BOOK REVIEW: AN EXTENSION OF CASSON'S INVARIANT BY KEVIN WALKER

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An *integral* (respectively *rational*) homology 3-sphere is a closed 3-manifold whose first integral (respectively rational) homology group vanishes. (We henceforth abbreviate these terms as $\mathbb{Z}$HS and $\mathbb{Q}$HS respectively.) These represent an important and slippery class of 3-manifolds — including, for example, those 3-manifolds for which Thurston's tantalizing *Geometrization Conjecture* has yet defied proof. Furthermore such 3-manifolds are as plentiful as knots: Given any knot $K \subset S^3$, the various *Dehn fillings* obtained by attaching to the complement $S^3 - K$ solid tori are all homology spheres. Before 1985 the principal invariant of a $\mathbb{Z}$HS Σ^3 was the $\mathbb{Z}/2$-valued *Rohlin invariant* $\mu(\Sigma^3)$, derived from the signature of a bounding 4-manifold). At that time it seemed plausible that the Rohlin invariant could detect a possible counterexample to the Poincaré conjecture.

In 1985 Andrew Casson found a powerful $\mathbb{Z}$-valued invariant $\lambda(\Sigma^3)$ of a $\mathbb{Z}$HS Σ^3, which laid these hopes to rest. From its existence and basic properties follow some remarkable results in geometric topology, such as: *there exist topological 4-manifolds which are not homeomorphic to simplicial complexes.* A closely related result concerns the Poincaré conjecture: Casson's invariant is defined in terms of the fundamental group and reduces modulo two to Rohlin's invariant. In particular Rohlin's invariant cannot detect a counterexample to the Poincaré conjecture.

(For an exposition of Casson's original results, presented in three lectures at MSRI in Spring 1985, we recommend the notes [[1]] of Akbulut and McCarthy. For other generalizations similar to Walker's invariant, see [4], [3], [2], [5], [7])

Shortly thereafter, Cliff Taubes found a highly suggestive analytic interpretation ([8]) of Casson's $\lambda(\Sigma^3)$. Let Σ^3 be an oriented $\mathbb{Z}$HS and consider the space $\mathcal{A}$ of all connections on a fixed (trivial) $\mathbf{SU}(2)$-bundle P over Σ^3. The group $\mathcal{G}$ of gauge transformations of P is isomorphic to the group of all maps from Σ^3 into $SO(3)$. Then the tangent space to $\mathcal{A}$ at A is the space $\Omega^1(\Sigma^3, \mathfrak{g})$ of 1-forms on Σ^3 taking values in

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the bundle of Lie algebras associated to P. The curvature $F(A)$ of A is an exterior 2-form with values in $\mathfrak{g}$, an element of $\Omega^2(\Sigma^3, \mathfrak{g})$. Using the invariant inner product on the Lie algebra of $\mathbf{SU}(2)$ and a volume form on Σ^3, the spaces $T_A\mathcal{A} = \Omega^1(\Sigma^3, \mathfrak{g})$ and $\Omega^2(\Sigma^3, \mathfrak{g})$ are canonically dual. Thus the operation assigning to a connection $A \in \mathcal{A}$ its curvature $F(A) \in \Omega^2(\Sigma^3, \mathfrak{g})$ is a 1-form Φ on $\mathcal{A}$. Furthermore $\mathcal{G}$ acts on $\mathcal{A}$ preserving Φ. The zeroes of Φ evidently correspond to flat connections. $\mathcal{G}$-orbits of flat connections correspond to equivalence classes of representations of the fundamental group into $\mathbf{SU}(2)$. Taubes defines a Poincaré-Hopf index of the 1-form Φ on $\mathcal{A}$ and shows that the resulting Euler characteristic equals Casson's invariant.

This invariant depends on the Fredholm structure of the moduli space of flat $SU(2)$-connections on Σ^3. The "instanton homology" (see [6]) invented by A.Floer is a homology theory associated to Σ^3 whose Euler characteristic equals $\lambda(\Sigma^3)$. However, the Floer homology has been rather difficult to compute except in the simplest examples, while $\lambda(\Sigma^3)$ can be computed simply in terms of a surgery formula discovered by Casson.

Given the power of this invariant, it was natural to try to extend it in various directions. The book under review (essentially the author's Berkeley dissertation) describes an extension of $\lambda(\Sigma^3)$ to the case when Σ^3 is a $\mathbb{Q}$HS. If $K \subset S^3$ is a knot, then its "complement" is a compact 3-manifold N^3 with boundary a 2-torus (whose interior is homeomorphic to $S^3 - K$). One obtains a rational homology sphere $N_{p/q}$ by attaching to M^3 a solid torus $T \approx S^1 \times D^2$ by glueing the meridian $(\{s\} \times \partial D^2$ of T to a (p, q)-curve on ∂N^3. (A (p, q)-curve on ∂N^3 is defined as any curve in the homology class corresponding to $(p, q) \in \mathbb{Z} \oplus \mathbb{Z} \cong H_1(\partial N)$ which is the linear combination: p times a generator $(1, 0)$ of the image of

$$\mathbb{Z} \cong H_1(N) \hookrightarrow H_1(\partial N)$$

plus q times an element $(0, 1)$ projecting to a generator under

$$H_1(\partial N) \longrightarrow H_1(\partial N)/H_1(N) \cong \mathbb{Z}$$

In particular $N_{1/q}$ is a $\mathbb{Z}$HS for each $q \in \mathbb{Z}$ while *every* $N_{p/q}$ is a $\mathbb{Q}$HS.

Casson's original definition and Walker's generalization involve both the moduli space $\mathbf{Hom}(\pi_1(M), \mathbf{SU}(2))/\mathbf{SU}(2)$ of equivalence classes of representations of the fundamental group $\pi_1(M)$ into $\mathbf{SU}(2)$. The representations themselves form a real algebraic set upon which the inner automorphisms of $\mathbf{SU}(2)$ act algebraically. Accordingly the moduli space is a real analytic space, and stratifies into analytic submanifolds. Roughly, Casson's invariant counts — *algebraically* — the points in $\mathbf{Hom}(\pi_1(M), \mathbf{SU}(2))/\mathbf{SU}(2)$ where M^3 is a closed oriented $\mathbb{Z}$HS.

The actual counting is a bit more involved. Following Casson, one decomposes the 3-manifold Σ^3 as a union of two solid handlebodies $W_1 \cup_F W_2$ of genus $g \geq 1$ along a closed surface F and computes the space $\mathbf{Hom}(\pi_1(\Sigma^3), \mathbf{SU}(2))/\mathbf{SU}(2)$ in terms of this

Heegaard decomposition. The space of interest $\mathbf{Hom}(\pi_1(\Sigma^3), \mathbf{SU}(2))/\mathbf{SU}(2)$ is embedded in the larger space $\mathbf{Hom}(\pi_1(F), \mathbf{SU}(2))/\mathbf{SU}(2)$ corresponding to the surface $F \subset \Sigma^3$ under the natural restriction map, but there is considerably more space (and geometry) in $\mathbf{Hom}(\pi_1(F), \mathbf{SU}(2))/\mathbf{SU}(2)$ to study $\mathbf{Hom}(\pi_1(\Sigma^3), \mathbf{SU}(2))/\mathbf{SU}(2)$. Let $X = \mathbf{Hom}(\pi_1(F), \mathbf{SU}(2))/\mathbf{SU}(2)$. Then the space of equiavalence class of representations

$$\mathbf{Hom}(\pi_1(\Sigma^3), \mathbf{SU}(2))/\mathbf{SU}(2) \subset X$$

(alternately, the space of gauge-equivalence classes of flat connections) is the intersection of the images Q_i of

$$\mathbf{Hom}(\pi_1(W_i), \mathbf{SU}(2))/\mathbf{SU}(2) \hookrightarrow X$$

The orientation of Σ^3 induces orientations of $W_1, W_2, F \subset \Sigma^3$, which in turn induce orientations on the various spaces $Q_1, Q_2 \subset X$ and Casson's invariant $\lambda(\Sigma^3)$ is a kind of oriented intersection number $Q_1 \cdot Q_2$.

To develop intuition for the nature of this invariant, consider the following simple analogue. Replace $\mathbf{SU}(2)$ by the circle group $\mathbf{U}(1)$. If Σ^3 is a $\mathbb{Z}$HS, every representation $\pi_1(\Sigma^3) \longrightarrow U(1)$ is trivial, but for a more general $\mathbb{Q}$HS, (say one with cyclic fundamental group $\mathbb{Z}_m$), nontrivial representations $\pi_1(\Sigma^3) \longrightarrow U(1)$ will exist. It suffices to consider the case that Σ^3 is a *lens space:* glue two solid tori W_1 and W_2 along a common bounding 2-torus T. The inclusions of the boundary $T \longrightarrow \partial W_j \subset W_j$ induce homomorphisms

$$\begin{aligned} \mathbb{Z} \oplus \mathbb{Z} \cong \pi_1(T) &\longrightarrow \pi_1(W_j) \cong \mathbb{Z} \\ (x, y) &\longmapsto a_j x + b_j y \end{aligned}$$

where a_j and b_j are relatively prime integers. Then the analogue X' of X is the torus $\mathbf{Hom}(\pi_1(T), \mathbf{U}(1))$ dual to $\pi_1(T)$. Explicitly X' consists of representations of the form

$$\begin{aligned} \mathbb{Z} \oplus \mathbb{Z} &\longrightarrow \mathbf{U}(1) \\ (x, y) &\longmapsto e^{i(\alpha x + \beta y)} \end{aligned}$$

where $\alpha, \beta \in \mathbb{R}$. The analogues of the Q_j are the circles Q'_j (one-parameter subgroups with respect to the structure of X as a group) defined by

$$\alpha = t a_j, \qquad \beta = t b_j$$

All of the intersections of Q'_1 and Q'_2 are transverse and have the same oriented intersection number. The cardinality of $Q'_1 \cap Q'_2$ is the determinant $m = a_1 b_2 - a_2 b_1$, the order of $\pi_1(\Sigma^3)$. Thus $\#\pi_1(\Sigma^3)$ may be regarded as a "abelian" analogue of Casson's invariant.

The book under review is a monograph describing Walker's extension of Casson's invariant to $\mathbb{Q}$HS. This was Walker's 1989 Berkeley doctoral thesis and a summary of this work may be found in his research announcement [9]. The idea of Walker's generalization is similar to Casson's definition: sum local expressions over the intersections of Q_1 and Q_2 in X, after perhaps making a generic perturbation to make them transverse. However, the presence of singularities in these spaces produces significant technical problems. Using the symplectic geometry of X (in which the Q_j are Lagrangian), Walker defines local invariants which give a meaningful invariant. §1 is background on the moduli spaces of representations and its symplectic geometry. §2 contains the definition of $\lambda(\Sigma^3)$ and §3 develops its elementary properties. Many of these were proved by Casson in $\mathbb{Z}$HS case. §4 proves the surgery formula: how $\lambda(\Sigma^3)$ behaves when surgery is performed on a knot in a $\mathbb{Q}$HS. Conversely, uniqueness of Walker's invariant is proved in §5: any function of rational homology spheres vanishing on S^3 and satisfying the surgery formula must equal $\lambda(\Sigma^3)$. §6 contains various consequences of this theory. For example Walker's λ is additive under connected sum and relates to the Rohlin invariant of a spin $\mathbb{Q}$HS. Two appendices summarize results concerning Dedekind sums and Alexander polynomials which are needed in the main body of the monograph.

This is a fascinating subject and Walker's book is informative and well written. Together with [1], it makes a rather pleasant introduction to a very active area in geometric topology.

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