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Reviewer: W.M.Goldman

Reviewer number: 014521

Address:

Math Dept. University of Maryland,College Park, Maryland 20742

Author: Loftin, John C.

Short title: Affine spheres and convex $\mathbb{RP}^n$-manifolds

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Review text:

The present paper relates affine differential geometry to the study of (flat) real projective structures on manifolds.

An $\mathbb{RP}^n$-manifold is a manifold M with a distinguished atlas of coordinate charts to $\mathbb{RP}^n$ with coordinate changes in the projective group $\mathbf{PGL}(n+1, \mathbb{R})$. Such a structure determines a projective equivalence class of projectively flat connections. Apart from exceptional cases (constant curvature Riemannian metrics) such a projectively flat connection will *not* be the Levi-Civita connection for a Riemannian structure. However, in some cases, one can find a canonical Riemannian metric on M with strong analytic properties related to the $\mathbb{RP}^n$-structure.

An $\mathbb{RP}^n$-manifold is *convex* if it is equivalent to the quotient Ω/Γ where $\Omega \subset \mathbb{RP}^n$ is a properly convex domain and $\Gamma \subset \mathbf{PGL}(n+1, \mathbb{R})$ is a discrete subgroup acting properly on Ω. The present paper identifies convex $\mathbb{RP}^n$-structures with structures arising in affine differential geometry and applies this correspondence to the moduli of $\mathbb{RP}^n$-structures on surfaces when $n = 2$.

The basic object is an *affine sphere,* classically studied by Blaschke and others in the early twentieth century. An affine sphere is a strictly convex hypersurface in affine space satisfying a differential equation of Monge-Ampère type analogous to the Kähler-Einstein condition for Hermitian metrics on complex manifolds. The basic existence, uniqueness and regularity of solutions of this equation are due to Calabi, Loewner-Nirenberg, Cheng and Yau and others. In particular Cheng and Yau proved that every convex domain $\Omega \subset \mathbb{RP}^n$ determines an affine sphere asymptotic to the cone over Ω in $\mathbb{R}^{n+1}$ (*Complete affine hyperspheres I. The completeness of affine metrics,* Comm. Pure Appl. Math. **33** (1986), 839–866). (This settles a conjecture of Calabi.) Using this correspondence, the author proves (Theorem 1) that every convex $\mathbb{RP}^n$-manifold carries a canonical complete Riemannian metric h and two canonical projectively flat affine connections ∇ and $\bar{\nabla}$. The projectively flat connection ∇ is due to Blaschke, and the average $\hat{\nabla} = (\nabla + \bar{\nabla})/2$ is the Levi-Civita connection of h. Furthermore ∇ and $\bar{\nabla}$ correspond to dual $\mathbb{RP}^n$-structures, that is modelled on dual convex domains $\Omega, \bar{\Omega}$ in dual projective spaces. The two connections agree precisely when Ω is the interior of a hyperquadric, in which case the $\mathbb{RP}^n$-structure is a hyperbolic structure, via the Klein-Beltrami projective model of hyperbolic geometry.

In dimension $n = 2$, the reviewer has defined (*Convex real projective structures on compact surfaces,* J. Diff. Geo. **31** (1990), 791–845) a deformation space $\mathcal{G}(S)$ parametrizing equivalence classes of convex $\mathbb{RP}^n$-structures on a compact surface S, and proved it to be diffeomorphic to $\mathbb{R}^{16g-16}$ if S is closed orientable, and of genus $g > 1$. Using the analytic machinery, this paper contains a sharp analytic proof of this result.

With the conformal structure induced by the canonical Riemannian metric h, the convex $\mathbb{RP}^n$-manifold M becomes a Riemann surface Σ. Following C.-P. Wang (*Some examples of complete hyperbolic affine 2-spheres in* $\mathbb{R}^3$, Global Differential Geometry and Global Analysis, Lecture Notes in Math. vol. 1481, Springer-Verlag, New York (1991) 271-280), the affine differential geometry determines a holomorphic cubic differential $C \in H^0(\Sigma; K^3)$ (where K is the canonical bundle of Σ). Namely *Pick form* $\hat{\nabla} - \nabla$ is a tensor field in

$$\mathrm{Hom}(T\Sigma, \mathrm{End}(T\Sigma)) \cong T^*\Sigma \otimes (T^*\Sigma \otimes T\Sigma)$$

which corresponds to a holomorphic cubic differential. As explained to us by F. Labourie, dualizing the Pick form with respect to h produces a symmetric 3-tensor in $\Gamma(S^3 T^*\Sigma)$. The holomorphic cubic differential C is then the (3,0)-Hodge component of the complexified 3-tensor.

Thus the deformation space $\mathcal{G}(S)$ of convex $\mathbb{RP}^2$-structures on S identifies with the total space of the holomorphic $\mathbb{C}^{5g-5}$-bundle over the Teichmüller space $\mathcal{T}(S)$ whose fiber over a point $\sigma \in \mathcal{T}(S)$ is the vector space $H^0(\Sigma; K^3)$ where Σ is the marked Riemann surface representing σ.

This approach closely relates to Hitchin's description of a component $\mathfrak{C}$ of the deformation space

$$\mathcal{D} = \mathrm{Hom}(\pi_1(S), \mathrm{SL}(3; \mathbb{R}))/\mathrm{SL}(3; \mathbb{R})$$

flat $\mathrm{SL}(3;\mathbb{R})$-bundles (*Lie groups and Teichmüller space,* Topology **31** (3), (1992), 449–473). For a fixed conformal structure on S, Hitchin finds a component of $\mathcal{D}$ which identifies with the complex vector space $H^0(\Sigma, K^2) \oplus$

$H^0(\Sigma, K^3)$. Choi and the reviewer showed that $\mathfrak{C}$ contains the equivalence classes of holonomy representations $\phi : \pi_1(\Sigma) \longrightarrow \mathrm{SL}(3;\mathbb{R})$ of convex $\mathbb{RP}^2$-structures, that is, $\mathfrak{C}$ equals $\mathcal{G}(S)$. (*Convex real projective structures on closed surfaces are closed,* Proc. A.M.S. **118** no.2 (1993), 657–661) Thus ϕ determines an element

$$Q \oplus C \in H^0(\Sigma, K^2) \oplus H^0(\Sigma, K^3).$$

The holomorphic quadratic differential Q is the Hopf differential of the (unique) ϕ-equivariant harmonic map of the universal covering space of Σ into the symmetric space of $\mathrm{SL}(3;\mathbb{R})$. When Σ is the conformal structure induced by the affine metric h, then this map is holomorphic and the holomorphic cubic differential is the one induced by the Pick form above.