COMPLETE FLAT LORENTZIAN 3-MANIFOLDS WITH FREE FUNDAMENTAL GROUP

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Introduction

In 1983, G.A.Margulis proved the following startling result, answering a question posed by Milnor [5] (see also [2]):

Theorem (Margulis)**.** *There exists a nonabelian free discrete group* Γ *of affine transformations which acts properly and freely on* $\mathbb{R}^3$*.*

In this paper we give a new proof of this result, using ideas from Margulis' original proof [3, 4] as well as geometric conditions due to the first author [1]. Let $\mathbb{R}^{2,1}$ denote a 3-dimensional real vector space with Lorentzian inner product

$$\mathbb{B}(u,v) = u_1v_1 + u_2v_2 - u_3v_3$$

and $\mathbf{SO}(2,1)$ its group of unimodular linear automorphisms. A *Schottky subgroup of* $\mathbf{SO}(2,1)$ is a discrete free subgroup of $\mathbf{SO}(2,1)$ such that every element other than the identity has distinct real eigenvalues. (The usual definition is more geometric, although it is equivalent to this one.) Let $\mathbb{E}$ denote the affine space corresponding to $\mathbb{R}^{2,1}$; as (flat) Minkowski space, it is a simply connected complete flat Loretzian manifold and as such is unique up to Lorentz isometry. We denote the group of translations of $\mathbb{E}$ by $\mathbf{V}$; as a vector space is naturally

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identified with $\mathbb{R}^{2,1}$. Let $\mathbf{G}$ be the identity component of $\mathbf{SO}(2,1)$ and $\mathbf{H} = \mathbf{G} \rtimes \mathbf{V}$ denote the identity component in the group of isometries of $\mathbb{E}$ and $\mathbb{L} : \mathbf{H} \longrightarrow \mathbf{G}$ be the homomorphism assigning to $h \in \mathbf{H}$ its linear part. If $\Lambda \subset \mathbf{G}$ is a subgroup, an *affine deformation of* Λ is a representation $\rho : \Lambda \longrightarrow \mathbf{H}$ such that $\mathbb{L}(\rho(g)) = g$ for all $g \in \Lambda$. Since such a homomorphism ρ is necessarily an embedding, we shall also refer to its image $\rho(\Lambda)$ as an affine deformation of Λ.

Theorem. *Every Schottky group of genus $k > 1$ inside $\mathbf{SO}(2,1)$ possesses an affine deformation which acts freely and properly discontinuously on $\mathbb{E}$.*

(This answers a question raised in Margulis [3].) The set of affine deformations of Λ is the space $Z^1(\Lambda, \mathbb{R}^{2,1})$ of 1-cocycles and our proof provides a nonempty open cone of 1-cocycles determining properly discontinuous affine deformations.

In [5] it is shown that a connected group of affine transformations acts properly only if it is amenable; thus it seemed natural to conjecture that a closed subgroup (in particular a discrete subgroup) of the group of affine transformations which acts properly must also be amenable. In [5] Milnor showed this question is equivalent to whether a free group of rank two can act properly discontinuously by affine transformations. In [2] it is shown that a discrete group Γ of affine transformations acting properly on $\mathbb{R}^3$ must be amenable, provided that the quotient space is compact; indeed it is shown there that unless Γ is amenable it preserves a parallel Lorentz metric. In [1] fundamental polyhedra for these actions are constructed explicitly from which it is shown that the quotient manifolds are solid handlebodies.

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1. The Geometry of $\mathbb{R}^{2,1}$.

The set $\{x \in \mathbb{R}^{2,1} \mid \mathbb{B}(x,x) = 0\}$ of *null vectors* forms the *the light cone* and its two components form the *time-orientations* of $\mathbb{R}^{2,1}$. We choose one component as the *positive time-orientation* and denote it:

$$\mathbb{W}^+ = \{x \in \mathbb{R}^{2,1} \mid \mathbb{B}(x,x) = 0, \quad x_3 > 0\}$$

If $u \in \mathbb{R}^{2,1}$ we denote by $\|u\|$ the usual Euclidean norm of u and the Euclidean distance by $\rho(u,v) = \|u - v\|$. The Euclidean unit sphere S^2 consists of u such that $\|u\| = 1$.

The *Lorentzian unit sphere* (one-sheeted hyperboloid) in $\mathbb{R}^{2,1}$ consists of vectors $u \in \mathbb{R}^{2,1}$ satisfying $\mathbb{B}(u,u) = 1$ and will be denoted by $\mathfrak{S}$. If $u \in \mathbb{R}^{2,1}$, its *Lorentzian orthogonal complement* $u^\perp$ is the linear subspace $\{v \in \mathbb{R}^{2,1} \mid \mathbb{B}(u,v) = 0\}$. If $u \in \mathfrak{S}$, there are uniquely determined null vectors $\mathbf{x}^+(u), \mathbf{x}^-(u) \in \mathbb{W}^+ \cap S^2 \cap u^\perp$ such that

$$\{u, \mathbf{x}^-(u), \mathbf{x}^+(u)\}$$

is a positively oriented basis of $\mathbb{R}^{2,1}$. Explicitly, given $u \in \mathfrak{S}$ (so that $u_1^2 + u_2^2 = 1 + u_3^2$) we have:

$$\mathbf{x}^-(u) = \frac{1}{\sqrt{2}} \begin{bmatrix} (u_1u_3 - u_2)/(u_1^2 + u_2^2) \\ (u_2u_3 + u_1)/(u_1^2 + u_2^2) \\ 1 \end{bmatrix},$$

$$\mathbf{x}^+(u) = \frac{1}{\sqrt{2}} \begin{bmatrix} (u_1u_3 + u_2)/(u_1^2 + u_2^2) \\ (u_2u_3 - u_1)/(u_1^2 + u_2^2) \\ 1 \end{bmatrix}.$$

The two rulings of the hyperboloid $\mathfrak{S}$ consist of the lines

$$\{u \in \mathfrak{S} \mid \mathbf{x}^+(u) = p\} \quad (\text{respectively}\{u \in \mathfrak{S} \mid \mathbf{x}^-(u) = p\})$$

as p varies over the circle $\mathbb{P} = \mathbb{W}^+ \cap S^2$ of (positively time-oriented null directions). Alternatively, there exist unique polar coordinates $r \geq 1, 0 \leq \theta < 2\pi$ and a sign such that

$$u = \begin{bmatrix} r\cos(\theta) \\ r\sin(\theta) \\ \pm\sqrt{r^2-1} \end{bmatrix} = R_\theta \begin{bmatrix} r \\ 0 \\ \pm\sqrt{r^2-1} \end{bmatrix}$$

where

$$R_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \mathbf{SO}(2,1) \cap \mathbf{O}(3)$$

is an isometry of $\mathbb{R}^{2,1}$ preserving the Euclidean geometry as well. Then

$$S^2 \cap \mathbb{W}^+ \cap u^\perp = \{\mathbf{x}^-(u), \mathbf{x}^+(u)\}$$

where

$$\mathbf{x}^+(u) = R_\theta \frac{1}{\sqrt{2}r} \begin{bmatrix} \pm\sqrt{r^2-1} \\ -1 \\ r \end{bmatrix},$$

$$\mathbf{x}^-(u) = R_\theta \frac{1}{\sqrt{2}r} \begin{bmatrix} \pm\sqrt{r^2-1} \\ +1 \\ r \end{bmatrix}.$$

and these two vectors lie at distance $\sqrt{2}/r$ from each other.

Let $\epsilon > 0$. We say that $u \in \mathfrak{S}$ is *ϵ-spacelike* if

$$\rho(\mathbf{x}^+(u), \mathbf{x}^-(u)) \geq \epsilon.$$

We denote the space of ϵ-spacelike vectors by $\mathfrak{S}_\epsilon$. If $\epsilon > \epsilon' > 0$, then $\mathfrak{S}_\epsilon \subset \mathfrak{S}_{\epsilon'}$. If $\epsilon > \sqrt{2}$ then $\mathfrak{S}_\epsilon$ is empty and if $\epsilon < \sqrt{2}$ then the above calculation shows that $\mathfrak{S}_\epsilon$ is a compact annulus and $\mathfrak{S}$ may be written as the increasing union:

$$\mathfrak{S} = \bigcup_{\sqrt{2} > \epsilon > 0} \mathfrak{S}_\epsilon$$

Lemma 1.1. *Let J^+ and J^- are disjoint closed nondegenerate intervals in the circle $\mathbb{P}$ of null directions. Then the set*

$$J = \{u \in \mathfrak{S} \mid \mathbf{x}^+(u) \in J^+, \quad \mathbf{x}^-(u) \in J^-\}$$

is a nondegenerate quadrilateral embedded in $\mathfrak{S}$ (whose edges are straight lines, segments of the rulings of $\mathfrak{S}$). which lies in an open half-space.

In particular the "dual" of J, the set of all $v \in \mathbb{R}^{2,1}$ such that $\mathbb{B}(v, u) > 0$ for all $u \in J$, is a compact convex cone in $\mathbb{R}^{2,1}$ having nonempty interior.

Proof. That J is a quadrilateral follows from the definition of $\mathbf{x}^+$ and $\mathbf{x}^-$ in terms of the rulings. For the other assertions, note that the "Lorentzian cross-product"

$$\boxtimes : \mathbb{R}^{2,1} \times \mathbb{R}^{2,1} \longrightarrow \mathbb{R}^{2,1}$$
$$\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \right) \mapsto \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_2 b_1 - a_1 b_2 \end{bmatrix}$$

is a bilinear mapping defined by:

$$\mathbb{B}(a \boxtimes b, c) = \det(a, b, c)$$

where $a \wedge b \wedge c \mapsto \det(a, b, c)$ is the isomorphism $\wedge^3 \mathbb{R}^{2,1} \longrightarrow \mathbb{R}$ corresponding to the fixed orientation of $\mathbb{R}^{2,1}$. Elementary consequences of the definition are:

- $\mathbb{B}(a, a \boxtimes b) = \mathbb{B}(b, a \boxtimes b) = 0$;
- $a \boxtimes b = -b \boxtimes a$;
- If $a, b \in \mathbb{P}$ are linearly independent, then $a \boxtimes b$ is spacelike, $\mathbf{x}^-(a \boxtimes b) = a$, and $\mathbf{x}^-(a \boxtimes b) = b$;
- $\mathbb{B}(a \boxtimes b, a \boxtimes b) = \mathbb{B}(a, a)\mathbb{B}(b, b) - \mathbb{B}(a, b)^2$

In terms of this operation, J can be described as the image $J^- \boxtimes J^+$. We can find two points which divide $\mathbb{P}$ into two open intervals I^- and

I^+ containing J^- and J^+ respectively and by applying an element of $\mathbf{G}$ may assume these intervals consist of vectors

$$\begin{bmatrix} \sin(\theta) \\ \cos(\theta) \\ 1 \end{bmatrix}$$

where $\sin(\theta) > 0$ and $\sin(\theta) < 0$ respectively. Now

$$\mathbb{B}\left(\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, I^- \boxtimes I^+ \right) > 0$$

whence $J = J^- \boxtimes J^+ \subset I^- \boxtimes I^+$ lies in a half-space. The proof of the lemma is complete. ☐

Lemma 1.2. *Let $u \in \mathfrak{S}$ be ϵ-spacelike. Then for any vector $v \in \mathbb{R}^{2,1}$,*

$$|\mathbb{B}(v, u)| \leq 2\epsilon^{-1}\|v\|$$

Proof. By applying R_θ we may assume that

$$u = \begin{bmatrix} r \\ 0 \\ \pm\sqrt{r^2 - 1} \end{bmatrix}$$

and since $u \in \mathfrak{S}_\epsilon$,

$$r \leq \frac{\sqrt{2}}{\epsilon}.$$

Now

$$\begin{aligned} |\mathbb{B}(v, u)| &= |rv_1 - \sqrt{r^2 - 1}\, v_3| \\ &\leq r|v_1 + v_3| \\ &\leq \frac{\sqrt{2}}{\epsilon}\sqrt{2}\|v\| \end{aligned}$$

as claimed. ☐

An element $g \in \mathbf{G}$ is *hyperbolic* if it has three distinct real eigenvalues; in this case the eigenvalues are positive and form the sequence

$$\lambda(g) < 1 < \lambda(g)^{-1}.$$

We denote the set of hyperbolic elements of $\mathbf{G}$ by $\mathcal{H}$. Associated to every hyperbolic element $g \in \mathcal{H}$ is its *characteristic frame* $\mathbf{x}(g)$ defined as follows. Let $\mathcal{A}^-(g)$ denote the $\lambda(g)$-eigenspace of g and $\mathcal{A}^+(g)$ denote the $\lambda(g)^{-1}$-eigenspace of g. Then there exists a unique $\mathbf{x}^0(g) \in \mathfrak{S}$ such that

$$\mathbf{x}^+(g) = \mathbf{x}^+(\mathbf{x}^0(g)),$$
$$\mathbf{x}^-(g) = \mathbf{x}^-(\mathbf{x}^0(g)).$$

In summary, given $g \in \mathcal{H}$, $\mathbf{x}(g) = \{\mathbf{x}^0(g), \mathbf{x}^-(g), \mathbf{x}^+(g)\}$ is the unique positively oriented basis of $\mathbb{R}^{2,1}$ such that:

- $g\mathbf{x}^0(g) = \mathbf{x}^0(g)$;
- $g\mathbf{x}^-(g) = \lambda(g)\mathbf{x}^-(g)$ and $\mathbf{x}^-(g) \in S^2 \cap \mathbb{W}^+$;
- $g\mathbf{x}^+(g) = \lambda(g)^{-1}\mathbf{x}^+(g)$ and $\mathbf{x}^+(g) \in S^2 \cap \mathbb{W}^+$.

Lemma 1.3. *Let $g \in \mathbf{G}, h \in \mathcal{H}$ and $\xi = \pm 1$. Then*

$$\mathbb{B}(g(\xi v), \mathbf{x}^0(h)) = \mathbb{B}(v, \mathbf{x}^0(g^{-1}h^\xi g)).$$

Proof. Since $\mathbf{x}^\pm(h^{-1}) = \mathbf{x}^\mp(h)$, it follows that $\mathbf{x}^0(h^\xi) = \xi\mathbf{x}^0(h)$. Furthermore uniqueness of $\mathbf{x}^\pm(h)$ implies $\mathbf{x}^+(g^{-1}hg) = g^{-1}\mathbf{x}^+(h)$ and $\mathbf{x}^-(g^{-1}hg) = g^{-1}\mathbf{x}^-(h)$. Thus $\mathbf{x}^0(g^{-1}hg) = g^{-1}\mathbf{x}^0(h)$ and

$$\begin{aligned}\mathbb{B}(g(\xi v), \mathbf{x}^0(h)) &= \xi\mathbb{B}(v, g^{-1}\mathbf{x}^0(h))\\ &= \mathbb{B}(v, \xi g^{-1}\mathbf{x}^0(h))\\ &= \mathbb{B}(v, \mathbf{x}^0(g^{-1}h^\xi g))\end{aligned}$$

as desired. □

Let $\epsilon > 0$. A hyperbolic element $g \in \mathcal{H}$ is said to be *ϵ-hyperbolic* if $\mathbf{x}^0(g)$ is ϵ-spacelike. Equivalently g is ϵ-hyperbolic if

$$\rho(\mathbf{x}^+(g), \mathbf{x}^-(g)) \geq \epsilon.$$

We denote the set of ϵ-hyperbolic elements by $\mathcal{H}_\epsilon$. For each $0 < \epsilon < \sqrt{2}$ the function

$$\lambda : \mathcal{H}_\epsilon \longrightarrow (0,1)$$

is a proper map and decomposes $\mathcal{H}_\epsilon$ into a fibration over the open interval with fiber a compact annulus. Furthermore

$$\mathcal{H} = \bigcup_{\sqrt{2}>\epsilon>0} \mathcal{H}_\epsilon.$$

Now let $h \in \mathbf{H}$ be an isometry of $\mathbb{E}$. We say that h is *hyperbolic* (respectively *ϵ-hyperbolic*) if its linear part $\mathbb{L}(h)$ is hyperbolic (respectively ϵ-hyperbolic). We shall write

$$\mathbf{x}^0(h) = \mathbf{x}^0(\mathbb{L}(h)),$$

$$\mathbf{x}^+(h) = \mathbf{x}^+(\mathbb{L}(h)),$$

$$\mathbf{x}^-(h) = \mathbf{x}^-(\mathbb{L}(h)).$$

If $h \in \mathbf{H}$ is hyperbolic and $u \in \mathbb{E}$ is arbitrary, then (following Margulis) consider the following real number:

$$\alpha(h, u) = \mathbb{B}(h(u) - u, \mathbf{x}^0(h))$$

The following facts are established in Margulis[3, 4]:

Lemma 1.4. *Let $h \in \mathbf{H}$ be hyperbolic. Then:*

- *$\alpha(h, u)$ is independent of u;*
- *There exists a line $\mathfrak{C}(h)$ invariant under h, parallel to $\mathbf{x}^0(h)$, such that the restriction of h is translation by $\alpha(h)\mathbf{x}^0(h)$;*
- *h acts freely on $\mathbb{E} \Leftrightarrow \alpha(h) \neq 0$, in which case $\mathfrak{C}(h)$ is the unique h-invariant line;*
- *If $n \in \mathbb{Z}$, $\alpha(h^n) = |n|\alpha(h)$.*

Since $\alpha(h, u)$ is independent of u, henceforth we write $\alpha(h) = \alpha(h, u)$.

Proof. Let $\lambda = \lambda(\mathbb{L}(h))$. Every $y \in \mathbb{R}^{2,1}$ may be decomposed in terms of the characteristic frame of h as follows:

$$\begin{aligned} y = &\ \mathbb{B}(y, \mathbf{x}^+(h))\mathbf{x}^+(h) \\ &+ \mathbb{B}(y, \mathbf{x}^-(h))\mathbf{x}^-(h) \\ &+ \mathbb{B}(y, \mathbf{x}^0(h))\mathbf{x}^0(h) \end{aligned}$$

Then

$$\begin{aligned} h(y) = &\ \mathbb{B}(\lambda y + h(0), \mathbf{x}^-(h))\mathbf{x}^-(h) \\ &+ \mathbb{B}(\lambda^{-1} y + h(0), \mathbf{x}^+(h))\mathbf{x}^+(h) \\ &+ \mathbb{B}(y + h(0), \mathbf{x}^0(h))\mathbf{x}^0(h) \end{aligned}$$

whence

$$\begin{aligned} \alpha(h, y) &= \mathbb{B}(h(y) - y, \mathbf{x}^0(h)) \\ &= \mathbb{B}(h(0), \mathbf{x}^0(h)) \\ &= \alpha(h, 0) \end{aligned}$$

is independent of y. Let

$$\begin{aligned} y_0 = &\ \frac{1}{1-\lambda}\mathbb{B}(h(0), \mathbf{x}^-(h))\mathbf{x}^-(h) \\ &+ \frac{1}{1-\lambda^{-1}}\mathbb{B}(h(0), \mathbf{x}^+(h))\mathbf{x}^+(h) \end{aligned}$$

then for $y = y_0 + t\mathbf{x}^0(h)$,

$$\begin{aligned} h(y) &= \left(\frac{\lambda}{1-\lambda} + 1\right) \mathbb{B}(h(0), \mathbf{x}^-(h))\mathbf{x}^-(h) \\ &\quad + \left(\frac{\lambda^{-1}}{1-\lambda^{-1}} + 1\right) \mathbb{B}(h(0), \mathbf{x}^+(h))\mathbf{x}^+(h) \\ &\quad + (t + \mathbb{B}(h(0), \mathbf{x}^0(h))\mathbf{x}^0(h) \\ &= y + \alpha(h)\mathbf{x}^0(h) \end{aligned}$$

Thus the line

$$\mathfrak{C}(h) = y_0 + \mathbb{R}\mathbf{x}^0(h)$$

is invariant under h, is parallel to $\mathbf{x}^0(h)$, and h acts on it by translation by $\alpha(h)\mathbf{x}^0(h)$. Furthermore h^n acts on $\mathfrak{C}(h)$ by translation by

$$n\alpha(h)\mathbf{x}^0(h);$$

since $\mathbf{x}^0(h^{-1}) = -\mathbf{x}^0(h)$ and $\mathbf{x}^0(h^n) = \mathbf{x}^0(h)$ for positive n, it follows that $\alpha(h^n) = |n|\alpha(h)$.

Suppose that h does not act freely. For any y fixed under h, $\alpha(h) = \alpha(h, y) = 0$. Conversely, if $\alpha(h) = 0$, then every point on $\mathfrak{C}(h)$ is fixed under h so that h does not act freely. To see that $\mathfrak{C}(h)$ is the unique h-invariant line if h acts freely, observe that any h-invariant line is parallel to one of the eigenvectors of the linear part $\mathbb{L}(h)$ and the restriction will be an affine transformation whose linear part is the corresponding eigenvalue. An affine transformation of a line which acts freely must have linear part equal to 1. Thus every h-invariant line must be parallel to $\mathbf{x}^0(h)$; however if there were more than one such line, there would be a plane upon which h acts by translations, contradicting the one-dimensionality of the 1-eigenspace of $\mathbb{L}(h)$. □

One can concretely see the dynamics of such a cyclic group as follows. After a change of coordinates, we may write

$$h = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (\lambda^{-1}+\lambda)/2 & (\lambda^{-1}-\lambda)/2 \\ 0 & (\lambda^{-1}-\lambda)/2 & (\lambda^{-1}+\lambda)/2 \end{bmatrix} \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix}$$

with characteristic frame

$$\mathbf{x}(h) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \right\}$$

and $\mathfrak{C}(h)$ is the x-axis. In particular if h acts freely, the quotient $\mathbb{E}/\langle h\rangle$ is a complete flat Lorentz 3-manifold diffeomorphic to a solid torus having $\mathfrak{C}(h)/\langle h\rangle$ as a core (spacelike) closed geodesic. Margulis' invariant $\alpha(h) = t$ represents a kind of "normalized signed Lorentzian length" of the core closed geodesic on the cylinder $\mathbb{E}/\langle h\rangle$.

We denote the free group of rank k by $\mathbb{F}_k$ and we denote the length of the word $w \in \mathbb{F}_k$ by $\ell(w)$.

Lemma 1.5 (Margulis)**.** *Suppose that $h_1, \dots, h_k \in \mathbf{H}$ satisfy the property that there exists $\epsilon > 0$, $g \in \mathbf{G}$, and a constant $C > 0$ such that for each $w \in \mathbb{F}_k$,*

- *$gw(h_1, \dots, h_k)$ is ϵ-hyperbolic;*
- *$|\alpha(gw(h_1, \dots, h_k))| \geq C\ell(w)$.*

Then the subgroup Γ of $\mathbf{G}$ generated by $h_1, \dots, h_k$ acts properly discontinuously on E.

Proof. It suffices to show that for each compact set $K \subset \mathbb{E}$ the set

$$\Gamma(K) = \{\gamma \in \Gamma \mid \gamma K_\cap K \neq \emptyset\}$$

is finite. If $w \in \mathbb{F}_k$, let $\gamma = w(h_1, \dots, h_k)$. Now for any $u \in \mathbb{E}$ and $w \in \mathbb{F}_k$,

$$\begin{aligned} C\ell(w) &\leq |\alpha(g\gamma)| \\ &= |\mathbb{B}(g\gamma(u) - u, \mathbf{x}^0(g\gamma))| \leq \frac{2}{\epsilon}\|g\gamma(u) - u\| \\ &= \frac{2}{\epsilon}\rho(u, g\gamma(u)) \end{aligned}$$

by Lemma 1.2 Let $K \subset \mathbb{E}$ be a compact set and suppose that $\gamma = w(h_1, \dots, h_k) \in \Gamma$ satisfies $K \cap \gamma^{-1}K \neq \emptyset$. Let $u \in K \cap \gamma^{-1}K$. Then $g\gamma(u) \in gK$ and $\rho(g\gamma(u), u) \leq \operatorname{diam}(K \cup gK)$, and by the previous assertion,

$$\begin{aligned} \ell(w) &\leq \frac{2}{\epsilon C}\rho(g\gamma(u), u) \\ &\leq \frac{2}{\epsilon C}\operatorname{diam}(K \cup gK). \end{aligned}$$

Therefore

$$\Gamma(K) \subset \left\{ w(h_1, \dots, h_k) \in \Gamma \;\middle|\; w \in \mathbb{F}_k,\ \ell(w) \leq \frac{2}{\epsilon C}\operatorname{diam}(K \cup gK) \right\}$$

is finite as desired. ∎

We observe that in general we cannot expect that $w(h_1, \dots, h_k)$ itself is ϵ-hyperbolic. Consider a sequence of words $\gamma_n = w_1^n w_2 w_1^{-n}$ where w_1 is hyperbolic; then both $\mathbf{x}^+(\gamma_n) = w_1^n x^+(w_2)$ and $\mathbf{x}^-(\gamma_n) = w_1^n x^-(w_2)$ converge to the same null vector $\mathbf{x}^+(w_1)$. For sufficiently large n, $\mathbf{x}^+(\gamma_n)$ and $\mathbf{x}^-(\gamma_n)$ will be arbitrarily close and γ_n will not be

ϵ-hyperbolic. The following discussion will illustrate the importance of considering the coset $g\Gamma$ rather than Γ itself. If γ corresponds to a reduced word it will be possible to determine the locations of the null eigenvectors $\mathbf{x}^{\pm}(\gamma)$ from its initial and terminal letters (Lemma 2.1). Although γ_n itself may correspond to a reduced word, its "rearrangement"

$$\gamma'_n = w_1^{n-1} w_2 w_1^{-n} w_1$$

will generally not be reduced if the initial and terminal letters of γ_n happen to be inverses.

2. Schottky groups in $\mathbf{SO}(2,1)$

Let $\mathbb{P}$ denote the set of rays in the positive nappe $\mathbb{W}^+$ of the light cone; $\mathbb{P}$ is diffeomorphic to a circle upon which the group $\mathbf{G}$ acts by projective transformations. We realize $\mathbb{P}$ concretely as the cross-section $S^2 \cap \mathbb{W}^+$; then the action of $g \in \mathbf{G}$ on $\mathbb{P}$ is given by

$$g : u \mapsto \frac{g(u)}{\|g(u)\|}$$

where $g \in \mathbf{G}$ and $u \in S^2 \cap \mathbb{W}^+$.

Choose $2k+2$ disjoint intervals $A_0^{\pm}, A_1^{\pm}, \dots, A_k^{\pm} \subset \mathbb{P}$, and elements $g = g_0, g_1, \dots, g_k \in \mathbf{G}$ such that

$$g_i(A_i^-) = \mathbb{P} - \bar{A}_i^+$$

and choose $\epsilon > 0$ less than the distance between any of the endpoints of these intervals. It follows that $g_i^{-1}(A_i^+) = \mathbb{P} - \bar{A}_i^-$ and the Brouwer fixed point theorem applied to the restriction of g_i to $\mathbb{P} - \bar{A}_i^-$ implies that

$$\mathbf{x}^+(g_i) \in A_i^+$$

(since $g_i(\mathbb{P} - \bar{A}_i^-) \subset A_i^+$). Similarly the Brouwer fixed point theorem applied to the restriction of g_i^{-1} to $\mathbb{P} - \bar{A}_i^+$ implies that

$$\mathbf{x}^-(g_i) \in A_i^-.$$

Let $w = w(x_0, \dots, x_k) \in \mathbb{F}_{k+1}$, then we write

$$w(x_0, \dots, x_k) = \prod_{i=1}^{l} x_{m_i}^{\xi_i}$$

where $\xi_i = \pm 1$ and $m_i = 0, 1, \dots, k$ for $i = 1, \dots, l = \ell(w)$. Then w is a reduced word if and only if

$$m_i = m_{i+1} \Longrightarrow \xi_i = \xi_{i+1}.$$

If $1 \leq j \leq l$, the *left-, right-subwords* are defined by

$$w_j^L(x_0, \dots, x_k) = \prod_{i=1}^{j-1} x_{m_i}^{\xi_i}$$
$$w_j^R(x_0, \dots, x_k) = \prod_{i=j+1}^{l} x_{m_i}^{\xi_i}$$

so that

$$w(x_0, \dots, x_k) = w_j^L(x_0, \dots, x_k) x_{m_j}^{\xi_j} w_j^R(x_0, x_1, \dots, x_k).$$

Lemma 2.1. *Let w be a reduced word as above. Then*

$$w(g_0, \dots, g_k)(\mathbb{P} - A_{m_l}^{-\xi_l}) \subset \bar{A}_{m_1}^{\xi_1}$$
$$w(g_0, \dots, g_k)^{-1}(\mathbb{P} - A_{m_1}^{\xi_1}) \subset \bar{A}_{m_l}^{-\xi_l}$$

and

$$\mathbf{x}^+(w(g_0, \dots, g_k)) \in \bar{A}_{m_1}^{\xi_1},$$
$$\mathbf{x}^-(w(g_0, \dots, g_k)) \in \bar{A}_{m_l}^{-\xi_l}.$$

Proof. Induction on l. If $l = 1$, then $w(g_0, \dots, g_k) = g_{m_1}^{\xi_1}$ and

$$w(g_0, \dots, g_k)(\mathbb{P} - A_{m_1}^{-\xi_1}) = \bar{A}_{m_1}^{\xi_1}$$
$$w(g_0, \dots, g_k)^{-1}(\mathbb{P} - A_{m_1}^{\xi_1}) = \bar{A}_{m_1}^{-\xi_1}$$

by hypothesis. Inductively we assume that

$$w_1^R(g_0, \dots, g_k)(\mathbb{P} - A_{m_l}^{-\xi_l}) \subset \bar{A}_{m_2}^{\xi_2}.$$

Now $\bar{A}_{m_2}^{\xi_2} \subset \mathbb{P} - A_{m_1}^{-\xi_1}$ (since w is a reduced word) and

$$g_{m_1}^{\xi_1}(\mathbb{P} - A_{m_1}^{-\xi_1}) = \bar{A}_{m_1}^{\xi_1}$$

imply that

$$\begin{aligned} w(g_0, \dots, g_k)(\mathbb{P} - A_{m_l}^{-\xi_l}) &= g_{m_1}^{\xi_1} w_1^R(g_0, g_1, \dots, g_k)(\mathbb{P} - A_{m_l}^{-\xi_l}) \\ &\subset g_{m_1}^{\xi_1} \bar{A}_{m_2}^{\xi_2} \\ &\subset g_{m_1}^{\xi_1}(\mathbb{P} - A_{m_1}^{-\xi_1}) \\ &= \bar{A}_{m_1}^{\xi_1} \end{aligned}$$

as desired. Replacing w by w^{-1} the second assertion follows. By applying the Brouwer fixed point theorem to the restriction of $w(g_0, \dots, g_k)$ to $\mathbb{P} - A_{m_l}^{-\xi_l}$ we see that $w(g_0, \dots, g_k)$ has a fixed point in $\mathbb{P} - A_{m_l}^{-\xi_l}$; this fixed point must lie in $w(g_0, \dots, g_k)(\mathbb{P} - A_{m_l}^{-\xi_l}) \subset A_{m_1}^{\xi_1}$. The final assertion also follows upon replacing w by w^{-1}. $\square$

Applying this lemma to a word of the form $x_0w(x_1,\dots,x_k)$ we obtain:

$$\begin{aligned} &\mathbf{x}^+(gw(g_1,\dots,g_k)) \in \bar{A}_0^+ \\ &\mathbf{x}^-(gw(g_1,\dots,g_k)) \in \bar{A}_{m_l}^{-\xi_l} \end{aligned} \tag{2.1}$$

and we conclude:

Corollary 2.2. *Let $w = w(x_1,\dots,x_k) \in \mathbb{F}_k$. Then $gw(g_1,\dots,g_k)$ is ϵ-hyperbolic.*

Lemma 2.3. *Let*

$$\delta_i = \frac{\xi_i - 1}{2} = \begin{cases} 0 & \text{if } \xi_j = +1 \\ -1 & \text{if } \xi_j = -1 \end{cases}$$

and write

$$h_i(x) = g_i(x) + v_i$$

for $v_i \in \mathbb{E}$. Then

$$h_i^{\xi_j}(x) = g_i^{\xi_j}(x) + \xi_j g_i^{\delta_j} v_i$$

and if $w = w(x_1,\dots,x_k) \in \mathbb{F}_k$, then

$$w(h_1,\dots,h_k)(0) = \sum_{j=1}^{l} w_j^L(g_1,\dots,g_k) g_{m_j}^{\delta_j}(\xi_j v_{m_j}).$$

Proof. The first assertion follows from

$$h_i^{-1}(x) = g_i^{-1}(x) - g_i^{-1} v_i$$

which is easily checked. The second assertion is proved inductively on l. If $l = 0$, the sum is empty and there is nothing to prove. Inductively we assume that

$$w_1^R(h_1,\dots,h_k)(0) = \sum_{j=2}^{l} \bar{w}_j^L(g_1,\dots,g_k) g_{m_j}^{\delta_j}(\xi_j v_{m_j}).$$

where

$$\bar{w}_j^L(x_1,\dots,x_k) = \prod_{i=2}^{j-1} x_{m_i}^{\xi_i}.$$

is the j-th left-subword of w_1^R. Now

$$\begin{aligned} w(h_1,\dots,h_k)(0) &= h_{m_1}^{\xi_1} w_1^R(h_1,\dots,h_k)(0) \\ &= g_{m_1}^{\xi_1}\Big(\sum_{j=2}^{l} \bar{w}_j^L(g_1,\dots,g_k) g_{m_j}^{\delta_j}(\xi_j v_{m_j})\Big) + \xi_1 g_{m_1}^{\delta_1}(v_{m_1}) \\ &= \sum_{j=2}^{l} w_j^L(g_1,\dots,g_k) g_{m_j}^{\delta_j}(\xi_j v_{m_j}) + \xi_1 g_{m_1}^{\delta_1}(v_{m_1}) \end{aligned}$$

as desired. □

3. Drumm's geometric conditions for proper discontinuity

Consider $g_0, g_1, \dots, g_k \in \mathbf{G}$ as above. Then the set defined by

$$\mathfrak{U}_i = \left\{ u \in \mathfrak{S} \,\middle|\, \mathbf{x}^+(u) \in \bar{A}_i^+,\ \mathbf{x}^-(u) \in \bar{A}_i^- \cup \bigcup_{j\neq i} \bar{A}_j^\pm \cup \bar{A}_0^\pm \right\}$$

is compact and is contained in an open half-space (Lemma 1.1). Thus there exists $v_i \in \mathbb{R}^{2,1}$ such that $\mathbb{B}(v_i, u) > 0$ for all $u \in \mathfrak{U}_i$. Furthermore (for this choice $(v_1, \dots, v_k)$) there exists $C > 0$ such that $\mathbb{B}(v_i, u) \geq C$ for $u \in \mathfrak{U}_i$. In particular if $f \in \mathcal{H}$ satisfies $\mathbf{x}^0(f) \in \mathfrak{U}_i$, then

$$\mathbb{B}(v_i, \mathbf{x}^0(f)) \geq C.$$

Letting $h_i(x) = g_i(x) + v_i$ as above, we have:

Theorem 3.1. *The group Γ generated by $h_1, \dots, h_k$ acts properly on $\mathbb{E}$.*

Proof. Let $\gamma = w(h_1, \dots, h_k)$ for $w \in \mathbb{F}_k$. We compute $\alpha(g\gamma)$ using Lemma 2.3:

$$\begin{aligned} \alpha(g\gamma) &= \mathbb{B}(g\gamma(0), \mathbf{x}^0(g\gamma)) \\ &= \sum_{j=1}^{l} \mathbb{B}(g w_j^L(g_1,\dots,g_k) g_{m_j}^{\delta_j}(\xi_j v_{m_j}), \mathbf{x}^0(g w(g_1,\dots,g_k))) \\ &= \sum_{j=1}^{l} \mathbb{B}(v_{m_j}, \mathbf{x}^0(f_j)) \end{aligned}$$

where

$$f_j = (g w_j^L(g_1,\dots,g_k) g_{m_j}^{\delta_j})^{-1} (g w(g_1,\dots,g_k))^{\xi_j} (g w_j^L(g_1,\dots,g_k) g_{m_j}^{\delta_j})$$

by Lemma 1.3. Now

$$\begin{aligned} f_j^{\xi_j} &= g_{m_j}^{-\delta_j} w_j^L(g_1,\dots,g_k)^{-1} g^{-1} g w(g_1,\dots,g_k) g w_j^L(g_1,\dots,g_k) g_{m_j}^{\delta_j} \\ &= g_{m_j}^{-\delta_j} g_{m_j}^{\xi_j} w_j^R(g_1,\dots,g_k) g w_j^L(g_1,\dots,g_k) g_{m_j}^{\delta_j} \end{aligned}$$

so

$$f_j = \begin{cases} g_{m_j} w_j^R(g_1,\dots,g_k) g w_j^L(g_1,\dots,g_k) & \text{if } \xi_j = 1 \\ g_{m_j} w_j^L(g_1,\dots,g_k)^{-1} g^{-1} w_j^R(g_1,\dots,g_k)^{-1} & \text{if } \xi_j = -1 \end{cases}$$

is the rearrangement of $(gw(g_1,\dots,g_k))^{\xi_j}$ with initial letter g_{m_j}.

In particular f_j is a reduced word since $w(x_1,\dots,x_k)$ and $w(x_1,\dots,x_k)^{-1}$ are reduced words, and by Lemma 2.1,

$$\mathbf{x}^+(f_j) \in A_{m_j}^+$$

and

$$\mathbf{x}^-(f_j) \in \begin{cases} A_{m_{j-1}}^{-\xi_{j-1}} & \text{if } \xi_j = 1 \text{ and } 1 < j < l \\ A_{m_{j+1}}^{\xi_{j+1}} & \text{if } \xi_j = -1 \text{ and } 1 < j < l \\ A_0^{-\xi_j} & \text{if } j = 1 \\ A_0^{\xi_j} & \text{if } j = l \end{cases}$$

again by Lemma 2.1. Thus $\mathbf{x}^0(f_j) \in \mathfrak{U}_j$ and

$$\begin{aligned} \alpha(g\gamma) &= \sum_{j=1}^{l} \mathbb{B}(v_{m_j}, \mathbf{x}^0(f_j)) \\ &\geq C\ell(w) \end{aligned}$$

establishing condition (2) of Lemma 1.5. The first condition of Lemma 1.5 follows from (2.1). Applying Lemma 1.5, Γ acts properly on $\mathbb{E}$ as desired. $\square$

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