A Lagrangian Central Scheme for Multi-Fluid Flows*

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Abstract

We develop a central scheme for multi-fluid flows in Lagrangian coordinates. The main contribution is the derivation of a special equation of state to be imposed at the interface in order to avoid non-physical oscillations. The proposed scheme is validated by solving several tests concerning one-dimensional hyperbolic interface problems.

1 Introduction.

We study a high-resolution central scheme for the system of conservation laws describing two gases separated by an interface. We consider a piston problem as described by Fazio and LeVeque [1]: a tube contains two different gases separated by an interface at some point $I(t) < L(t)$ where $I(t)$ and $L(t)$ are interface and piston position. The governing equations are Euler equations of gas dynamics,

$$\frac{\partial \mathbf{q}}{\partial t} + \frac{\partial}{\partial x}\left[\mathbf{f}(\mathbf{q})\right] = \mathbf{0} \ , \tag{1}$$

with

$$\begin{aligned} \mathbf{q} &= [\rho, \rho u, E]^T \\ \mathbf{f}(\mathbf{q}) &= [\rho u, \rho u^2 + p, (E+p)u]^T \ , \end{aligned} \tag{2}$$

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and with the constitutive law for ideal gases

$$p = (\gamma(x,t) - 1)\left(E - \frac{1}{2}\rho u^2\right) , \tag{3}$$

where ρ, u, E, and p denote density, velocity, total energy density per unit volume, and pressure of the gas. The polytropic constant $\gamma(x,t)$ takes the value γ_1 on $0 \leq x < I(t)$, and γ_2 on $I(t) < x \leq L(t)$. The motion of the piston is driven by Newton's equation

$$\frac{d^2L}{dt^2} = \frac{A}{m}(p(L(t),t) - p_{out}(t)) , \tag{4}$$

where A is the area of the piston, m is its mass and $p_{out}(t)$ is the external pressure.

Note that by setting $A = 0$ we resume, from the piston model above, the classical tube problem with an interface and fixed boundaries.

2 Lagrangian formulation.

By introducing the Lagrangian coordinate ξ given by

$$\xi = \int_{x_0(t)}^{x} \rho(z,t)dz ,$$

where $x_0(t)$ denotes the Eulerian coordinate of a given fluid particle, the Euler equations (1)-(2) can be transformed in Lagrangian form

$$\frac{D\mathbf{q}}{Dt} + \frac{\partial}{\partial \xi}[\mathbf{f}(\mathbf{q})] = \mathbf{0} , \tag{5}$$

which is also in conservation form with

$$\begin{aligned} \mathbf{q} &= [V, u, \mathcal{E}]^T , \\ \mathbf{f}(\mathbf{q}) &= [-u, p, up]^T , \end{aligned} \tag{6}$$

here the time derivative is the Lagrangian derivative

$$D/Dt = \partial/\partial t + u\partial/\partial x ,$$

the new field variables are defined by $V(\xi,t) = \rho^{-1}, \mathcal{E} = E/\rho$, and the equation of state (3) becomes

$$p = (\gamma(\xi,t) - 1)\left(\mathcal{E} - \frac{1}{2}u^2\right)/V . \tag{7}$$

The inverse transformation of coordinate is

$$x = x_0(t) + \int_0^\xi V(z,t)dz \ ,$$

where $x_0(t)$ denotes the Eulerian coordinate of the first particle of the domain, and satisfies the equation

$$\frac{d}{dt}x_0 = u(\xi = 0, t) \ .$$

Hence, $0 \leq \xi \leq \xi_{\max}$ will be our "computational domain" in which we have a fixed uniform grid with $\xi_i = (i - 1/2)\Delta\xi$ for $i = 1, 2, \ldots, N$ denoting the center of i-th cell, and $\Delta\xi = \xi_{\max}/N$.

3 The Nessyahu and Tadmor central scheme.

The Nessyahu and Tadmor central scheme [6] has the form of a predictor-corrector scheme

$$\begin{aligned}
\mathbf{q}_j^{n+1/2} &= \mathbf{q}_j^n - \frac{\lambda}{2}\mathbf{f}_j' \ , \\
\mathbf{q}_{j+1/2}^{n+1} &= \frac{1}{2}(\mathbf{q}_j^n + \mathbf{q}_{j+1}^n) + \frac{1}{8}(\mathbf{q}_j' - \mathbf{q}_{j+1}') - \lambda\left(\mathbf{f}(\mathbf{q}_{j+1}^{n+1/2}) - \mathbf{f}(\mathbf{q}_j^{n+1/2})\right) \ ,
\end{aligned}$$

where $\lambda = \Delta t/\Delta\xi$. The time step Δt must satisfy the stability condition

$$\lambda \max_j \rho(A(u(\xi_j, t)) < \frac{1}{2}$$

where $\rho(A(u(\xi_j, t))$ denotes the spectral radius of the Jacobian matrix

$$A = \left[\frac{\partial f_i}{\partial q_k}\right] \ .$$

This condition ensures that the generalized Riemann problems with piece-wise smooth data at time t_n do not interact during the time step Δt.

$\mathbf{q}_j'/\Delta\xi$ and $\mathbf{f}_j'/\Delta\xi$ are a first order approximation of the space derivatives of the field and of the flux and can be computed in several ways. The simplest choice is

$$\begin{aligned}
\mathbf{q}_j' &= \mathrm{MM}(\mathbf{q}_{j+1} - \mathbf{q}_j, \mathbf{q}_j - \mathbf{q}_{j-1}), \\
\mathbf{f}_j' &= \mathrm{MM}(\mathbf{f}_{j+1} - \mathbf{f}_j, \mathbf{f}_j - \mathbf{f}_{j-1}),
\end{aligned}$$

where $\mathrm{MM}(v, w)$ is the min-mod limiter

$$\mathrm{MM}(v,w) = \begin{cases} \mathrm{sgn}(v)\cdot\min(|v|,|w|) & \text{if } \mathrm{sgn}(v) = \mathrm{sgn}(w) \\ 0 & \text{otherwise} \ . \end{cases}$$

Several other choices of the limiters are possible, as discussed in [6].

4 Balancing the pressure at the interface.

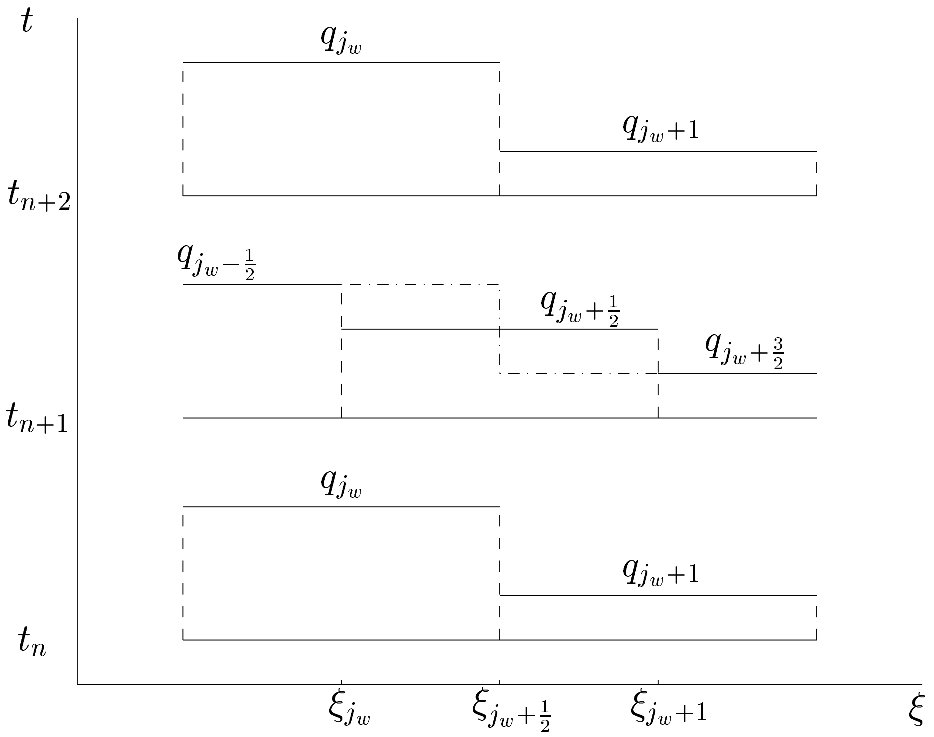

Figure 1: Evolution of field quantities at the interface for a steady solution.

The development of this section follows the simple idea that due to the presence of the interface several quantities (such as density and energy) may be discontinuous, but pressure and velocity have to be continuous across the interface.

At the initial time we assume that the interface is located at the boundary between cell j_w and cell j_w+1. Because of the use of Lagrangian coordinates, the interface will always separate cell j_w from cell j_w+1 at even time steps, and it will be in the middle of cell $j_w+1/2$ at odd time steps (see Figure 1).

Let us denote by subscript 1 and 2 the values of the field variables on the two sides of the interface cell at odd time steps. The balance of the pressure on the two sides of the interface, $p_1 = p_2$, gives

$$\frac{\gamma_1 - 1}{V_1}(\mathcal{E}_1 - \frac{1}{2}u^2) = \frac{\gamma_2 - 1}{V_2}(\mathcal{E}_2 - \frac{1}{2}u^2) \ . \tag{8}$$

We assume that on the two sides of the interface there are two different gases, whose ratio η of densities is equal to the ratio of molecular masses:

$$V_2 = \eta V_1 \tag{9}$$

The interface values of specific volume and energy are

$$V = \frac{V_1 + V_2}{2}, \quad \mathcal{E} = \frac{\mathcal{E}_1 + \mathcal{E}_2}{2} . \tag{10}$$

Here V and $\mathcal{E}$ denote the cell average of specific volume and energy of interface cell $j_w + 1/2$. Making use of above relations (8, 9, 10) we get the following formula for the pressure $p = p_1 = p_2$ in terms of the field variables at the interface

$$p(V, u, \mathcal{E}) = \frac{(1+\eta)(\gamma_1 - 1)(\gamma_2 - 1)}{\eta\gamma_1 + \gamma_2 - 1 - \eta} \left(\mathcal{E} - \frac{1}{2}u^2\right) /V , \tag{11}$$

Note that by setting $\eta = 1$ and $\gamma_1 = \gamma_2 = \gamma$ we recover from (11) the classical equation of state for a polytropic gas. Once the interface quantities are known, the values of the field variables on the two sides can be computed

$$V_1 = \frac{2}{1+\eta}V, \quad \mathcal{E}_1 = \frac{p(V,u,\mathcal{E})V_1}{\gamma_1 - 1} + \frac{1}{2}u^2$$
$$V_2 = \frac{2\eta}{1+\eta}V, \quad \mathcal{E}_2 = \frac{p(V,u,\mathcal{E})V_2}{\gamma_2 - 1} + \frac{1}{2}u^2 .$$

Figure 1 shows the evolution of field variables on two time steps for a steady solution. Note how the scheme maintains static solutions.

5 Numerical tests

In this section we report several numerical tests we performed in oder to validate the Lagrangian scheme.

5.1 Adiabatic approximation.

In this subsection we consider the adiabatic approximation for the piston problem of Section 1. In the adiabatic approximation the pressures of the two gases are equal and depend only on time. Furthermore, the entropies of both gases are constants, equal to their initial values. The above conditions alone are enough to describe the motion of the interface and of the piston by a single ordinary differential equation of second order.

The above hypotheses are verified when the piston motion is slow compared with the sound speed and this is the case for a sufficiently high piston mass. In the case of polytropic gases, the ODE for the interface can be written explicitly

$$\frac{d^2 I}{dt^2}(1 + c_1 \varrho I^{\varrho-1}) + c_1 \varrho(\varrho - 1) I^{\varrho-2} \left(\frac{dI}{dt}\right)^2 = \frac{A}{m}(c_2 I^{-\gamma_1} - p_{out}(t)) \ , \quad (12)$$

with initial conditions $I(0) = I_0$ and $dI/dt(0) = v_0$ and

$$c_1 = (L(0) - I(0)) \left(\frac{1}{I(0)}\right)^{\varrho} ,$$
$$\varrho = \gamma_1/\gamma_2 ,$$
$$c_2 = p_0 [I(0)]^{\gamma_1} \ .$$

p_0 is the initial pressures on both sides of the interface. The piston position can be obtained from the interface position via the relation

$$L(t) = (L(0) - I(0)) \left(\frac{I(t)}{I(0)}\right)^{\varrho} + I(t) \ .$$

We solve Eq. (12) with the following initial conditions

$$I(0) = 0.3 \ , \quad L(0) = 1 \ , \quad \frac{dL}{dt}(0) = 0 \ ,$$

and then we solve Eq. (5) with the corresponding initial and boundary conditions

$$\begin{cases} \mathbf{q}_1(x,0) = (\rho_1, u_1, p_1)^T = (1.0, 0.0, 1.0)^T & 0 \le x \le I(0) \\ \mathbf{q}_2(x,0) = (\rho_2, u_2, p_2)^T = (1.0, 0.0, 1.0)^T & I(0) \le x \le L(0) \ , \end{cases}$$

The position of the interface is computed from the numerical solution by solving the equation

$$\frac{dI}{dt}(t) = u_1(I(t), t) = u_2(I(t), t) \ .$$

The remaining data are $A/m = 0.01, p_{out}(t) = 2, \gamma_1 = 1.4, \gamma_2 = 2.8$.

Figure 2 shows the good agreement between the adiabatic approximation and the numerical results obtained by setting $\Delta\xi = 0.005$.

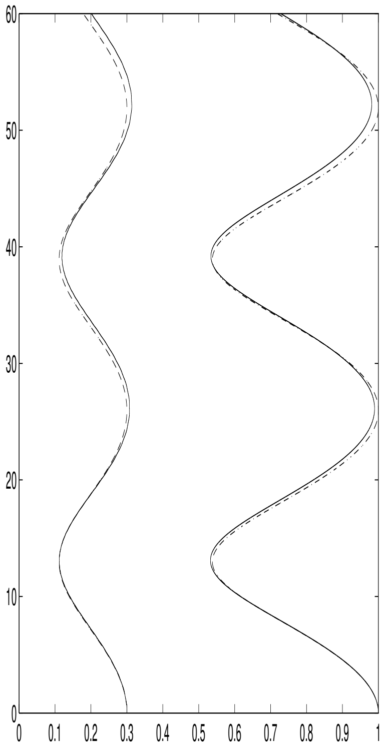

Figure 2: Adiabatic approximation: interface and piston positions in the $x - t$ plane, dashed line: adiabatic solution, solid line: numerical solution with 200 mesh-cells.

5.2 S. Karni test problems.

The five tests described here were proposed by S. Karni in order to test her hybrid approach to multi-fluid flows [5]. Moreover, variations of those tests were used by Jenny, Müller and Thomann [4] and by Fedkiw et al. [2].

Test A is a classical tube problem with the interface initially at $x = 0.5$ and the following initial conditions

$$\begin{aligned} \mathbf{q}_1 &= (\rho_1, u_1, p_1)^T = (1.0, 0.0, 1.0)^T \qquad 0 \le x < 0.5 \\ \mathbf{q}_2 &= (\rho_2, u_2, p_2)^T = (0.125, 0.0, 0.1)^T \qquad 0.5 < x \le 1. \end{aligned}$$

with $\gamma_1 = 1.4$ and $\gamma_2 = 1.2$. Figure 3 illustrates the numerical results.

The physical setup for the other tests is to consider the interaction of a weak (strong) shock at Mach number of 1.1952 (3.6055) moving from left to right in air with an air/helium (Tests B and D, Case 1) or air/R22 (Tests C and D, Case 2) interface. Inflow (outflow) boundary conditions at the left (right) boundary were applied to all tests in this subsection.

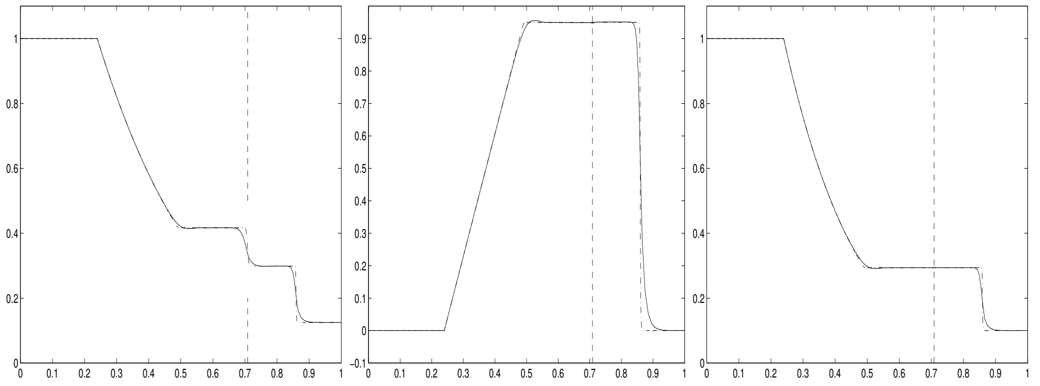

Figure 3: S. Karni's Test A. Left: density. Center: velocity. Right: pressure.

The initial conditions for Test B are as follows

$$
\begin{aligned}
\mathbf{q}_1 &= (\rho_1, u_1, p_1)^T = \begin{cases} (1.3333, 0.3535, 1.5)^T & \text{post-shock, air} \quad 0 \le x < 0.3 \\ (1.0, 0.0, 1.0)^T & \text{pre-shock, air} \quad 0.3 < x < 0.5 \end{cases} \\
\mathbf{q}_2 &= (\rho_2, u_2, p_2)^T = (0.1379, 0.0, 1.0)^T \qquad \text{pre-shock, helium} \quad 0.5 < x \le 1.
\end{aligned}
$$

with $\gamma_1 = 1.4$ and $\gamma_2 = 1.67$. The initial conditions for Test C are given by $\mathbf{q}_1$ as in Test B and

$$
\mathbf{q}_2 = (\rho_2, u_2, p_2)^T = (3.1538, 0.0, 1.0)^T \qquad \text{pre-shock, R22} \quad 0.5 < x \le 1.
$$

with $\gamma_1 = 1.4$ and $\gamma_2 = 1.249$. The Figures 4 and 5 illustrate the numerical results.

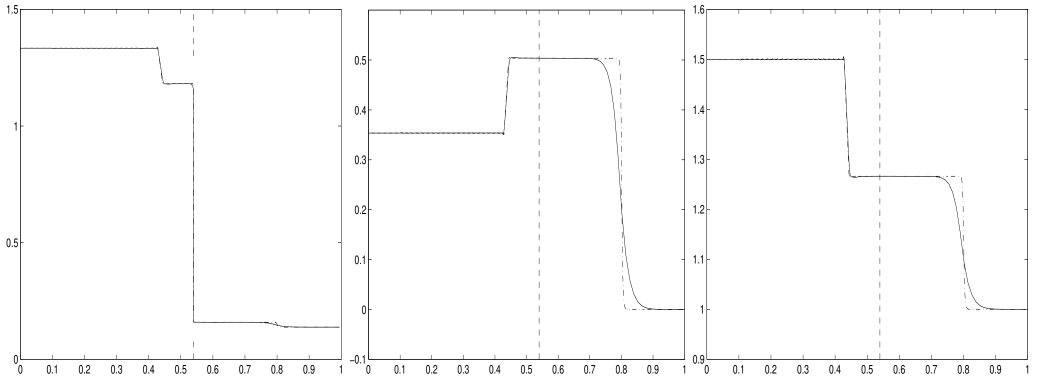

Figure 4: S. Karni's Test B. Left: density. Center: velocity. Right: pressure.

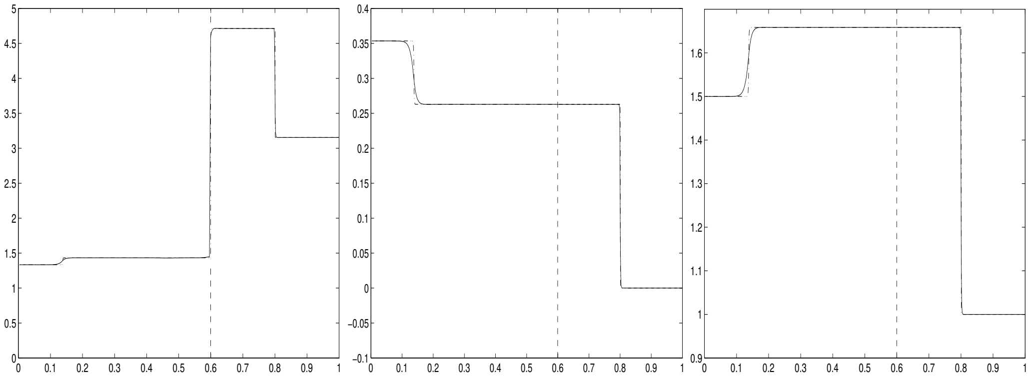

Figure 5: S. Karni's Test C. Left: density. Center: velocity. Right: pressure.

Within the Figures 3, 4 and 5 the solid lines and dashed-dotted lines are obtained with 400 and 4000 mesh-cells, respectively, and a dashed line indicates the position of the interface.

Tests D, Case 1 and Test D, Case 2 differ from Tests B and C, respectively, because the post-shock state is (in both cases) given by

$$\mathbf{q}_1 \quad = \quad (\rho_1, u_1, p_1) = (4.3333, 3.2817, 15.0) \qquad \text{post-shock, air} \quad 0 \leq x < 0.3 \; .$$

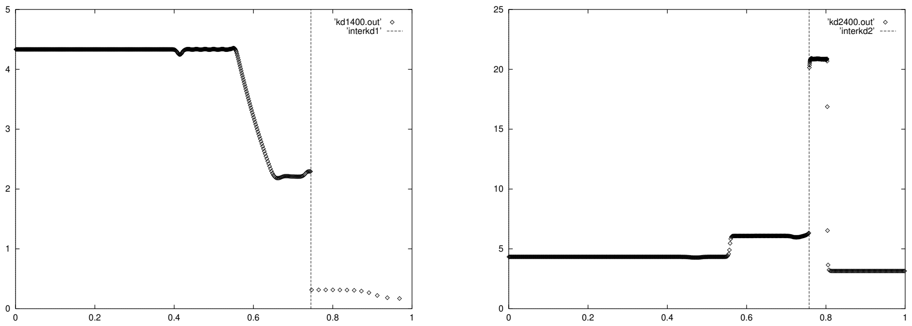

Figure 6: S. Karni's Density for Test D. Left: Case 1. Right: Case 2. The interface is marked by a dashed line.

Figure 6 shows the numerical results obtained for the density with 400 mesh-cells for the Test D, Case 1 and Test D, Case 2. The spacing of grid points is uniform in Lagrangian coordinates, but in the physical domain it is determined by the density values. Note how, by setting a constant cell-width $\Delta\xi$, we get only 13 nodes on the right of the interface on the left frame of Figure 6. A comparison can be made with the number of cells in the case of the right frame of Figure 6. Moreover, the over-shoot (under-shoot) on the left of the interface on the left frame of Figure 6 (right frame of the same Figure) is due to the inability of the central scheme to resolve sharply a contact discontinuity. Several options are available to deal with the mentioned inaccuracy. Let us mention here the possibility of using Harten Artificial compression [3] or applying central scheme on all but the contact field so that only the contact wave is separated but the other waves are treated component-wise [7]. Another possibility is to impose that the interface is adiabatic, i.e. that no heat flows through it (and therefore the entropy is locally flat). Moreover, work to improve the performance of central schemes at contact discontinuities is in progress. The above possibilities are subject of present investigation.

Note that, in spite of the abovementioned difficulty and the lack of resolution on the low density regions, the scheme provides an accurate position of the waves.

References

[1] R. Fazio and R. J. LeVeque. Moving-mesh methods for one-dimensional hyperbolic problems using CLAWPACK. Preprint on Conservation Laws n. 1998-020: available at the URL: http://www.math.ntnu.no/conservation/1998/020.html, 1998.

[2] R. P. Fedkiw, T. Aslam, B. Merriman, and S. Osher. A non-oscillatory eulerian approach to interfaces in multimaterial flows (the ghost fluid method). *J. Comput. Phys.*, 152:457–492, 1999.

[3] A. Harten. The artificial compression method for computation of shocks and contact discontinuities. i. single conservation laws. *Comm. Pure Appl. Math.*, 30:611–638, 1977.

[4] P. Jenny, B. Müller, and H. Thomann. Correction of conservative euler solvers for gas mixture. *J. Comput. Phys.*, 132:91–107, 1997.

[5] S. Karni. Hybrid multifluid algorithms. *SIAM J. Sci. Comput.*, 17:1019–1039, 1996.

[6] H. Nessyahu and E. Tadmor. Non-oscillatory central differencing for hyperbolic conservation laws. *J. Comput. Phys.*, 87:408–463, 1990.

[7] E. Tadmor, April 2000. Private Communication.