This is a review text file submitted electronically to MR.

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Short title: Geometrical aspects of Schlesinger's equation

Control number: 1 484 592

Primary classification: 32G34

Secondary classification(s): 53C 81R25

Review text:

This very interesting paper interprets the classical differential equation of Schlesinger in terms of a projective structure on a complex surface and relates its solutions to the symplectic geometry of the moduli space of representations of the fundamental group of a quadruply-punctured sphere S in $\mathrm{SL}(2,\mathbb{C})$.

A projective structure on a manifold M can be defined as a projective equivalence class of affine connections. Two affine connections are said to be projectively equivalent if they possess the same unparametrized geodesics. Thus a projective structure gives a well-defined 1-dimensional foliation of the projectivized tangent bundle PTM. For a geodesically convex neighborhood $U \subset M$ the leaf space U^* is a manifold of dimension $2\dim(M) - 2$ and possesses an interesting geometry of its own.

In dimension two, É. Cartan proved that the geodesics of a projective structure are defined by a differential equation of the form

$$\frac{d^2y}{dx^2} = F\left(\frac{dy}{dx}\right)$$

where $F(u)$ is a cubic polynomial in u whose coefficients are functions of local coordinates (x, y).

The present paper considers the case when M is a complex surface and the projective structure is *holomorphic,* that is, the affine connections are holomorphic affine connections on M. In that case, the points of U determine lines in U^* of self-intersection number 1, generalizing the duality between points and lines in the projective plane. The geometry of U^* with this family of lines determines the projective structure on U.

A significant feature of this correspondence is that although U has only a local rigid geometry (a projective structure), the geometry of its dual U^* is locally trivial (U^* is just a complex manifold) but its global structure (the collection of lines) reflects the geometry of U.

This paper discusses an important example of this construction. Start with a connection on the trivial rank two trivial bundle over $\mathbb{C} \times V$ where $V \subset \mathbb{C}^3$ is an open subset. The connection is defined by a meromorphic matrix-valued 1-form

$$A = \sum_{k=1}^{3} A_k \frac{dz - dx_k}{z - x_k}$$

where $z \in \mathbb{C}$ and $(x_1, \ldots, x_3) \in V$ and $A_k : V \longrightarrow \mathfrak{gl}(2, \mathbb{C})$ is holomorphic. For fixed $(x_1, \ldots, x_3) \in V$, the 1-form A has simple poles at x_1, x_2, x_3 with residues $A_1(x)$, $A_2(x)$, $A_3(x)$ and ∞ with residue $A_4(x)$. By the vanishing of the sum of the residues of a meromorphic 1-form on a closed Riemann surface,

$$A_1 + A_2 + A_3 + A_4 = 0$$

Then *Schlesinger's equation* is the condition that the associated connection is flat: $dA + \frac{1}{2}[A, A] = 0$. Although this construction works in general, for any rank m and number of points n, one obtains a two-dimensional moduli space for $m = 2$ and $n = 3$.

To obtain a surface M with an affine connection (with a resulting projective structure), normalize the parameter $x \in V$ to $(x_1, \ldots, x_3) = (0, 1, t)$ by applying an affine transformation of $\mathbb{C}$. Then A may be regarded a connection on a holomorphic rank two vector bundle over $S = \mathrm{P}^1 - \{0, 1, t, \infty\}$. Schlesinger's equation reduces to the system

$$\begin{aligned}
\frac{dA_1}{dt} &= \frac{[A_3, A_1]}{t}, \\
\frac{dA_2}{dt} &= \frac{[A_3, A_2]}{t - 1}, \\
\frac{dA_3}{dt} &= \frac{[A_1, A_3]}{t} + \frac{[A_2, A_3]}{t - 1}, \\
\frac{dA_4}{dt} &= 0
\end{aligned}$$

By writing each of these equations in Lax form, one sees that each $A_k(t)$ (for $k = 1, 2, 3, 4$) lies in a single conjugacy class $\mathcal{O}_k \in \mathfrak{sl}(2, \mathbb{C})$. The moduli space M is thus determined by four adjoint orbits

$$\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_4 \in \mathrm{SL}(2, \mathbb{C})$$

and the two-dimensional moduli space M is the quotient of

$$\{(A_1, A_2, A_3, A_4) \in \mathcal{O}_1 \times \mathcal{O}_2 \times \mathcal{O}_3 \times \mathcal{O}_4 \mid A_1 + A_2 + A_3 + A_4 = 0\}$$

(which has dimension five) by $\mathrm{SL}(2, \mathbb{C})$. Schlesinger's equation then determines flows (varying with t) on M (assuming no A_k is zero). The functions

$$x = \mathrm{tr}(A_1 A_3)$$
$$y = \mathrm{tr}(A_2 A_3)$$

define local coordinates on M. By differentiating the above system, one obtains a system of equations on M equivalent to the geodesic equations for an affine connection on M. Thus (Proposition 1), Schlesinger's equation defines a projective structure on M.

This equation is equivalent to Painlevé's sixth equation and cannot be solved explicitly. However, U^* possesses a Poisson geometry which can be used to study the geometry resulting from Schlesinger's equation. The Wronskian of two solutions of the linearization of Cartan's equation for $y(x)$ (tangent vectors to U^*) is seen to be constant, from which one obtains an exterior 2-form on U^*. This exterior 2-form is singular along a divisor $D \subset U^*$ with three components. The paper explicitly determines this 2-form for the system arising from Schlesinger's equation.

For a solution $z(t)$ of Schlesinger's equation, the monodromy matrices $A_k(z_t)$ remain in a fixed conjugacy class, and hence solutions define *isomonodromic representations* of the fundamental group of a quadruply-punctured sphere. (A family of representations of $\pi_1(S)$ is said to be *isomonodromic* if for each boundary component $b_k \subset \partial S$ the representations of $\pi_1(b_k)$ remain in a single conjugacy class.) Furthermore the braid group $H(3)$ (which appears as a mapping class group of S) acts on the space $\mathcal{M}$ of representations whose restriction to each boundary component remains in a fixed conjugacy class C_k in $\mathrm{SL}(2, \mathbb{C})$. In this way the complement $U^* \backslash D$ identifies with a quotient of an open subset of $\mathcal{M}$ by $H(3)$.

For fixed conjugacy classes $C_k \in \mathrm{SL}(2, \mathbb{C})$, the space of equivalences classes of representations of $\pi_1(S)$ taking $\pi_1(b_k)$ to C_k has a natural symplectic structure. This structure induces a symplectic structure on $\mathcal{M}$, which is shown to be equivalent to the Poisson structure on U^* arising from the Wronskian constructed previously.

The paper concludes with some speculative remarks concerning degeneration. The projective structure on M gives, indirectly, an extension of the space of flat connections to a Poisson manifold U^* containing rational curves and the author proposes a more direct construction using flat connections.