REVIEW OF "ALGORITHMIC AND COMPUTER METHODS FOR THREE-MANIFOLDS," BY A. T. FOMENKO AND S. V. MATVEED.

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This book is an attractive and comprehensive introduction to three-dimensional topology. This extremely active subject has been revolutionized in the last forty years and is idiosyncratic due to its highly visual and intuitive nature. Accordingly, geometric topology is very difficult to learn unless one "sees the picture." Once one does see the picture, many proofs become visually obvious and the feeling of understanding exhilarates. Not seeing the picture can be extremely frustrating since traditional proofs written in words are often painfully inadequate in conveying the ideas. Furthermore, 3-manifolds lend themselves to pictorial representations having human meaning. Important examples of 3-manifolds arise as the complements of knots. A knot can be represented by projecting it to a plane and then, at each crossing, label whether the crossing is an under-crossing or an over-crossing, rendering it 3-dimensional. This kind of local description of 3-dimensional topological phenomenon is intuitively clear but rather hard to communicate accurately and efficiently in words. Hence the rather large amount of hand-waving which goes into explaining this subject.

The authors recognize the above difficulty in the subject and address it admirably. In its 334 pages, the book contains 316 illustrations. Many of the important fundamental results in 3-manifold topology are skillfully explaining with the use of Fomenko's pictures.

The book concentrates on the algorithmic aspects of 3-dimensional topology. Topology has, ever since its inception over a century ago, been very combinatorial in nature. Although analytic methods have become powerful in the subject, working with specific examples usually necessitates combinatorial calculations. The book is full of specific examples and detailed descriptions of how to calculate with 3-manifolds presented in various forms (identification spaces of polyhedra, Heegaard diagrams, surgeries on links, etc.). The book contains several algorithms for performing calculations with 3-manifolds.

Date: February 12, 1999.

The book begins with a preliminary chapter on general topology, construction of spaces as quotient spaces of polyhedra, basic algebraic topology (3-manifold topology really only uses the fundamental group and homology, cohomology and homotopy groups up to degree 2 — however it uses these functors quite heavily). The second chapter deals with surfaces: classification of surfaces, surgery and Dehn's lemma/Loop theorem. The third chapter is a detailed introduction to the homeotopy (or *mapping class*) group of surfaces, including the generation of the homeotopy group by Dehn twists and a discussion of the homeotopy group of a handlebody. This latter material is used later on in the discussion of Heegaard splittings.

Chapter 4 discusses the presentation of 3-manifolds as identification spaces of polyhedra. Lens spaces are discussed in detail. Chapter 5 discusses Heegaard decompositions; the main theorem is the Reidemeister-Singer theorem that two Heegaard decompositions are stably equivalent. (This is a key ingredient in the original proof that the Casson invariant is independent of the Heegaard decomposition used to define it.) Heegaard diagrams are combinatorial objects effectively representing 3-manifolds given by Heegaard decompositions. *Wave transformations* are introduced as moves which simplify Heegaard decompositions of the same 3-manifold. This algorithmic theory is described in detail for genus 2. Chapter 6 discusses the enumeration problem for 3-manifolds. Using the theory developed in the preceding chapter, an algorithm for recognizing S^3 among 3-manifolds with genus 2 Heegaard decompositions is given. It is described how to realize this algorithm as a computer program, and some of the data produced is given as an illustration of this theory.

Chapter 7 deals with connected sums, and proves the classical Prime Decomposition Theorem of Kneser-Milnor. Chapter 8 is a survey of knot and link theory, and knot invariants are developed in terms of *distributive groupoids.* The Alexander polynomial is developed algebraically, and then reinterpreted in terms of Conway's approach (which is now part of the theory of skein modules). This leads to the Jones polynomial and its 2-variable generalization, the HOMFLY polynomial. Chapter 9 is devoted to surgery along links, and in particular develops the *Kirby calculus* for calculating link surgery and 4-manifolds obtained by attaching 2-handles to the 4-cell. Chapter 10 gives the theory of Seifert manifolds, a class of 3-manifolds which behave much like 2-manifolds, and admit a classification based on the classification of surfaces. Seifert manifolds can be glued to one another to obtain more complicated 3-manifolds, called *graph manifolds* or *manifolds of class H*. They form the subject of Chapter 11, where among other things, they are classified combinatorially. Chapter 12 discusses Haken's method of classifying 3-manifolds in terms of their incompressible surfaces, which has been one of the most successful methods for analyzing 3-manifolds recently.

The book is readable and inviting. Its many illustrations make it particularly accessible. Not treating any differential geometry, this book complements the recently published book of Thurston, "Three-Dimensional Geometry and Topology" (Princeton Mathematical Series 35, Princeton University Press 1997). This book promises to be a valuable text and reference for an exciting area of mathematics.

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