α to obtain a surface-with-boundary $M|\alpha$ with an identification map:

$$i_\alpha : M|\alpha \longrightarrow M$$

Reidentify the representation ρ over $\pi_1(M|\alpha)$ (which is mapped injectively into $\pi_1(M)$) by a one-parameter subgroup of G centralizing $\rho(\alpha)$.

Also associated to α is the *Dehn twist* $\tau_\alpha \in \Gamma_M$ about α. The Dehn twist is represented by a diffeomorphism of M supported in a tubular neighborhood of α in M. The induced map on $X(M)$ leaves invariant the function s_α. Almost every level set of s_α is a closed orbit of ξ_α, upon which τ_α acts by an irrational rotation. An irrational rotation of the circle is ergodic with respect to Lebesgue measure. Disintegrating the measure on $X(M)$ over s_α gives measures on the level sets of s_α invariant under τ_α. In particular the action of τ_α on almost every level set of s_α is ergodic with respect to this measure.

These constructions extend easily to a collection of disjoint simple loops $\alpha = \{\alpha_1, \dots, \alpha_n\}$ on M. Assume no α_i is null-homotopic and no two α_i are homotopic. The map

$$s_\alpha : \mathrm{Hom}(\pi, G)/G \longrightarrow [-2, 2]^n$$
$$[\rho] \longmapsto \begin{bmatrix} \mathrm{tr}(\rho(\alpha_1)) \\ \vdots \\ \mathrm{tr}(\rho(\alpha_n)) \end{bmatrix}$$

is a moment map for a Hamiltonian $\mathbb{R}^n$-action ξ_α on $X(M)$. For each $i = 1, \dots, n$ let $\tau_i \in \Gamma_M$ correspond to the Dehn twist about α_i. The subgroup $\Gamma_\alpha \subset \Gamma_M$ generated by $\tau_1, \dots, \tau_n$ is free abelian with basis $\tau_1, \dots, \tau_n$. Let $X'(M|\alpha)$ denote the subset of $X(M|\alpha)$ such that $\chi(\alpha_+) = \chi(\alpha_-)$ where $\alpha_+, \alpha_- \in \pi_1(M|\alpha)$ correspond to $i_\alpha^{-1}(\alpha) \subset \partial M|\alpha$. The identification map i_α induces a map

$$(i_\alpha)_* : X(M) \longrightarrow X(M|\alpha)$$

with image $X'(M|\alpha)$. Disintegrate the measure μ corresponding to symplectic volume on $X(M)$ as a direct integral

$$\mu = \int_{[-2,2]^n} \mu_t \, d\mu_\alpha(t)$$

of measures μ_t on the level sets $s_\alpha^{-1}(t)$ with respect to the pushforward measure $\mu_\alpha = (s_\alpha)_*\mu$ on $[-2, 2]^n$..

The first ingredient of the proof involves a detailed analysis of the cases when M is a one-holed torus or a four-holed sphere. For a surface with nonempty boundary the restriction map $\partial^\# : X(M) \longrightarrow X(\partial M)$

associates to an equivalence class $[\rho]$ of a representation $\rho : \pi_1(M) \longrightarrow G$ the product

$$\{[\rho|_{\pi_1(\partial_i M)}]\}_{1\leq i\leq m} \in \prod_{i=1}^{m} X(\partial_i M) = X(\partial M)$$

taken over the m components $\partial_1(M), \ldots \partial_m(M)$ of ∂M. The boundary data in $X(\partial M)$ is recorded by a collection $\mathcal{C} = (\mathcal{C}_1, \ldots, \mathcal{C}_m)$ of conjugacy classes in G. The fibers of $\partial^{\#}$ are the relative character varieties $X_{\mathcal{C}}(M)$ of M. When M is either a one-holed torus or a four-holed sphere, $X_{\mathcal{C}}(M)$ are two-dimensional; in trace coordinates they are cubic surfaces in affine space. If $\gamma \subset M$ is a simple closed curve, and τ is the corresponding Dehn twist, then the ergodic decomposition of τ acting on $X_{\mathcal{C}}(M)$ is given by the trace mapping

$$s_\gamma : X(M) \longrightarrow [-2, 2]$$

This trace mapping exhibits the cubic surface $X_{\mathcal{C}}(M)$ as a pencil of ellipses $s_\gamma^{-1}(x)$, for $x \in [-2, 2]$. Let $\gamma_1, \gamma_2 \subset M$ be a pair of homotopically nontrivial curves and let $\mathcal{F}_1, \mathcal{F}_2$ be the corresponding pair of foliations of $X_{\mathcal{C}}(M)$ by ellipses. If γ_1 and γ_2 intersect, then $\mathcal{F}_1$ and $\mathcal{F}_2$ *span* all of $X(M)$ in the following sense: every measurable subset of $X(M)$ which is a both a union of leaves of $\mathcal{F}_1$ and a union of leaves of $\mathcal{F}_2$ must be null or conull.

The second ingredient in the proof is the following:

Theorem 1.9. *Let α be a maximal collection of homotopically distinct and nontrivial simple closed curves on M. For almost every $t \in [-2, 2]^n$, $s_\alpha^{-1}(t)$ is an orbit of the Hamiltonian $\mathbb{R}^n$-action diffeomorphic to an n-torus and Γ_α is ergodic on $s_\alpha^{-1}(t)$. For every Γ_α-invariant measurable function $f : X(M) \longrightarrow \mathbb{R}$, there exists a measurable function $f' : X'(M|\alpha) \longrightarrow \mathbb{R}$ such that $f = f' \circ (i_\alpha)_*$ almost everywhere.*

The functions s_α define coordinates on $X(M)$. By applying Dehn twists about curves α, we show that any invariant measurable function is a function only of the coordinates s_α.

The two ingredients combine to prove the theorem. Consider a maximal collection $\alpha = \{\alpha_1, \ldots, \alpha_N\}$ of null-homotopic simple closed curves, no two of which are homotopic. α corresponds to a "pants decomposition" of M, since each component of $M - \alpha$ is a "pants," a three-holed sphere. The total number of curves in the decomposition equals

$$N = 3g - 3 + 2m$$

where g equals the genus of M and m equals the number of boundary components. By Theorem 1.9, a Γ_M-invariant function depends only on the trace functions s_{α_i} for $i = 1, \ldots, N$.

Eliminate each variable s_{α_i} individually: For each α_i, consider an embedding of a surface A, homeomorphic to a one-holed torus or to a four-holed sphere, such that $\alpha_i \subset \mathrm{int}(A)$ and $\partial A \subset \alpha$. The embedding $f : \mathrm{int}(A) \hookrightarrow M$ induces an embedding of mapping class groups $f_* : \Gamma_A \longrightarrow \Gamma_M$. The corresponding mapping of character varieties $X(M) \longrightarrow X(A)$ is Γ_A-equivariant respecting f_*. Since Γ_A is ergodic on the relative character varieties $X_{\mathcal{C}}(A)$, an invariant function depends on $s_{\alpha_1}, \dots, s_{\alpha_{i-1}}, s_{\alpha_{i+1}}, \dots, s_{\alpha_N}$: a Γ_M-invariant subset is (up to a null set) a union of level sets of s_{α_i}. This argument applied to each α_i as $i = 1, \dots, N$, implies that a Γ_M-invariant function is almost everywhere constant, or alternatively that a Γ_M-invariant subset is either null or conull.

1.7. **Acknowledgements.** Conversations with S. Adams, R. Benedetto, M. Boyle, S. Donaldson, D. Fried, H. Furstenberg, D. Heicklein, L. Jeffrey, D. Kazhdan, C. McMullen, J. Millson, W. Neumann, D. Rudolph, G. Stuck, J. Weitsman, S. Wolpert, and R. Zimmer were useful in understanding this subject. I especially wish to thank S. Adams and G. Stuck for comments on preliminary versions. Critical comments by S. Adams were particularly useful in clarifying ideas in the proof. I apologize for the long delay in this manuscript, the results of which were obtained in 1982 (and announced in [12]). Finally I would like to thank the referee for several helpful suggestions.

2. Preliminaries

2.1. **Moduli spaces.** In this section we summarize, for the reader's convenience, basic facts about moduli spaces of representations. For details and proofs we refer to [7], [12],[16], [27],[29], [34],[35],[38].

Let K be a compact Lie group. K may be given the structure of a real algebraic Lie group, that is, a Zariski-closed subgroup of $\mathrm{GL}(n, \mathbb{R})$. Let $K_{\mathbb{C}}$ denote its complexification, which is an algebraic subgroup of $\mathrm{GL}(n, \mathbb{C})$. For any finitely generated group π, the set $\mathrm{Hom}(\pi, K_{\mathbb{C}})$ has the structure of a complex affine variety, that is, the set of complex points of an affine scheme defined over $\mathbb{Z}$. The real variety $\mathrm{Hom}(\pi, K)$ is the subset consisting of $\mathbb{R}$-points.

Let G be a reductive algebraic group defined over a field k; for our purposes G will either be K or $K_{\mathbb{C}}$ and $k = \mathbb{R}$ or $k = \mathbb{C}$, respectively. An explicit description of $\mathrm{Hom}(\pi, G)$ involves a choice of generators. If $\gamma_1, \dots, \gamma_N \in \pi$ is a generating set, then the evaluation map

$$\begin{aligned}\mathrm{Hom}(\pi, G) &\longrightarrow G^N\\ \rho &\longmapsto (\rho(\gamma_1), \dots, \rho(\gamma_N))\end{aligned}$$

is injective and its image consists of all

$$(g_1, \dots, g_N) \in G^N$$

satisfying

$$R(g_1, \dots, g_N) = I \tag{2.1}$$

for each relation R satisfied by the generating set $\gamma_1, \dots, \gamma_N$:

$$R(\gamma_1, \dots, \gamma_N) = I$$

The equations (2.1) are polynomial equations in the matrix entries of $(g_1, \dots, g_N)$ and thus the image of the evaluation map is Zariski-closed in G^N. Furthermore two different generating sets define isomorphic affine variety structures on $\mathrm{Hom}(\pi, G)$.

The Cartesian product $\mathrm{Aut}(\pi) \times \mathrm{Aut}(G)$ acts on $\mathrm{Hom}(\pi, G)$ by composition, preserving the algebraic structure. In particular G acts on $\mathrm{Hom}(\pi, G)$ by inner automorphisms. In general the quotient space $\mathrm{Hom}(\pi, G)/G$ is not an algebraic variety. Rather there is a G-invariant Zariski-open subset $\mathrm{Hom}(\pi, G)^-$ (consisting of the *semistable points*) such that the orbit space

$$\mathrm{Hom}(\pi, G)^-/G$$

is the set of k-points of an affine varitey

$$\mathrm{Hom}(\pi, G)//G$$

(the *character variety*) whose coordinate ring equals the subring of G-invariant functions in the coordinate ring of $\mathrm{Hom}(\pi, G)$. In practice, $\mathrm{Hom}(\pi, G)$ will consist of semisimple ("completely reducible") representations. In the case of a compact group K, every representation is semisimple and the map $\mathrm{Hom}(\pi, K)/K \longrightarrow \mathrm{Hom}(\pi, K_{\mathbb{C}})/K_{\mathbb{C}}$ induced by inclusion $K \longrightarrow K_{\mathbb{C}}$ is a bijection onto a component of the set of $\mathbb{R}$-points.

Since the subgroup $\mathrm{Inn}(\pi)$ of $\mathrm{Aut}(\pi)$ consisting of the inner automorphisms $\pi \to \pi$ preserves each G-orbit on $\mathrm{Hom}(\pi, G)$, it acts trivially on the quotient space

$$\mathrm{Hom}(\pi, G)/G.$$

Hence the action of $\mathrm{Aut}(\pi)$ induces an action of the quotient group $\mathrm{Out}(\pi) = \mathrm{Aut}(\pi)/\mathrm{Inn}(\pi)$ on $\mathrm{Hom}(\pi, G)/G$.

When π is the fundamental group of a closed orientable surface of genus $g > 1$, there is the presentation

$$\pi = \langle A_1, B_1, \dots, A_g, B_g \mid [A_1, B_1] \dots [A_g, B_g] = I \rangle$$

where $[X, Y] = XYX^{-1}Y^{-1}$. Then $\mathrm{Hom}(\pi, G)$ identifies with the set of all

$$(a_1, b_1, \dots, a_g, b_g) \in G^{2g}$$

satisfying the equation (in G):

$$[a_1, b_1] \dots [a_g, b_g] = I \tag{2.2}$$

2.2. **Twisted moduli spaces.** Consider an element $\zeta_0 \in G$ in the center $Z(G)$ of G. For simplicity suppose that $Z(G)$ is discrete. As π admits the presentation

$$\pi = \langle A_1, B_1, \dots A_g, B_g \mid [A_1, B_1] \dots [A_g, B_g] = I \rangle$$

representations

$$\phi \in \mathrm{Hom}(\pi, G)$$

correspond to $2g$-tuples $(a_1, b_1, \dots, a_g, b_g) \in G^{2g}$ satisfying (2.2) by

$$a_i = \phi(A_i), \qquad b_i = \phi(B_i)$$

A ζ_0-*twisted* representation is defined to be a $2g$-tuple

$$(a_1, b_1, \dots, a_g, b_g) \in G^{2g}$$

satisfying

$$[a_1, b_1] \dots [a_g, b_g] = \zeta_0.$$

Denote the subspace of G^{2g} consisting of ζ_0-twisted representations by $\mathrm{Hom}_{\zeta_0}(\pi, G)$. Since ζ_0 is central, the diagonal action of G by inner automorphisms preserves $\mathrm{Hom}_{\zeta_0}(\pi, G)$ and we denote the quotient by $\mathrm{Hom}_{\zeta_0}(\pi, G)/G$.

Alternatively $\mathrm{Hom}_{\zeta_0}(\pi, G)$ relates to the quotient group $G/Z(G)$ as follows. The quotient homomorphism $\Pi : G \longrightarrow G/Z(G)$ induces a map

$$\Pi_* : \mathrm{Hom}_{\zeta_0}(\pi, G) \longrightarrow \mathrm{Hom}(\pi, G/Z(G)) \tag{2.3}$$

whose image is open and closed in $\mathrm{Hom}(\pi, G/Z(G))$. The restriction of Π_* to $\mathrm{Hom}_{\zeta_0}(\pi, G)$ is a covering space. Every fiber of this map admits a simply transitive action of the abelian group $\mathrm{Hom}(\pi, Z(G))$ under multiplication of homomorphisms: if $\zeta \in \mathrm{Hom}(\pi, Z(G))$ and $\phi \in \mathrm{Hom}_{\zeta_0}(\pi, G)$, then

$$\zeta \cdot \phi : \gamma \mapsto \zeta(\gamma)\phi(\gamma)$$

also lies in $\mathrm{Hom}_{\zeta_0}(\pi, G)$. Therefore (2.3) is a principal $\mathrm{Hom}(\pi, Z(G))$-bundle.

2.3. **Ergodic decomposition.** The basic tool for proving ergodicity is the following general lemma from measure theory:

Lemma 2.1. *Let $(X, \mathfrak{B}, \mu)$ be a standard measure space and Y, Z be Borel spaces and $F : X \longrightarrow Y$ a measurable map. Suppose that Γ is a group of automorphisms of $(X, \mathfrak{B}, \mu)$ such that F is Γ-invariant. Let μ_y be the measures on $F^{-1}(y)$ obtained by disintegrating μ over F. Suppose that the action of Γ is ergodic on $(F^{-1}(y), \mu_y)$ for $F_*\mu$-almost every $y \in Y$. Let $h : X \longrightarrow Z$ be an Γ-invariant measurable function. Then there exists a measurable function $H : Y \longrightarrow Z$ such that $h = H \circ F$ almost everywhere.*

The proof follows readily from disintegration theory, as in Furstenberg [10] (Theorem 5.8). The decomposition of the measure μ on X into Γ-ergodic components is precisely the disintegration of μ over h. (See also [11], §3.5 of Rudolph [40], Varadarajan [41],§4 and also §13 of [8].) Recall that this means that there is a family μ_y of measures, parametrized by $y \in Y$, such that μ_y is a measure on $F^{-1}(y)$ and a measure $F_*\mu$ (the *pushforward of μ by F*), such that for every measurable function f on Y,

$$\int_X f d\mu = \int_Y \left(\int_{F^{-1}(y)} f d\mu_y \right) d(F_*\mu)$$

In that case μ is said to be the *direct integral* of the measures μ_y and is written

$$\mu = \int_Y \mu_y d(F_*\mu)$$

2.4. **Surfaces with boundary.** Let M be a compact surface of genus $g \geq 0$ with boundary components $C_1, \ldots, C_m$. Suppose that

$$\chi(M) = 2 - 2g - m < 0.$$

For each $j = 1, \ldots, m$ choose a basepoint $x_j \in C_j$. The inclusion $C_j \hookrightarrow M$ induces a monomorphism $\pi_1(C_j, x_j) \longrightarrow \pi_1(M, x_j)$ and so we regard each fundamental group $\pi_1(C_j)$ as an infinite cyclic subgroup of $\pi_1(M)$. Choose a basepoint x_0 in M and paths from x_0 to x_j. Denote by σ_j the corresponding generator of $\pi_1(C_j) \subset \pi_1(M)$. We systematically confuse simple closed curves on M with based loops and their homotopy classes in the fundamental group. In particular choose a collection

$$\sigma_1, \ldots, \sigma_m, \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$$

of simple loops such that the intersection of any two of them is x_0. Then $\pi_1(M)$ admits a presentation:

$$\pi_1(M) \cong \langle \sigma_1, \ldots, \sigma_m, \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g \mid \sigma_1 \ldots \sigma_m [\alpha_1, \beta_1] \ldots [\alpha_g, \beta_g] = I \rangle$$

For a compact surface M with nonempty boundary, its *mapping class group* Γ_M is the group of connected components of the group of diffeomorphisms of M whose restriction to ∂M is the identity. That is, Γ_M consists of equivalence classes of diffeomorphisms of M which pointwise fix the boundary. Two such diffeomorphisms are equivalent if they are joined by an isotopy fixing ∂M pointwise. Alternatively, Γ_M is the subgroup of $\mathrm{Out}(\pi)$ preserving the conjugacy class of each $\pi_1(C_j) \subset \pi_1(M)$.

Let $G = \mathrm{SU}(2)$ and write $X(M) = \mathrm{Hom}(\pi, G)/G$. Let μ be the measure on $X(M)$ induced by the symplectic structure. Consider the map

$$\begin{aligned} \partial_j^\# : X(M) &\longrightarrow \mathbb{R} \\ [\rho] &\longmapsto \mathrm{tr}(\rho(\sigma_j)) \end{aligned}$$

and let

$$\partial^\# = (\partial_1^\#, \ldots, \partial_m^\#) : X(M) \longrightarrow \mathbb{R}^m.$$

For almost every $t \in \mathbb{R}^m$, the map q is submersive on ${\partial^\#}^{-1}(t)$ and the disintegration of μ with respect to q is a smooth measure μ_t on the submanifold ${\partial^\#}^{-1}(t)$. Here is the slightly more precise reformulation of Theorem 1.5 in the form which we shall prove:

Theorem 2.2. *For each t, the action of Γ_M on the fiber ${\partial^\#}^{-1}(t)$ is ergodic with respect to μ_t. For every Γ_M-invariant measurable function $f : X(M) \longrightarrow \mathbb{R}$, there exists a (Lebesgue-) measurable function $\tilde{f} : \mathbb{R}^m \longrightarrow \mathbb{R}$ such that $f = \tilde{f} \circ q$ almost everywhere.*

3. Decompositions of surfaces and character varieties

Let $C \subset M$ be a homotopically nontrivial simple closed curve. Let $M|C$ be the surface M *split along* C, defined as the surface-with-boundary $M|C$ for which there is an identification map $i_C : M|C \longrightarrow M$ satisfying:

- The restriction of i_C to $i_C^{-1}(M - C)$ (which contains the interior of $M|C$) is a diffeomorphism;
- $i_C^{-1}(C)$ consists of two components $C_+, C_- \subset \partial(M|C)$, to each of which the restriction of i_C is a diffeomorphism onto C.

The character variety $X(M|C)$ is "simpler" than $X(M)$ and $X(M)$ is reconstructedfrom $X(M|C)$. There are two cases, depending on whether C is *nonseparating* ($M|C$ is connected) or C *separates* ($M|C$ is disconnected), respectively.

3.1. **Nonseparating curves.** Suppose that $M|C$ is connected. Then M is obtained by gluing two boundary components C_+, C_- of $A = M|C$. The identification map $i_C : A \longrightarrow M$ is injective along $A - (C_- \cup C_+)$ and is two-to-one along

$$i_C^{-1}(C) = C_- \cup C_+$$

Then $\pi_1(M)$ is obtained from $\pi_1(A)$ by an *HNN-construction:* $\pi_1(M)$ is the quotient of the free product $\pi_1(A) \amalg \langle\tau\rangle$ of $\pi_1(A)$ with a cyclic group $\langle\tau\rangle \cong \mathbb{Z}$ by the normal subgroup generated by the set

$$\{\iota_-(\gamma)^{-1}\tau\iota_+(\gamma)\tau^{-1} \mid \gamma \in \pi_1(C)\} \subset \pi_1(A) \amalg \langle\tau\rangle$$

where

$$\iota_\pm : \pi_1(C) \longrightarrow \pi_1(C_\pm) \xrightarrow{(i_C)_*} \pi_1(M)$$

denote the embeddings induced by inclusion $C \hookrightarrow C_\pm \hookrightarrow M$. The new generator τ corresponds to a loop in M which crosses C. Its preimage in $A = M|C$ is an arc α with one endpoint on C_+ and one endpoint on C_-. Choose for a basepoint x_0 the endpoint of α on C_-

Let γ_- be the element of $\pi_1(A, x_0)$ corresponding to C_-. Let γ_+ denote the element of $\pi_1(A, x_0)$ corresponding to the loop $\alpha^{-1} \star C_+ \star \alpha$ where C_+ is regarded as a loop based on the endpoint of α on C_+. A representation $\rho : \pi_1(A) \longrightarrow G$ extends to a representation of $\pi_1(M)$ if and only if there exists a $t \in G$ such that

$$\rho(\gamma_-) = t\rho(\gamma_+)t^{-1} \tag{3.1}$$

that is, if $\rho(\gamma_-)$ is conjugate to $\rho(\gamma_+)$. Since two elements of G are conjugate if and only if they have the same trace, the image of the restriction map $X(M) \longrightarrow X(A)$ consists of all $[\rho]$ such that if

$$\mathrm{tr}(\rho(\gamma_-)) = \mathrm{tr}(\rho(\gamma_+)).$$

The fiber of $X(M) \longrightarrow X(A)$ identifies with the set of all $t \in G$ (corresponding to $\rho(\tau)$) satisfying (3.1). The set of all such t is a right-coset of the centralizer $Z(\rho(\gamma_-))$. In particular $Z(\rho(\gamma_-))$ acts simply transitively on the fibers of $X(M) \longrightarrow X(A)$ by the action defined by

$$\zeta \cdot \rho : \begin{cases} \alpha & \longmapsto \rho(\alpha) \quad \text{for } \alpha \in \pi_1(A) \\ \gamma & \longmapsto \rho(\gamma)\zeta \end{cases}$$

3.2. **Separating curves.** Let M be a compact surface decomposed into closed subsurfaces A and B along a simple closed curve $C = A \cap B$. Assume that $\chi(A) < 0$ and $\chi(B) < 0$. Then the inclusions

$$\begin{tikzcd} C \arrow[r] \arrow[d] & A \arrow[d] \\ B \arrow[r] & M \end{tikzcd}$$

induce monomorphisms

$$\begin{tikzcd} \pi_1(C) \arrow[r] \arrow[d] & \pi_1(A) \arrow[d] \\ \pi_1(B) \arrow[r] & \pi_1(M) \end{tikzcd}$$

and $\pi_1(M)$ is represented as the amalgamated free product

$$\pi_1(M) = \pi_1(A) \amalg_{\pi_1(C)} \pi_1(B)$$

corresponding to the above pushout diagram.

Define $X(M; A, B, C)$ as the pullback in the diagram

$$\begin{tikzcd} X(C) & X(A) \arrow[l] \\ X(B) \arrow[u] & X(M; A, B, C) \arrow[l] \arrow[u] \end{tikzcd}$$

that is, $X(M; A, B, C)$ consists of all pairs $([\alpha], [\beta]) \in X(A) \times X(B)$ such that

$$[\alpha|_{\pi_1(C)}] = [\alpha|_{\pi_1(C)}] \in X(C)$$

The natural map $j : X(M; A, B, C) \longrightarrow X(M)$ is surjective: if $[\rho] \in X(M)$, for a representation ρ, then the restrictions of ρ to $\pi_1(A)$ (respectively $\pi_1(B)$) are equivalent to representations ρ_A (respectively ρ_B) such that $[\rho_A] \in X(A)$ and $[\rho_B] \in X(B)$ and $\rho_A|\pi_1(C) = \rho_B|\pi_1(C)$. Let

$$([\alpha], [\beta]) \in X(M; A, B, C)$$

and let ρ be a homomorphism such that

$$\rho|_{\pi_1(A)} \equiv \alpha, \qquad \rho|_{\pi_1(B)} \equiv \beta.$$

Elements ζ of the centralizer $Z = Z(\rho, C)$ of $\rho(\pi_1(C))$ act on the fiber $j^{-1}([\alpha], [\beta])$ by the rule:

$$\zeta \cdot \rho : \gamma \longmapsto \begin{cases} \alpha(\gamma) & \text{for } \gamma \in \pi_1(A) \\ \zeta\beta(\gamma)\zeta^{-1} & \text{for } \gamma \in \pi_1(B) \end{cases}$$

where $\rho \in j^{-1}([\alpha], [\beta])$.

4. Three-holed spheres

The basic building block of surfaces of negative Euler characteristic is the three-holed sphere ("pair-of-pants"). Let A, B, C denote its three boundary components. Its fundamental group admits the geometric presentation

$$\pi_1(M) = \langle A, B, C \mid ABC = 1\rangle \tag{4.1}$$

The following basic fact concerning representations of its fundamental group is discussed in [14] and the references given there:

Lemma 4.1. *Let M be a surface of genus 0 with three boundary components A, B, C. Then the trace map*

$$\begin{aligned} \chi : X(M) &\longrightarrow \mathbb{R}^3 \\ [\rho] &\longmapsto \begin{bmatrix} \mathrm{tr}(\rho(A)) \\ \mathrm{tr}(\rho(B)) \\ \mathrm{tr}(\rho(C)) \end{bmatrix} \end{aligned}$$

identifies $X(M)$ with the set

$$\mathfrak{B} = \{(a, b, c) \in \mathbb{R}^3 \mid -2 \le a, b, c \le 2, a^2 + b^2 + c^2 - abc \le 4\}$$

Characters of reducible representations form the boundary

$$\partial\mathfrak{B} = \{(a, b, c) \in \mathbb{R}^3 \mid -2 \le a, b, c \le 2, a^2 + b^2 + c^2 - abc = 4\}$$

Since an element of G is determined up to conjugacy by its trace, the mapping χ is the restriction map $X(M) \longrightarrow X(\partial M)$. Lemma 4.1 asserts that this mapping is injective and hence the *relative* character varieties are trivial. The mapping class group Γ_M is trivial. This is the first – and the trivial — case of Theorem 2.2. (For groups with simple factors other than $\mathrm{SU}(2)$ and $\mathrm{U}\,1$, the relative character variety is nonatomic and hence an alternative method for proving Conjecture 1.3 is required. Compare §9.)

5. The one-holed torus

In this section we consider the case when M is a one-holed torus. Since its fundamental group is free of rank two, Lemma 4.1 gives explicit coordinates for $X(M)$ as in the case of a three-holed sphere. We analyze the action of a Dehn twist on $X(M)$, explicitly examining Theorem 2.2 in these coordinates. Applying Dehn twists about two loops generating $\pi_1(M)$ demonstrates ergodicity of Γ_M on the relative character varieties.

5.1. **Trace coordinates.** Let M be a compact surface of genus 1 with one boundary component, a *one-holed torus.* Its fundamental group admits the (geometric) presentation

$$\langle X, Y, K \mid XYX^{-1}Y^{-1} = K\rangle \tag{5.1}$$

where K corresponds to the generator of $\pi_1(\partial M)$. Its *character variety,* which parametrizes equivalence classes of representations $\rho : \pi_1(M) \longrightarrow \mathrm{SL}(2,\mathbb{C})$, is isomorphic to affine space $\mathbb{C}^3$. Its global coordinates are given by:

$$x = \mathrm{tr}(\rho(X)), \quad y = \mathrm{tr}(\rho(Y)), \quad z = \mathrm{tr}(\rho(XY)) \tag{5.2}$$

An extremely important function on $X(M)$ is

$$k = \mathrm{tr}(\rho(K)) = x^2 + y^2 + z^2 - xyz - 2$$

Characters of representations $\pi_1(M) \longrightarrow \mathrm{SU}(2)$ have coordinates satisfying

$$-2 \leq x, y, z \leq 2, x^2 + y^2 + z^2 - xyz - 2 \leq 2$$

and, for fixed $k \in [-2, 2]$, comprise the *relative character variety*

(5.3)

$$X_1(k) = \{(x, y, z) \in \mathbb{R}^3 \mid -2 \leq x, y, z \leq 2, x^2 + y^2 + z^2 - xyz - 2 = k\}$$

We denote the full character variety by

$$X_1 = \bigcup_{-2\leq k\leq 2} X_1(k)$$

5.2. **Action of a Dehn twist.** The Dehn twist τ_X about X is given by the automorphism

$$X \mapsto X, \quad Y \mapsto YX$$

which induces the map

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} x \\ z \\ xz - y \end{bmatrix}$$

Clearly x and k are invariant functions under this transformation. For fixed values of x, k, the intersection

$$\begin{aligned} E_{x,k} &:= X_1(k) \cap (\{x\} \times \mathbb{R}^2) \\ &= \{x\} \times \left\{ (y, z) \in \mathbb{R}^2 \mid \frac{2+x}{4}(y+z)^2 + \frac{2-x}{4}(y-z)^2 = 2 + k - x^2 \right\} \end{aligned}$$

is an ellipse preserved by τ_X.

The change of coordinates (which is linear in y, z)

$$\begin{aligned}
\tilde{x} &= x \\
\tilde{y} &= \frac{\sqrt{2-x}+\sqrt{2+x}}{2}y + \frac{\sqrt{2-x}-\sqrt{2+x}}{2}z \\
\tilde{z} &= \frac{\sqrt{2-x}-\sqrt{2+x}}{2}y + \frac{\sqrt{2-x}+\sqrt{2+x}}{2}z
\end{aligned}$$

transforms $E_{x,k}$ to the circle

$$E_{x,k} = \{(\tilde{y}, \tilde{z}) \in \mathbb{R}^2 \mid \tilde{y}^2 + \tilde{z}^2 = 2 + k - x^2\}$$

τ_X transforms to the orthogonal rotation of $E_{x,k}$

$$\tau_X : \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{bmatrix}$$

by angle

$$\theta = \cos^{-1}(a/2).$$

In particular whenever $\cos^{-1}(x/2)$ is an irrational multiple of π, the automorphism τ_X is ergodic on $E_{x,k}$. Since $\cos^{-1}(x/2)/\pi$ is irrational for almost every $x \in [-2, 2]$, a τ_X-invariant measurable function $f : X_1 \longrightarrow \mathbb{R}$ of (x, y, k) must almost everywhere equal a function depending only on x, k.

Similarly, apply Dehn twists τ_Y about Y. Then f must be a function of k and y alone. The foliations defined by level sets of x, y span the level set $X_1(k)$: every subset of $X_1(k)$ which is almost everywhere a union of level sets of x and level sets of y is either null or conull. This establishes Theorem 2.2 in the case of a one-holed torus.

Think of this as "eliminating the variable y." Let $\eta : X_1 \longrightarrow [-2, 2]$ denote the mapping defined by the coordinate y. Then the function f is almost everywhere constant along the level set $\eta^{-1}(y)$ for almost every $y \in [-2, 2]$. The measure μ on X_1 is the direct integral

$$\int_{[-2,2]} \mu_y \, d(\eta_* \mu)$$

of the disintegrated measures μ_y on the fibers $\eta^{-1}(y)$ with respect to the pushforward measure $\eta_*\mu$ on $[-2, 2]$. Since τ_X is ergodic on $\eta^{-1}(y)$ with respect to μ_y, Lemma 2.1 implies that every τ_X-invariant function factors through η.

5.3. **The symplectic structure.** The symplectic structure on each $X_1(k)$ admits a simple expression in trace coordinates (Compare also Wolpert [42]). Clearly the mapping class group acts by polynomial automorphisms of affine space $\mathbb{C}^3$ and since a complex polynomial with no zeroes is constant, the mapping class group acts by unimodular polynomial automorphisms. In particular the exterior 3-vector field

$$\Xi = \frac{1}{4\pi^2}\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z}$$

is invariant under Γ_M. (The coefficient $1/4\pi^2$ is included, as before, to give an integral cohomology class and to agree with Jeffrey-Weitsman.) Similarly the trace-commutator map $k : \mathbb{C}^3 \longrightarrow \mathbb{C}$ is also Γ_M-invariant, and so its exterior derivative

$$dk = (2x - yz)dx + (2y - zx)dy + (2z - xy)dz$$

is also Γ_M-invariant. The exterior 2-vector field

$$\Xi \cdot dk = \frac{1}{4\pi^2}\Big(\,(2x - yz)\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z} \tag{5.4}$$

$$+ (2y - zx)\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial y} \tag{5.5}$$

$$+ (2z - xy)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}\,\Big) \tag{5.6}$$

is also Γ_M-invariant and defines a cosymplectic structure on each relative character variety $X_1(k)$.

6. The four-holed sphere

6.1. **Trace coordinates.** Next consider the four-holed sphere. Like the one-holed torus, its relative character varieties are two-dimensional. The one-holed torus doubly covers a disc with four branch points of order two; this branched covering leads to an embedding of the relative character variety of the one-holed torus in the relative character variety of the four-holed sphere with special boundary data (see §6.3 below). Their mapping class groups are commensurable. Analogous to the case of the one-holed torus, the mapping class group is ergodic on the relative character variety.

The fundamental group of the four-holed sphere admits the (geometric) presentation

$$\langle A, B, C, D \mid ABCD = I\rangle$$

where A, B, C, D correspond to the four boundary components. By Magnus [30] (compare also [3],[16], [17], [18]), the seven functions

a, b, c, d, x, y, z on $X_0(a, b, c, d)$ defined by

$$\operatorname{tr}(A) = a, \quad \operatorname{tr}(B) = b, \quad \operatorname{tr}(C) = c, \quad \operatorname{tr}(D) = d,$$
$$\operatorname{tr}(AB) = x, \quad \operatorname{tr}(BC) = y, \quad \operatorname{tr}(CA) = z$$

satisfy the defining equation

$$(x^2 + y^2 + z^2 + xyz) = (ab + cd)x + (ad + bc)y + (ac + bd)z - (a^2 + b^2 + c^2 + d^2 + abcd - 4) \tag{6.1}$$

The character variety of the boundary consists of

$$(a, b, c, d) \in [-2, 2]^4$$

and for fixed boundary data (a, b, c, d), the relative character variety $X_0(a, b, c, d)$ consists of all

$$(x, y, z) \in [-2, 2]^3$$

satisfying (6.1). Let Γ be the mapping class group of S. Theorem 2.2 asserts in this case, that for every $(a, b, c, d) \in [-2, 2]^4$, the action of Γ on $X_0(a, b, c, d)$ is ergodic.

When $x \neq \pm 2$, rewrite equation (6.1) as

$$\frac{2+x}{4}\left((y+z) - \frac{(a+b)(d+c)}{2+x}\right)^2 + \frac{2-x}{4}\left((y-z) - \frac{(a-b)(d-c)}{2-x}\right)^2 = \frac{(x^2 - abx + a^2 + b^2 - 4)(x^2 - cdx + c^2 + d^2 - 4)}{4 - x^2} \tag{6.2}$$

For fixed x with $-2 < x < 2$, the left-hand-side of (6.2) is a quadratic function of y, z with positive definite leading term. Since (a, b, x) and (c, d, x) are characters of representations into $\mathrm{SU}(2)$, both

$$x^2 - abx + a^2 + b^2 - 4 < 0$$

and

$$x^2 - cdx + c^2 + d^2 - 4 < 0$$

and the right-hand-side of (6.2) is nonnegative. It is positive whenever the representations of $\pi_1(P)$ and $\pi_1(P')$ are irreducible. Rewrite (6.2) as

$$Q_x(y - y_0(x), z - z_0(x)) = \frac{(x^2 - abx + a^2 + b^2 - 4)(x^2 - cdx + c^2 + d^2 - 4)}{4 - x^2} \tag{6.3}$$

where

$$\begin{aligned} y_0(x) &= ad + bc \\ z_0(x) &= ac + bd - x(ad + bc) \end{aligned}$$

and

$$\begin{aligned} Q_x(\eta, \zeta) &= \frac{2+x}{4}\left(\frac{\eta+\zeta}{2+x}\right)^2 + \frac{2-x}{4}\left(\frac{\eta-\zeta}{2-x}\right)^2 \\ &= \frac{\eta^2 + \zeta^2 - x\eta\zeta}{4 - x^2} \end{aligned}$$

is a positive definite quadratic form for all $-2 < x < 2$. This exhibits $X_0(a,b,c,d)$ as a family of ellipses parametrized by (a,b,c,d,x).

6.2. **Action of a Dehn twist.** The Dehn twist τ_X about the separating curve homotopic to AB induces the automorphism of $\pi_1(S)$ defined by:

$$\begin{aligned} A &\mapsto A \\ B &\mapsto B \\ C &\mapsto ABCB^{-1}A^{-1} \\ D &\mapsto ABDB^{-1}A^{-1} \end{aligned}$$

The separating simple closed curves corresponding to AB, BC, and CA transform by:

$$\begin{aligned} AB &\mapsto AB \\ BC &\mapsto BABCB^{-1}A^{-1} \\ CA &\mapsto ABCB^{-1} \end{aligned}$$

The transformation of $X_0(a,b,c,d)$ induced by the Dehn twist τ_X preserves the boundary traces (a,b,c,d) and leaves invariant the function $x : X_0(a,b,c,d) \longrightarrow [-2,2]$ but maps

$$\begin{bmatrix} y \\ z \end{bmatrix} \mapsto \begin{bmatrix} (ad+bc) - x(ac+bd) + x(xy+z)) - y \\ (ac+bd) - (xy+z) \end{bmatrix}$$

which in terms of y_0, z_0 appears as:

$$\begin{bmatrix} y \\ z \end{bmatrix} \mapsto \begin{bmatrix} y_0(x) \\ z_0(x) \end{bmatrix} + \begin{bmatrix} x^2-1 & x \\ -x & -1 \end{bmatrix} \cdot \left(\begin{bmatrix} y \\ z \end{bmatrix} - \begin{bmatrix} y_0(x) \\ z_0(x) \end{bmatrix} \right)$$

This transformation is a rotation of angle $2\cos^{-1}(x/2)$ on the ellipse $E_x(a,b,c,d)$ defined by (6.3) for given (a,b,c,d,x). In particular, for fixed boundary traces (a,b,c,d), and almost every x, the action of τ_X on the level set $E_x(a,b,c,d)$ is ergodic.

The Dehn twist τ_Y about the separating curve homotopic to BC induces the automorphism of $\pi_1(S)$ defined by:

$$\begin{aligned} A &\mapsto BCAC^{-1}B^{-1} \\ B &\mapsto B \\ C &\mapsto CABCB^{-1}A^{-1} \\ D &\mapsto BCDC^{-1}B^{-1} \\ AB &\mapsto BCAC^{-1} \\ BC &\mapsto BC \\ CA &\mapsto CBCAC^{-1}B^{-1} \end{aligned}$$

The corresponding transformation of $X_0(a,b,c,d)$ leaves invariant the functions a,b,c,d as well as y. It maps the point with coordinates z,x by:

$$\begin{bmatrix} z \\ x \end{bmatrix} \mapsto \begin{bmatrix} (ba+cd) - y(bd+ca) + y(yz+x)) - z \\ (bd+ca) - (yz+x) \end{bmatrix}$$

Just as for τ_X, the transformation induced by τ_Y on almost level set of y is ergodic. Hence any function invariant under both τ_X and τ_Y must be a function only of the boundary traces (a,b,c,d). This establishes 2.2 in the case M is a four-holed sphere.

As in the case of T_1, the exterior 2-vector field

$$\begin{aligned} \frac{1}{4\pi^2}\big((2x + yz - (ab+cd))&\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} \\ +(2y + zx - (ad+bc))&\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial y} \\ +(2z + xy - (ac+bd))&\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}\big) \end{aligned}$$

defines invariant cosymplectic structures on the relative character variety $X_0(a,b,c,d)$ for each (a,b,c,d).

6.3. **Relationship with the one-holed torus.** The character varieties of the one-holed torus T_1 and the four-holed sphere S_4 are intimately related by the *elliptic involution* on the torus. Consider the presentation (5.1) of $\pi_1(T_1)$. Let ι denote the elliptic involution of T_1, which induces the automorphism of $\pi_1(T_1)$:

$$\iota_*(X) = X^{-1}, \qquad \iota_*(Y) = Y^{-1}, \qquad \iota_*(K) = (YX)^{-1}K(YX)$$

Let O denote the quotient orbifold: the surface underlying O is a 2-disc. O has three singularities, each a branch point of order two. Its fundamental group admits the presentation

$$\pi_1(O) = \langle A', B', C', D' \mid (A')^2 = (B')^2 = (C')^2 = A'B'C'D' = I \rangle$$

and the covering projection $\phi : T_1 \longrightarrow O$ induces the homomorphism ϕ_* of orbifold fundamental groups

$$X \longmapsto A'B', \qquad Y \longmapsto B'C'$$

and $\phi_*(K) = (C'B'A')^2 = A'D'^2 A'^{-1}$.

Every semistable representation $\phi : \pi_1(T_1) \longrightarrow \mathrm{PSL}(2, \mathbb{C})$ arises from a representation $\pi_1(O) \longrightarrow \mathrm{PSL}(2, \mathbb{C})$. (Recall ϕ is *semistable* if the image of any lift $\tilde{\phi} : \pi_1(T_1) \longrightarrow \mathrm{SL}(2, \mathbb{C})$ is a direct sum of irreducible representations. Compare §2.1.) Therefore the representation variety of $\pi_1(T_1)$ embeds in the representation variety of $\pi_1(O)$. A lift $\tilde{\phi}$ always exists, since $\pi_1(T_1)$ is free; however a representation of $\pi_1(O)$ to $\mathrm{SL}(2, \mathbb{C})$ will almost never exist, due to 2-torsion. Specifically, if $\tilde{\phi} : \pi_1(O) \longrightarrow \mathrm{SL}(2, \mathbb{C})$ is a homomorphism, then

$$\tilde{\phi}(A), \tilde{\phi}(B), \tilde{\phi}(C)$$

have order two in $\mathrm{SL}(2, \mathbb{C})$ and must equal $\pm I$; this gives very few representations.

To embed the full character variety of $\pi_1(T_1)$ inside a character variety related to $\pi_1(O)$, consider the following $\mathbb{Z}/2$-central extension $\tilde{\pi} \longrightarrow \pi_1(O)$:

$$\tilde{\pi} = \langle \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}, \zeta \mid [\tilde{A}, \zeta] = [\tilde{B}, \zeta] = [\tilde{C}, \zeta] = \tilde{A}\tilde{B}\tilde{C}\tilde{D} = \zeta^2 = I, \\ \tilde{A}^2 = \tilde{B}^2 = \tilde{C}^2 = \zeta \rangle$$

The center of $\tilde{\pi}$ is generated by ζ. The quotient homomorphism $\tilde{\pi} \longrightarrow \pi_1(O)$ maps

$$\tilde{A} \mapsto A$$
$$\tilde{B} \mapsto B$$
$$\tilde{C} \mapsto C$$
$$\tilde{D} \mapsto D$$

The homomorphism $\Psi : \pi_1(T_1) \longrightarrow \tilde{\pi}$ defined by:

$$\Psi(X) = \tilde{A}\tilde{B}, \qquad \Psi(Y) = \tilde{B}\tilde{C}$$

maps K to $\zeta \tilde{A}\tilde{D}^2\tilde{A}^{-1}$ Furthermore if $\rho : \pi_1(T_1) \longrightarrow \mathrm{SL}(2, \mathbb{C})$ is a semistable representation, then there exists a representation $\tilde{\rho} : \tilde{\pi} \longrightarrow \mathrm{SL}(2, \mathbb{C})$ such that $\rho = \tilde{\rho} \circ \Psi$. (Compare Theorem 10.2.)

Consider an irreducible representation $\tilde\rho : \tilde\pi \longrightarrow \mathrm{SL}(2,\mathbb{C})$. Then $\tilde\rho(\zeta)$ is a central element of $\mathrm{SL}(2,\mathbb{C})$ (Schur's lemma), either $\pm I$. If $\tilde\rho(\zeta) = I$, then, as mentioned above, $\tilde\rho$ maps the generators $\tilde A, \tilde B, \tilde C$ to $\pm I$, and the representation is reducible, a contradiction. Hence $\tilde\rho(\zeta) = -I$ and $\tilde\rho(\tilde A)$, $\tilde\rho(\tilde B)$, and $\tilde\rho(\tilde C)$ are elements of $\mathrm{SL}(2,\mathbb{C})$ having order four. Thus

$$\mathrm{tr}(\tilde\rho(\tilde A)) = \mathrm{tr}(\tilde\rho(\tilde B)) = \mathrm{tr}(\tilde\rho(\tilde C)) = 0$$

Let $\Phi : \pi_1(S) \longrightarrow \tilde\pi$ denote the homomorphism defined by

$$\begin{aligned}\Phi(A) &= \tilde A\\ \Phi(B) &= \tilde B\\ \Phi(C) &= \tilde C\\ \Phi(D) &= \tilde D\end{aligned}$$

then the character $(\tilde a, \tilde b, \tilde c, \tilde d; \tilde x, \tilde y, \tilde z)$ of $\tilde\rho \circ \Phi$ satisfies

$$\tilde a = \tilde b = \tilde c = 0$$

Applying (6.1), the character variety has coordinates $\tilde d, \tilde x, \tilde y, \tilde z$ given by

$$\mathrm{tr}(\tilde\rho(\tilde D)) = \tilde d, \quad \mathrm{tr}\,\tilde\rho(\tilde A\tilde B) = \tilde x, \quad \mathrm{tr}\,\tilde\rho(\tilde B\tilde C) = \tilde y, \quad \mathrm{tr}\,\tilde\rho(\tilde C\tilde A) = \tilde z$$

satisfying

$$\tilde x^2 + \tilde y^2 + \tilde z^2 + \tilde x\tilde y\tilde z = 4 - \tilde d^2$$

There is an isomorphism between the character varieties of $\pi_1(T_1)$ and $\tilde\pi$ as follows. The coordinates x, y, z, k on $X(\pi_1(T_1))$ given by (5.2) relate to the coordinates $\tilde x, \tilde y, \tilde z, \tilde d$ by

$$\begin{aligned}x &= \mathrm{tr}(\phi(X)) = \mathrm{tr}(\tilde\phi\circ\Psi(X)) = \mathrm{tr}(\tilde\phi\circ\Psi(\tilde A\tilde B)) = \tilde x,\\ y &= \mathrm{tr}(\phi(Y)) = \mathrm{tr}(\tilde\phi\circ\Psi(Y)) = \mathrm{tr}(\tilde\phi\circ\Psi(\tilde B\tilde C)) = \tilde y,\\ z &= \mathrm{tr}(\phi(XY)) = \mathrm{tr}(\tilde\phi\circ\Psi(Z)) = \mathrm{tr}(\tilde\phi\circ\Psi(\zeta\tilde A\tilde C)) = -\tilde z,\\ k &= \mathrm{tr}(\phi(K)) = \mathrm{tr}(\tilde\phi\circ\Psi(K)) = \mathrm{tr}(\tilde\phi\circ\Psi(\zeta\tilde A(\tilde D)^2\tilde A^{-1})) = -(d^2-4)\end{aligned}$$

Thus the relative character varieties of the one-holed torus T_1 embeds in the relative character varieties of a four-holed sphere S.

7. Pants decompositions

The Fenchel-Nielsen coordinates on Teichmüller space $\mathcal{T}_g$ describe $\mathcal{T}_g$ explicitly as a cell. This description starts from a decomposition of M into *pants,* (three-holed spheres). The N curves in the decomposition define a map from $\mathcal{T}_g$ to $\mathbb{R}^N$ the fibers of which are submanifolds of dimension N. Indeed, as shown by Wolpert [42], the map $\mathcal{T}_g \longrightarrow \mathbb{R}^N$ is a moment map for a Hamiltonian $\mathbb{R}^N$-action and its fibers are precisely the orbits. (See Abikoff [1], Wolpert [42] and Ratcliffe [39],§9.)

An analogous picture for $X(M)$ has been extensively studied in a series of papers by Jeffrey-Weitsman [23], [24],[25]. A pants decomposition $\mathcal{P}$ of M determines a map $X(M) \longrightarrow \mathbb{R}^N$ whose generic fiber is an N-torus. While global topological coordinates are impossible since $X(M)$ is compact, *as a measure space* $X(M)$ is a T^N-bundle over an N-cell.

M is a compact surface of genus g with m boundary components. Its Euler characteristic equals $\chi(M) = 2 - 2g - m$. Decompose M into *pants,* surfaces of genus 0 with three boundary curves each. Denote by $\mathcal{P}$ the collection of pants in the decomposition; since the Euler characteristic of a pants equals -1, the cardinality of $\mathcal{P}$ is $2g - 2 + m$. The union $\mathcal{B}$ of all ∂P_i $(i = 1, \dots, 2g - 2 + m)$ consists of

$$N = 3g - 3 + 2m$$

disjoint simple closed curves. (If $\chi(M) \leq -2$, then we may assume that each union of an adjacent pair of subsurfaces is itself a genus 0 subsurface with four boundary components.)

7.1. **The character mapping of a pants decomposition.** For each $b_i \in \mathcal{B}$, consider the mapping

$$\begin{aligned} s_i : X(M) &\longrightarrow \mathbb{R} \\ [\rho] &\longmapsto \operatorname{tr} \rho(b_i) \end{aligned}$$

and let $s_{\mathcal{P}}$ denote the mapping

$$X(M) \longrightarrow \mathbb{R}^N$$

defined by $s_{\mathcal{P}} = (s_1, \dots, s_N)$.

Lemma 7.1. *The image $\mathfrak{I}_{\mathcal{P}}$ of $s_{\mathcal{P}}$ consists of all $t \in [-2,2]^N$ such that*

$$t_i^2 + t_j^2 + t_k^2 - t_i t_j t_k \leq 4 \tag{7.1}$$

for each $P_l \in \mathcal{P}$ such that $\partial P_l = \{b_i, b_j, b_k\}$.

Proof. The character (t_i, t_j, t_k) of the restriction of ρ to $\pi_1(P_l)$ lies in $X(P_l)$ and so by Lemma 4.1, satisfies (7.1). Conversely suppose that $t \in [-2,2]^N$ satisfies (7.1) for each triple (i,j,k) corresponding to a pants $P_l \in \mathcal{P}$. Choose a representation $\rho_l : \pi_1(P_l) \longrightarrow G$ with character (t_i, t_j, t_k); then for each $b_i = \partial P_l \cap \partial P_q$, there is g_i such that $\rho_l(b_i) = g_i \rho_q(b_i) g_i^{-1}$. Continue this for all the pants in $\mathcal{P}$. ∎

Apply Lemma 4.1 to the mapping $s_{\mathcal{P}}$. The equivalence class of a representation of each $\pi_1(P_l)$ is determined by the three boundary traces t_i, t_j, t_k. The fiber $s_{\mathcal{P}}^{-1}(t)$ corresponds to the various amalgamations of the representations of the $\pi_1(P_l)$ and $\pi_1(P_q)$, where P_l and P_q are adjacent subsurfaces in the pants decomposition. Let

$$b_j = \partial P_l \cap \partial P_q \in \mathcal{B};$$

then the inclusions $b_j \hookrightarrow P_l$ and $b_j \hookrightarrow P_q$ induce monomorphisms $\pi_1(b_j) \longrightarrow \pi_1(P_l)$ and $\pi_1(b_j) \longrightarrow \pi_1(P_l)$ respectively. Denote the elements of $\pi_1(P_l)$ and $\pi_1(P_q)$ corresponding to a generator of $\pi_1(b_j)$ by γ_j as well. By van Kampen's theorem, the fundamental group of $P_l \cup P_q$ admits the decomposition

$$\pi_1(P_l \cup P_q) = \pi_1(P_l) \amalg_{\gamma_j} \pi_1(P_q).$$

Let ρ_l and ρ_q be representations of $\pi_1(P_l)$ and $\pi_1(P_q)$ respectively in G such that $\rho_l(\gamma_j) = \rho_q(\gamma_j)$. Elements of the centralizer $Z(\rho(\gamma_j))$ of $\rho(\gamma_j)$ determine equivalence classes $[\rho]$ of representations $\rho : \pi_1(P_l \cup P_q) \longrightarrow G$ such that $[\rho]$ restricts to $[\rho_l]$ on $\pi_1(P_l)$ and to $[\rho_q]$ on $\pi_1(P_q)$ respectively. For every $\zeta \in Z(\rho(\gamma_j))$,

$$\gamma \longmapsto \begin{cases} \rho(\gamma) & \text{for } \gamma \in \pi_1(P_l) \\ \zeta\rho(\gamma)\zeta^{-1} & \text{for } \gamma \in \pi_1(P_q) \end{cases}$$

defines a representation

$$\zeta \cdot \rho : \pi_1(P_l \cup P_q) \longrightarrow G \tag{7.2}$$

Moreover (7.2) defines an action of $Z(\rho(\gamma_j))$ on $X(P_l \cup P_q)$. This action is trivial on the center $Z(G) = \{\pm I\}$. The resulting action of $Z(\rho(\gamma_j))/Z(G)$ is effective if the restrictions of ρ to $\pi_1(P_l)$ and $\pi_1(P_q)$ are irreducible. If $\rho(\gamma_j) \neq \pm I$, then $Z(\rho(\gamma_j))$ is the unique one-parameter subgroup (a maximal torus in SU(2)) containing $\rho(\gamma_j)$. Thus the product over $\mathcal{B}$

$$\prod_{i=1}^{N} Z(\rho(\gamma_i)) \approx T^N$$

is isomorphic to an N-torus. The resulting T^N-action on the fiber $s_{\mathcal{P}}^{-1}(t)$ is simply transitive for almost every $t \in \mathfrak{I}_{\mathcal{P}}$.

7.2. **Automorphisms of pants decompositions.** The stabilizer $\Gamma_{\mathcal{P}}$ of $\mathcal{P}$ in Γ_M is generated by the Dehn twists τ_i about b_i. This free abelian group has basis $\tau_1, \dots, \tau_N$. Evidently $s_{\mathcal{P}} : X(M) \longrightarrow \mathfrak{I}_{\mathcal{P}}$ is $\Gamma_{\mathcal{P}}$-invariant. We summarize the preceding results:

Lemma 7.2. *Let $t \in \mathfrak{I}_{\mathcal{P}}$. Suppose that $t_j \neq \pm 2$ for $j = 1, \dots, N$. Let $\theta_j = \cos^{-1}(t_j/2)$. Then there is a $\Gamma_{\mathcal{P}}$-equivariant homeomorphism*

$$h : s_{\mathcal{P}}^{-1}(t) \longrightarrow T^N$$

such that for $\xi \in s_{\mathcal{P}}^{-1}(t)$, and for all $(n_1, \dots, n_N) \in \mathbb{Z}^n$,

$$h : \tau_1^{n_1} \dots \tau_N^{n_N} \xi \longmapsto \begin{bmatrix} e^{in_1\theta_1} h_1 \\ \vdots \\ e^{in_N\theta_N} h_N \end{bmatrix}$$

where

$$h(\xi) = \begin{bmatrix} h_1 \\ \vdots \\ h_N \end{bmatrix}$$

Furthermore h maps the measure on $s_{\mathcal{P}}^{-1}(t)$ obtained by disintegrating the symplectic volume on X over the map $s_{\mathcal{P}} : X \longrightarrow \mathbb{R}^N$ to a multiple of Haar measure on T^N.

The mapping $s_{\mathcal{P}} : X(M) \longrightarrow \mathbb{R}^N$ is a moment mapping for an $\mathbb{R}^N$-action (compare Goldman [13], Jeffrey-Weitsman [23]). It is a submersion except on the subset consisting of $[\rho] \in X(M)$ such that $s_j(\rho) = \pm 2$ for some $j = 1, \dots, N$. Its orbits are tori of dimension N. The isotropy group of the orbit in the fiber $s_{\mathcal{P}}^{-1}(t)$ is the lattice

$$\left\{ (n_1 \sin^{-1}(t_1/2), \dots, n_N \sin^{-1}(t_N/2) \mid n_1, \dots, n_N \in \mathbb{Z} \right\}$$

After discarding from $X(M)$ a null set, we obtain a set $X'(M)$ such that the restriction $s_{\mathcal{P}} : X'(M) \longrightarrow [-2, 2]^N$ satisfies:

- The fibers $s_{\mathcal{P}}^{-1}(t)$ are all N-dimensional tori;
- The action of $\Gamma_{\mathcal{P}}$ on $s_{\mathcal{P}}^{-1}(t)$ is ergodic.

Specifically let

$$X_{\mathbb{Q}} = \{[\rho] \in X(M) \mid \frac{\cos(s_j([\rho]))}{2} \in \mathbb{Q}\}$$

For each $[\rho] \notin X_{\mathbb{Q}}$, the mapping $s_{\mathcal{P}}$ is submersive (by the above paragraph) and hence the fiber $s_{\mathcal{P}}^{-1}(t)$, where $t = s_{\mathcal{P}}([\rho])$ is a smooth manifold of dimension N invariant under the locally free Hamiltonian $\mathbb{R}^N$-action. Since $s_{\mathcal{P}}^{-1}(t)$ is compact, this action is transitive and the fiber is a torus. Furthermore, since $[\rho] \notin X_{\mathbb{Q}}$, the action of $\Gamma_{\mathcal{P}}$ is a Cartesian product of irrational rotations. Therefore the action of $\Gamma_{\mathcal{P}}$ on $s_{\mathcal{P}}^{-1}(t)$ is ergodic. Since $X_{\mathbb{Q}}$ is a countable union of lower-dimensional subvarieties and the singular locus (which has zero measure), $X_{\mathbb{Q}}$ has measure zero. We therefore replace $X(M)$ by the $\Gamma_{\mathcal{P}}$-invariant subset $X'(M) = X(M) - X_{\mathbb{Q}}$ of full measure. Applying Lemma 2.1,

Lemma 7.3. *Let $f : X(M) \longrightarrow \mathbb{R}$ be a $\Gamma_{\mathcal{P}}$-invariant measurable function. Then there exists a function $F : \mathfrak{I}_{\mathcal{P}} \longrightarrow \mathbb{R}$ such that $f = F \circ s_{\mathcal{P}}$ almost everywhere.*

Theorem 1.9 follows readily since every system of curves α can be extended to a system of curves $\mathcal{B}$ arising from a pants decomposition.

7.3. **Relation to the work of Jeffrey-Weitsman.** Jeffrey and Weitsman consider the angular variable $\cos^{-1}(s_j/2)$ rather than the trace s_j itself. (See [23] and [24],§3.10.) The advantage of their approach is that the periods of the corresponding Hamiltonian flows are all equal. Hence these flows generate commuting circle actions. They combine to define an action of a compact Lie group (a torus). The disadvantage is that these actions are only defined on a dense open subset. Since the set upon which these periodic Hamiltonians are not defined has zero measure, the approach of Jeffrey and Weitsman would be perfectly appropriate here. The angular coordinates of [23],[24] are the compact analogue of length displacement functions on Teichmüller space. (Compare §1.12, §4.11 of [13]).

We can explicitly determine the pushforward $(s_{\mathcal{P}})_*\mu$ of the Liouville measure μ associated to the symplectic structure on $X(M)$. To be consistent with Jeffrey-Weitsman [23], [24], [25], we consider the symplectic structure associated with the bilinear form

$$\begin{aligned} B : \mathfrak{su}(2) \times \mathfrak{su}(2) &\longrightarrow \mathbb{R} \\ (X, Y) &\longmapsto \frac{1}{4\pi^2}\operatorname{tr}(XY) \end{aligned}$$

which has integral cohomology class in $X(M)$. Let $\gamma \in \pi$ correspond to a homotopically nontrivial simple loop. Then the Hamiltonian flow corresponding to the function

$$\begin{aligned} h_\gamma : X(M) &\longrightarrow \mathbb{R} \\ [\rho] &\longmapsto \frac{1}{\pi}\cos^{-1}\left(\frac{\operatorname{tr}(\rho(\gamma))}{2}\right) \end{aligned}$$

has period 1 (respectively 1/2) if γ represents a nonseparating (respectively separating) simple closed curve.

Define $h_{\mathcal{P}} : X(M) \longrightarrow \mathbb{R}^N$ by

$$h_{\mathcal{P}}([\rho]) = (h_{b_1}, \dots, h_{b_N})$$

(Jeffrey and Weitsman consider this moment map and their work motivates our discussion.) The pushforward $(h_{\mathcal{P}})_*(\mu)$ then equals 2^{-r} times Euclidean Lebesgue measure on $\mathbb{R}^n$, where r equals the number of separating curves in $\mathcal{B}$. By computing the derivative of h_c, the pushforward of μ under $s_{\mathcal{P}}$ is:

$$(s_{\mathcal{P}})_*(\mu) = 2^r(2\pi)^N \prod_{i=1}^{N} (4 - s_i^2)^{-1/2} \mu(s)$$

where $\mu(s)$ is Euclidean Lebesgue measure on $\mathbb{R}^N$.

8. Conclusion of the proof

The proof of Theorem 2.2 concludes by considering Dehn twists about curves which cross curves in the pants decomposition. The reduction associated with a pants decomposition $\mathcal{P}$ combines with the analysis of the one-holed torus and four-holed spheres to deduce ergodicity of all of Γ_M as follows. Suppose that $f : X(M) \longrightarrow \mathbb{R}$ is a measurable function invariant under Γ_M. Since f is invariant under the subgroup $\Gamma_{\mathcal{P}} \subset \Gamma_M$, it follows from Lemma 7.3 that f depends only on the traces $s_1, \dots, s_N$ of the curves in $\mathcal{B}$. Precisely, there is a function F defined on the image of $s_{\mathcal{P}}$ such that

$$f([\rho]) = F(s_1([\rho]), \dots, s_N([\rho]))$$

The choice of the pants decomposition $\mathcal{P}$ is really quite arbitrary. We next apply Dehn twists *not preserving* the pants decomposition, to show that the function f is actually independent of the functions s_j. Choose a curve $b_j \in \mathcal{B}$. Let $P, P' \in \mathcal{P}$ be the subsurfaces whose boundaries contain b_j and let $S = P \cup P' \subset M$. We distinguish two cases:

- b_j is nonseparating, in which case $P = P'$ and A is a one-holed torus;
- b_j separates, in which case A is a four-holed sphere.

In either case Γ_A embeds as a subgroup of Γ_M. The restriction map $X(M) \longrightarrow X(A)$ is Γ_A-equivariant.

Suppose inductively on

$$j = 1, \dots, N$$

that F depends only on $s_j, \dots, s_N$. We show that F is independent of s_j. That is, disintegrate the measure on $X(M)$ over the map $(s_1, \dots, s_{j-1})$ and suppose that F is almost everywhere constant on almost every level set of s_k for $1 \leq k < j$. The special cases of Theorem 2.2 for a one-holed torus or four-holed sphere (proved respectively in §5 and §6) imply that Γ_A is ergodic on almost every level set of the mapping $(s_1, \dots, s_{j-1}, s_{j+1}, \dots, s_N)$. By the induction hypothesis, F is almost everywhere constant on almost every level set of $(s_{j+1}, \dots, s_N)$.

Thus we have "eliminated the variable s_j" for $j = 1, \ldots, N$ and F is almost everywhere constant. It follows that Γ_M is ergodic on $X(M)$.

9. Appendix: Modifications for product groups

We describe the steps necessary to modify the proof of Theorem 2.2 from $G = \mathrm{SU}(2)$ to the general case when G is a compact Lie group locally isomorphic to a product of copies of SU(2) and an abelian group. Such a group G admits a finite covering space

$$\tilde{G} = \mathrm{SU}(2)^n \times \mathrm{U}(1)^m \longrightarrow G$$

and it suffices to prove Theorem 2.2 for $\tilde{G}$.

The outline of the proof for the product group $\tilde{G}$ is exactly the same as for SU(2). For any group π,

$$\mathrm{Hom}(\pi, \tilde{G}) = \mathrm{Hom}(\pi, \mathrm{SU}(2))^n \times \mathrm{Hom}(\pi, \mathrm{U}(1))^m$$

Begin with a pair-of-pants M, with presentation of its fundamental group given by (4.1). A representation of $\pi_1(M)$ into U(1) is given by $(\alpha, \beta, \gamma) \in \mathrm{U}(1)$ satisfying $\alpha\beta\gamma = 1$. The set of such triples is a 2-torus which we denote by $\mathfrak{t}$. A representation $\rho : \pi_1(M) \longrightarrow \tilde{G}$ is completely determined up to conjugacy by the characters of its factors

$$(\chi(\rho_1), \ldots, \chi(\rho_n), \rho_{n+1}, \ldots, \rho_{n+m}) \in \mathfrak{B}^n \times \mathfrak{t}^m$$

in the notation of §4. As in §4, the restriction map

$$\partial^\# : X(M) \longrightarrow X(\partial M)$$

is an embedding and the relative character varieties are points. (This property fails for other compact Lie groups, such as $\mathrm{SU}(n)$ for $n \geq 3$.)

The proof of ergodicity when M is a one-holed torus or a four-holed sphere goes through as before. For simplicity we only consider the case of the one-holed torus T_1. The restriction map in this case is

$$\partial^\# : X(M) \longrightarrow \mathfrak{B}^n \times \mathfrak{t}^m$$

The fibers of the character map associated to the interior curve $X \subset M$

$$x : X(M) \longrightarrow \mathfrak{B}^n \times \mathfrak{t}^m$$

are tori of dimension $n + m$ upon which the Dehn twist τ_X acts by rotation. For almost every t, the action of τ_X on $x^{-1}(t)$ is an ergodic rotation. (Although for an ergodic rotation r of the circle T^1, the diagonal action $T^2 \longrightarrow T^2$ defined by $(t_1, t_2) \mapsto (rt_1, rt_2)$ is not ergodic, for almost every pair (r_1, r_2) of rotations, the rotation $(t_1, t_2) \mapsto (r_1t_1, r_2t_2)$ is ergodic.)

The rest of the proof is identical to the SU(2) case.

10. Appendix: Effective actions of automorphisms

In this section we prove that the action of the mapping class group is effective. This is the only place in the paper in which we use the complex character variety. We also include several related facts about character varieties of small surfaces.

10.1. Effectiveness on nonabelian representations.

Theorem 10.1. *Let G be a compact nonabelian Lie group and let M be a closed surface of genus $g > 2$. Then Γ acts effectively on* $\mathrm{Hom}(\pi, G)$.

Proof. Since G contains $\mathrm{SU}(2)$ or $\mathrm{PU}(2)$ and $\mathrm{PU}(2)$ is a quotient of $\mathrm{SU}(2)$, there is a Γ-equivariant open mapping

$$\mathrm{Hom}(\pi, \mathrm{SU}(2)) \longrightarrow \mathrm{Hom}(\pi, G)$$

Thus it suffices to consider the case $G = \mathrm{SU}(2)$.

Suppose $\gamma \in \Gamma$ acts as the identity on $\mathrm{Hom}(\pi, G)$, then it must act as the identity on the complexification $\mathrm{Hom}(\pi, G)_{\mathbb{C}} \cong \mathrm{Hom}(\pi, \mathrm{SL}(2, \mathbb{C}))$, which is an irreducible variety (Goldman [14], compare also Rapinchuk, Benyash-Krivetz, and Chernousov [38]). There is a Γ-equivariant embedding $\mathcal{T}_g \longrightarrow \mathrm{Hom}(\pi, \mathrm{SL}(2, \mathbb{C}))$ of the Teichmüller space $\mathcal{T}_g$. By assumption γ fixes every point of $\mathcal{T}_M$. Since $g > 2$, then $\gamma = 1$. ☐

If $g = 2$, the only mapping class which preserves every conformal structure on M must be the hyperelliptic involution, and indeed since every genus 2 Riemann surface is hyperelliptic, the action of Γ on $\mathrm{Hom}(\pi, G)$ is not effective.

10.2. Hyperelliptic representations.

A closely related phenomenon involves representations of $\pi_1(M)$ which are invariant under a hyperelliptic involution of M. Let M be a compact surface of genus g with m boundary components. An involution $f : M \longrightarrow M$ is said to be *hyperelliptic* if the quotient surface is planar, that is, if the quotient orbifold is an m-holed sphere with $2 + 2g - m$ branch points of order two. Then a Riemann surface is hyperelliptic if it admits a hyperelliptic involution preserving the conformal structure. It is well known that every closed Riemann surface of genus at most 2 is hyperelliptic, and that every Riemann surface of genus 1 with at most two boundary components is hyperelliptic. Furthermore in these cases the hyperelliptic involution determines a central element F of the mapping class group.

Consider how the hyperelliptic mapping class acts on the character variety of M. Say that a representation

$$\rho \in \mathrm{Hom}(\pi_1(M), G)$$

is *hyperelliptic* if there exists a hyperelliptic involution f of M such that

$$[\rho \circ f_*] = [\rho] \in \rho \in \mathrm{Hom}(\pi_1(M), G)/G$$

The uniformization representation of a hyperelliptic Riemann surface is hyperelliptic. The Fuchsian representations of $\pi_1(M)$ in $\mathrm{SL}(2,\mathbb{C})$ form the Teichmüller space $\mathcal{T}_g$, which is fixed by f_*. Since the action of f_* on the variety of $\mathrm{SL}(2,\mathbb{C})$-characters $X_{\mathbb{C}}(M)$ is algebraic, the subset fixed by f_* is closed in the complex Zariski topology. Since $\mathcal{T}_g$ is dense in $X_{\mathbb{C}}(M)$ with the complex Zariski topology, it follows that f_* acts trivially on $X_{\mathbb{C}}(M)$, and hence on $X(M)$:

Theorem 10.2. *Let M be a closed surface of genus 2, or an m-holed torus with $m = 1, 2$.*

- *Then every representation $\pi_1(M) \longrightarrow G$ is invariant under the hyperelliptic involution.*
- *Let O be the quotient orbifold and let $\pi_1(M) \hookrightarrow \pi_1(O)$ be the corresponding embedding of $\pi_1(M)$ as an index-two subgroup of $\pi_1(O)$. Then every representation $\pi_1(M) \longrightarrow G$ extends to a representation $\pi_1(O) \longrightarrow G$.*

This theorem is false if G is replaced by other groups (such as $\mathrm{SU}(3)$ or $\mathrm{SL}(3)$).

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