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8. Second-Order Equations Reducible to First-Order Ones

8.1. **Reducible Second-Order Equations.** The normal form for second-order equations is

$$x'' = g(t, x, x')\,, \quad \text{or} \quad \ddot{x} = g(t, x, \dot{x})\,, \quad \text{or} \quad \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = g\left(t, x, \frac{\mathrm{d}x}{\mathrm{d}t}\right)\,. \tag{8.1}$$

The solution of such an equation cannot generally be analyzed using first-order methods. Here we study two special cases for which this can be done.

- When the dependent variable x is missing then (8.1) becomes

$$x'' = g(t, x')\,, \quad \text{or} \quad \ddot{x} = g(t, \dot{x})\,, \quad \text{or} \quad \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = g\left(t, \frac{\mathrm{d}x}{\mathrm{d}t}\right)\,. \tag{8.2a}$$

  In this case we can use the explicit reduction method, which seeks a reduction to an explicit equation.

- When the independent variable t is missing, then (8.1) becomes

$$x'' = g(x, x')\,, \quad \text{or} \quad \ddot{x} = g(x, \dot{x})\,, \quad \text{or} \quad \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = g\left(x, \frac{\mathrm{d}x}{\mathrm{d}t}\right)\,. \tag{8.2b}$$

  In this case we can use the autonomous reduction method, which seeks a reduction to an autonomous equation.

In each case the problem of solving the second-order equation is reduced to that of solving two first-order equations. Even when these first-order equations cannot both be solved, they often can be analyzed to gain insights into solutions of the second-order equation. We will show how to do this in the subsequent sections. We will apply the methods to equations of motion.

Remark. The autonomous case arises in most motion applications. The motion of an object of mass m being acted upon by a force F is governed by Newton's Law $ma = F$ where the acceleration a of the object is the second derivative of its position with respect to time and the force F can depend upon its position and velocity but often does not depend explicitly upon time.

8.2. **Explicit Reduction Method.** Second-order equations missing the dependent variable x have the normal form (8.2a), which is

$$x'' = g(t, x')\,. \tag{8.3}$$

Of course, other variables might be missing from the right-hand side too. The explicit reduction method reduces the problem of solving this second-order equation to that of solving an explicit first-order equation

$$x' = U(t)\,. \tag{8.4}$$

8.2.1. *Auxiliary and Reduced Equations.* For a solution of (8.4) to also be a solution of (8.3), the function $U(t)$ must be chosen with care. Because

$$x'' = U'(t)\,,$$

we see that $u = U(t)$ must satisfy the first-order equation

$$u' = U'(t) = x'' = g(t, x') = g(t, U(t)) = g(t, u)\,.$$

The converse also holds.

Fact 1. If $u = U(t)$ solves the first-order equation

$$u' = g(t, u)\,, \tag{8.5a}$$

and $x = X(t)$ solves the explicit first-order equation

$$x' = U(t)\,, \tag{8.5b}$$

then $x = X(t)$ solves (8.3). We call (8.5a) the *auxiliary equation* and call (8.5b) the *reduced explicit equation*. This is the explicit reduction method.

Reason. We can verify recipe (8.5) by a direct calculation. Because $x = X(t)$ solves the reduced explicit equation (8.5b), we see that

$$x' = U(t)\,, \quad \text{and} \quad x'' = U'(t)\,.$$

Because $u = U(t)$ solves the auxiliary equation (8.5a), we see that

$$U'(t) = g(t, U(t))\,.$$

By combining the foregoing equations, we see that

$$x'' = U'(t) = g(t, U(t)) = g(t, x')\,.$$

Therefore $x = X(t)$ solves the original second-order equation (8.3).

Remark. If the second-order equation (8.3) is linear then it can be put into the form

$$x'' + a(t)x' = f(t)\,,$$

in which case the auxiliary equation (8.5a) has the linear form

$$u' + a(t)u = f(t)\,.$$

This first-order linear equation can be solved by the methods of Chapter 2.

Remark. If the second-order equation (8.3) has the separable form

$$x'' = f(t)\,k(x')\,,$$

then the auxiliary equation (8.5a) has the separable form

$$u' = f(t)\,k(u)\,.$$

This first-order separable equation can be solved by the methods of Chapter 3.

Remark. There are other forms for which the second-order equation (8.3) can be solved by the explicit reduction method.

8.2.2. *Initial-Value Problems.* Every solution of (8.3) can be found by the explicit reduction method. Specifically, to solve the initial-value problem

$$x'' = g(t, x')\,, \qquad x(t_I) = x_o\,, \quad x'(t_I) = v_o\,,$$

we first let $u = U(t)$ solve the auxiliary initial-value problem

$$u' = g(t, u)\,, \qquad u(t_I) = v_o\,, \tag{8.6a}$$

and then let $x = X(t)$ solve the reduced explicit initial-value problem

$$x' = U(t)\,, \qquad x(t_I) = x_o\,. \tag{8.6b}$$

Example. We have already treated one example of this type. Recall the skydiver of mass 60 kg who jumped from an airplane and assumed a position with an aerodynamic cross-section of 0.1 m^2 in air with a density of 1.2 $\mathrm{kg/m}^3$. We were asked how far she had fallen after 10 s.

Solution. If we let $y(t)$ be the distance in meters that she had fallen at time t then by Newton's Law we have

$$\ddot{y} = g - k\dot{y}^2\,, \qquad y(0) = 0\,, \quad \dot{y}(0) = 0\,,$$

where $g = 9.8\ \mathrm{m/s}^2$ and

$$k = \frac{\rho_{\mathrm{air}} A}{m} = \frac{1.2 \cdot 0.1}{60} = \frac{1}{500} = .002\ 1/\mathrm{m}\,.$$

We let $u = U(t)$ solve the auxiliary initial-value problem

$$\dot{u} = g - ku^2\,, \qquad u(0) = 0\,,$$

which we solved to find

$$u = 70\,\frac{e^{0.28t} - 1}{e^{0.28t} + 1}\,.$$

We then let $y = Y(t)$ solve the reduced explicit initial-value problem

$$\dot{y} = 70\,\frac{e^{0.28t} - 1}{e^{0.28t} + 1}\,, \qquad y(0) = 0\,.$$

We thereby found that after 10 seconds she had fallen

$$Y(10) = \int_0^{10} 70\,\frac{e^{0.28t} - 1}{e^{0.28t} + 1}\,\mathrm{d}t = 500\log\left(\frac{e^{2.8} + 1}{2}\right) - 700\ \mathrm{m}\,.$$

8.3. **Autonomous Reduction Method.** Second-order equations missing the independent variable t have the normal form (8.2b), which we can express as

$$x'' = g(x, x')\,. \tag{8.7}$$

Of course, other variables might be missing from the right-hand side too. As with first-order equations, such equations are called autonomous. The autonomous reduction method reduces the problem of solving the second-order equation (8.7) to that of solving an autonomous first-order equation

$$x' = V(x)\,. \tag{8.8}$$

8.3.1. *Auxiliary and Reduced Equations.* For a solution of (8.8) to also be a solution of (8.7), the function $V(x)$ must be chosen with care. Because

$$x'' = V'(x)x' = V'(x)V(x)\,,$$

we see that $v = V(x)$ must satisfy the first-order equation

$$v\,\frac{\mathrm{d}v}{\mathrm{d}x} = V(x)\,V'(x) = x'' = g(x,x') = g(x,V(x)) = g(x,v)\,.$$

The converse also holds.

Fact 2. If $v = V(x)$ solves the first-order equation

$$v\,\frac{\mathrm{d}v}{\mathrm{d}x} = g(x,v)\,, \tag{8.9a}$$

and $x = X(t)$ solves the autonomous first-order equation

$$x' = V(x)\,, \tag{8.9b}$$

then $x = X(t)$ solves (8.7). We call (8.9a) the *auxiliary equation* and call (8.9b) the *reduced autonomous equation*. This is the autonomous reduction method.

Reason. We can verify recipe (8.9) by a direct calculation. Because $x = X(t)$ satisfies the autonomous reduction equation (8.9b), we see by the chain rule that

$$x' = V(x)\,, \quad \text{and} \quad x'' = V'(x)\,x' = V'(x)\,V(x)\,.$$

Because $v = V(x)$ solves the auxiliary equation (8.9a), we see that

$$V'(x)\,V(x) = g(x,V(x))\,.$$

By combining the foregoing equations, we see that

$$x'' = V'(x)\,V(x) = g(x,V(x)) = g(x,x')\,.$$

Therefore $x = X(t)$ solves the original second-order equation (8.7).

Remark. If the second-order equation (8.7) has the separable form

$$x'' = f(x)\,k(x')\,,$$

then the auxiliary equation (8.9a) has the separable form

$$\frac{\mathrm{d}v}{\mathrm{d}x} = f(x)\,\frac{k(v)}{v}\,.$$

When $k(v) = v$ this first-order equation is explicit. When $k(v) = v^2 + bv$ it is linear and can thereby be solved by the methods of Chapter 2. More generally, because it is separable, it can be solved by the methods of Chapter 3.

Remark. If the second-order equation (8.7) has the form

$$x'' = a(x)\,(x')^2 + b(x)\,x'\,,$$

then the auxiliary equation (8.9a) has the linear form

$$\frac{\mathrm{d}v}{\mathrm{d}x} = a(x)\,v + b(x)\,.$$

Because this first-order equation is linear, it can be solved by the methods of Chapter 2.

Remark. If the second-order equation (8.7) has the form

$$x'' = a(x)\,(x')^2 + c(x)\,,$$

then the auxiliary equation (8.9a) has the form

$$v\frac{\mathrm{d}v}{\mathrm{d}x} = a(x)\,v^2 + c(x)\,.$$

By introducing $w = v^2$ this can be brought into the linear form

$$\frac{\mathrm{d}w}{\mathrm{d}x} = a(x)\,w + c(x)\,.$$

Because this first-order equation is linear, it can be solved by the methods of Chapter 2.

Remark. There are other forms for which the second-order equation (8.7) can be solved by the autonomous reduction method.

Remark. Sometimes we will be able to find a solution of the auxiliary equation (8.9a), but the reduced autonomous equation (8.9b) will be either difficult or impossible to solve. However, we might be able to address the problem using a phase-line analysis of the reduced autonomous equation.

8.3.2. *Initial-Value Problems.* Every solution of (8.7) can be found by the autonomous reduction method so long as $x' \neq 0$. Specifically, to solve the initial-value problem

$$x'' = g(x, x')\,, \qquad x(t_o) = x_o\,, \quad x'(t_o) = v_o\,,$$

we first let $v = V(x)$ solve the auxiliary initial-value problem

$$v\,\frac{\mathrm{d}v}{\mathrm{d}x} = g(x, v)\,, \qquad v(x_o) = v_o\,, \tag{8.10a}$$

and then let $x = X(t)$ solve the reduced autonomous initial-value problem

$$x' = V(x)\,, \qquad x(t_o) = x_o\,. \tag{8.10b}$$

Notice that solutions of the auxiliary equation are undefined where $v = 0$, so we must require that $v_o \neq 0$ and that $V(X(t)) \neq 0$.

It is important to understand how we got the initial condition in the auxiliary initial-value problem (8.10a). Recall that the reduced autonomous equation (8.8) is $x' = V(x)$. If $x = X(t)$ is a solution of this equation that also satisfies the initial conditions $X(t_o) = x_o$ and $X'(t_o) = v_o$ then we see that

$$V(x_o) = V(X(t_o)) = X'(t_o) = v_o\,.$$

Because $v = V(x)$ solves the auxiliary equation, it must solve the auxiliary initial-value problem (8.10a).

Example. A puck with initial velocity $v_o > 0$ begins to slide on a surface that imparts a position-dependent frictional drag. Its position $x(t)$ is governed by the initial-value problem

$$\ddot{x} = -e^{-x}\dot{x}\,, \qquad x(0) = 0, \quad \dot{x}(0) = v_o > 0\,.$$

Find the smallest initial velocity v_o for which $x(t) \to \infty$ as $t \to \infty$.

Solution. This is a second-order autonomous initial-value problem. Its auxiliary initial-value problem is

$$v\,\frac{\mathrm{d}v}{\mathrm{d}x} = -e^{-x}v\,, \qquad v(0) = v_0\,.$$

The auxiliary differential equation is satisfied if and only if either

$$v = 0\,, \qquad \text{or} \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = -e^{-x}\,.$$

Because $v_o > 0$ we can eliminate the first case. The second case is an explicit differential equation for v. Its general solution is

$$v = e^{-x} + c\,.$$

The initial conditions imply that $v = v_o$ when $x = 0$, whereby

$$v_o = e^{-0} + c = 1 + c\,.$$

Hence, $c = v_o - 1$ and the solution of the auxiliary initial-value problem is

$$v = v_o - 1 + e^{-x}\,.$$

Therefore the resulting reduced autonomous initial-value problem is

$$\dot{x} = v_o - 1 + e^{-x}\,, \qquad x(0) = 0\,.$$

Rather than solving this reduced autonomous equation explicitly, we will analyze its phase-line portrait. Its stationary solutions must satisfy

$$0 = v_o - 1 + e^{-x}\,.$$

This has a solution if and only if $v_o < 1$, in which case the only stationary point is

$$x = x_\infty = \log\left(\frac{1}{1 - v_o}\right)\,.$$

Because $v_o > 0$ we see that $x_\infty > 0$. Therefore we have two cases to consider.

- When $0 < v_o < 1$ the reduced equation has the single stationary point x_∞ and its phase-line portrait is

$$\begin{array}{c} \qquad + \qquad\qquad\qquad\qquad - \\ \rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow \bullet \leftarrow\leftarrow\leftarrow\leftarrow\leftarrow\leftarrow\; x \\ x_\infty \end{array}$$

  Because $x(0) = 0 < x_\infty$, we see that $\dot{x}(t) > 0$ and that $x(t) \to x_\infty$ as $t \to \infty$. Therefore we do not have $x(t) \to \infty$ as $t \to \infty$ when $0 < v_o < 1$.

- When $v_o \geq 1$ the reduced equation has no stationary points and its phase-line portrait is

$$\begin{array}{c} + \\ \rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\; x \end{array}$$

  We see that $\dot{x}(t) > 0$ and that $x(t) \to \infty$ as$t \to \infty$. Therefore we have $x(t) \to \infty$ as $t \to \infty$ when $v_o \geq 1$.

Therefore $v_o = 1$ is the smallest initial velocity for which $x(t) \to \infty$ as $t \to \infty$.

Remark. The reduced equation may be solved explicitly, but that is not the best way to approach the problem. The explicit solution is

$$x(t) = \begin{cases} \log\left(\dfrac{1 - v_o e^{-(1-v_o)t}}{1 - v_o}\right) & \text{for } v_o < 1\,, \\ \log(1+t) & \text{for } v_o = 1\,, \\ \log\left(\dfrac{v_o e^{(v_o-1)t} - 1}{v_o - 1}\right) & \text{for } v_o > 1\,. \end{cases}$$

This shows that $v_o = 1$ is the smallest initial velocity for which $x(t) \to \infty$ as $t \to \infty$. To obtain this solution we have to first find an implicit general solution by integrating

$$\int \frac{1}{v_o - 1 + e^{-x}}\,\mathrm{d}x = t + c\,.$$

The integral on the left-hand side is easier to do after multiplying the numerator and denominator of the integrand by e^x.

Example. An ideal pendulum is a bob of mass m hung on a massless rod of length ℓ from a frictionless pivot that constrains it to planar motion. If θ is the angle the pendulum makes with the downward vertical then Newton's Law governs its motion by

$$m\ell\,\ddot{\theta} = -mg\sin(\theta)\,,$$

where g is gravitational acceleration. If the pendulum is initially displaced by an angle θ_I where $0 \leq \theta_I \leq \pi$ and is released with a angular velocity ω_I then what will be its maximum angular displacement?

Solution. This second-order differential equation is autonomous, so we can use the method of this section. The auxiliary equation (8.9a) for this differential equation is

$$v\,\frac{\mathrm{d}v}{\mathrm{d}\theta} = -\frac{g}{\ell}\,\sin(\theta)\,.$$

This equation is separable. Its separated differential form is

$$v\,\mathrm{d}v = -\frac{g}{\ell}\,\sin(\theta)\,\mathrm{d}\theta\,,$$

which yields

$$\int v\,\mathrm{d}v = -\frac{g}{\ell}\int \sin(\theta)\,\mathrm{d}\theta\,.$$

Therefore an implicit general solution of the auxiliary equation is

$$\tfrac{1}{2}v^2 = \frac{g}{\ell}\,\cos(\theta) + c\,.$$

Because at $t = 0$ we have $\theta = \theta_I$ and $v = \omega_I$, we see that c is given by

$$c = \tfrac{1}{2}\omega_I^2 - \frac{g}{\ell}\,\cos(\theta_I)\,,$$

whereby the solution of the auxiliary initial-value problem satisfies

$$v^2 = \omega_I^2 + 2\,\frac{g}{\ell}\,\big(\cos(\theta) - \cos(\theta_I)\big)\,.$$

Therefore if $\omega_I \neq 0$ then the reduced autonomous equation is

$$\dot{\theta} = \text{sign}(\omega_I)\sqrt{\omega_I^2 + 2\,\frac{g}{\ell}\,\big(\cos(\theta) - \cos(\theta_I)\big)}\,.$$

The maximum angular displacement will occur where $\dot{\theta} = 0$, which is at the stationary points of this equation. Such stationary points must satisfy

$$\cos(\theta) = \cos(\theta_I) - \frac{\ell}{2\,g}\,\omega_I^2\,.$$

We see that such stationary points exist if and only if

$$\cos(\theta_I) - \frac{\ell}{2\,g}\,\omega_I^2 \geq -1\,,$$

in which case the maximum angular displacement is

$$\cos^{-1}\left(\cos(\theta_I) - \frac{\ell}{2\,g}\,\omega_I^2\right)\,.$$

Otherwise the pendulum will rotate around its pivot.

Example. A projectile is shot straight up from the surface of the Earth with a velocity of magnitude v_o. How large does v_o have to be to insure that the projectile escapes the gravitational pull of Earth? (This is asking us to compute the escape velocity of Earth.) Assume that:

- Earth is a perfect sphere of radius r_{E},
- its gravitational acceleration at the surface is g,
- its gravitational acceleration obeys Newton's Universal Law of Gravitation, whereby it falls off like r^{-2}, where r is the distance from the center of Earth.

Express the answer in terms of r_{E} and g.

Solution. Because gravitational acceleration acts towards the center of Earth, has value g when $r = r_{\text{E}}$, and falls off like r^{-2}, the vertical accelleration of the projectile is governed by second-order differential equation

$$\ddot{r} = -g\frac{r_{\text{E}}^2}{r^2}\,.$$

The solutions we seek satisfy the initial conditions

$$r(0) = r_{\text{E}}\,, \qquad \dot{r}(0) = v_o\,.$$

The second-order differential equation is autonomous, so we can use the autonomous reduction method.

The auxiliary equation (8.9a) for this differential equation is

$$v\frac{\mathrm{d}v}{\mathrm{d}r} = -g\frac{r_{\text{E}}^2}{r^2}\,.$$

This equation is separable. Its separated differential form is

$$v\,\mathrm{d}v = -g\frac{r_{\text{E}}^2}{r^2}\,\mathrm{d}r\,,$$

which yields

$$\int v \, \mathrm{d}v = -g \int \frac{r_{\mathrm{E}}^2}{r^2} \, \mathrm{d}r \, .$$

Therefore an implicit general solution of the auxiliary equation is

$$\tfrac{1}{2} v^2 = g \frac{r_{\mathrm{E}}^2}{r} + c \, .$$

Because at $t = 0$ we have $r = r_{\mathrm{E}}$ and $v = v_o$, we see that c is given by

$$c = \tfrac{1}{2} v_o^2 - g r_{\mathrm{E}} \, ,$$

whereby the solution of the auxiliary iniitial-value problem satisfies

$$v^2 = v_o^2 - 2 g r_{\mathrm{E}} + 2g \frac{r_{\mathrm{E}}^2}{r} \, .$$

Therefore the reduced autonomous equation is

$$\dot{r} = \sqrt{v_o^2 - 2 g r_{\mathrm{E}} + 2g \frac{r_{\mathrm{E}}^2}{r}} \, .$$

The right-hand side will be positive for every $r > r_{\mathrm{E}}$ provided

$$v_o \geq \sqrt{2 g r_{\mathrm{E}}} \, .$$

Therefore the escape velocity is $v_{\mathrm{E}} = \sqrt{2 g r_{\mathrm{E}}}$.

Remark. For Earth we have $g \approx 9.8$ m/s^2 and $r_{\mathrm{E}} \approx 6.4 \cdot 10^6$ m. Therefore the escape velocity for Earth is approximately

$$v_{\mathrm{E}} \approx \sqrt{2 \cdot 9.8 \cdot 6.4 \cdot 10^6} = 1.12 \cdot 10^4 = 11,200 \text{ m/s} = 11.2 \text{ km/s} \, .$$

For comparison, orbital velocity for an Earth satellite is about 7.8 km/s and the speed of sound at sea level is about .343 km/s.

8.4. **Application: Projectile Height.** In this section we will address the question of how high a projectile will go that is shot straight up into the air. We will assume that the projectile stays low enough that we can treat the graviational field as uniform. We will study three models of aerodynamic drag.

- First, we will neglect drag.
- Second, we will include turbulent drag in air with uniform density.
- Third, we will include turbulent drag in air with variable density.

These models exhibit increasing complexity and will be used to illustrate methods from this and previous chapters.

8.4.1. *No Drag.* Let $h(t)$ be the height of the projectile as a function of time. We will assume that initially $h(0) = 0$ and $\dot{h}(0) = v_o$ where $v_o > 0$. The simplest model assumes that there is no drag. Then $h(t)$ is governed by the initial-value problem

$$\ddot{h} = -g\,, \qquad h(0) = 0\,, \quad \dot{h}(0) = v_o\,, \tag{8.11}$$

where g is the uniform gravitational accelleration. We will solve this problem three ways:

- as an explicit second-order equation,
- by the explicit reduction method,
- by the autonomous reduction method.

The first is the way that you might have seen this problem solved before. The other two are the methods from this chapter.

First, because the right-hand side of the differential equation in (8.11) is independent of both h and $\dot{h}$, we see that it can be integrated twice explicitly. Upon doing so while using the initial conditions we find that

$$\dot{h}(t) = v_o - gt\,, \qquad h(t) = v_o t - \tfrac{1}{2} g t^2\,.$$

Because $h(t)$ will reach its maximum value when $\dot{h}(t) = 0$, we see that this happens when

$$t = t_{\max} = \frac{v_o}{g}\,.$$

Therefore the maximum height of the projectile is

$$h_{\max} = h(t_{\max}) = \frac{v_o^2}{2g}\,.$$

Second, because the right-hand side of the differential equation in (8.11) is independent of h, we see that it can be solved by the explicit reduction method. The auxiliary initial-value problem for this method is

$$\dot{u} = -g\,, \quad u(0) = v_o\,,$$

which has solution

$$u(t) = v_o - gt\,.$$

Therefore the reduced explicit initial-value problem is

$$\dot{h} = v_o - gt\,, \quad h(0) = 0\,,$$

which has solution

$$h(t) = v_o t - \tfrac{1}{2} g t^2\,.$$

The maximum height of the projectile is then found as before.

Remark. We see that for this problem the explicit reduction method is essentially the same as the first method. This is always the case when the right-hand side of the second-order equation does not depend upon either the dependent variable or its derivative.

Third, because the right-hand side of the differential equation in (8.11) is independent of t, we see that it can be solved by the autonomous reduction method. The auxiliary initial-value problem for this method is

$$v\,\frac{\mathrm{d}v}{\mathrm{d}h} = -g\,, \qquad v(0) = v_o\,.$$

This differential equation is separable. It has the separated form

$$v\,\mathrm{d}v = -g\,\mathrm{d}h\,,$$

which can be intgrated to obtain

$$\tfrac{1}{2}v^2 = \tfrac{1}{2}v_o^2 - gh\,.$$

So long as the projectile is going up we have $v > 0$, so the solution of the auxiliary initial-value problem

$$v = \sqrt{v_o^2 - 2gh}\,.$$

Therefore the reduced autonomous initial-value problem is

$$\dot{h} = \sqrt{v_o^2 - 2gh}\,, \qquad h(0) = 0\,.$$

We know that the maximum height of the projectile is reached when $\dot{h} = 0$, which is when

$$v_o^2 - 2gh = 0\,.$$

Therefore we again obtain

$$h_{\max} = \frac{v_o^2}{2g}\,.$$

Remark. With this approach there is no need to find an explicit solution for $h(t)$.

8.4.2. *Turbulent Drag in Uniform Density.* It is natural to ask how turbulent drag will reduce the maximum height that a projectile will obtain. If this height is not so great that we can treat the density of air as uniform then while the projectile is ascending its height is governed by the initial-value problem

$$\ddot{h} = -g - k\dot{h}^2\,, \qquad h(0) = 0\,, \quad \dot{h}(0) = v_o\,, \tag{8.12}$$

where $k > 0$ is the turbulent drag coefficient. We know that the projectile should not go as high for this model as it did for the model without drag! The question is by how much.

Because the right-hand side of this differential equation depends upon $\dot{h}$, it cannot be integrated explicitly as we did for problem (8.11), in which drag was neglected. However, we can solve this problem two ways:

- by the explicit reduction method,
- by the autonomous reduction method.

These are the methods from this chapter.

First, because the right-hand side of the differential equation in (8.12) is independent of h, we see that it can be solved by the explicit reduction method. The auxiliary initial-value problem for this method is

$$\dot{u} = -g - ku^2\,, \quad u(0) = v_o\,.$$

This differential equation is autonomous, and thereby separable. Its separated differential form is

$$\frac{\mathrm{d}u}{g+ku^2} = -\mathrm{d}t\,.$$

Because

$$\int \frac{\mathrm{d}u}{g+ku^2} = \sqrt{\frac{1}{gk}}\,\tan^{-1}\!\left(\sqrt{\frac{k}{g}}u\right) + c\,,$$

we see that the solution of the auxiliary initial-value problem is given implicitly by

$$\tan^{-1}\!\left(\sqrt{\frac{k}{g}}u\right) = \tan^{-1}\!\left(\sqrt{\frac{k}{g}}v_o\right) - \sqrt{gk}\,t\,.$$

This can be solved to obtain the explicit solution

$$u = \sqrt{\frac{k}{g}}\,\tan\!\left(\tan^{-1}\!\left(\sqrt{\frac{k}{g}}v_o\right) - \sqrt{gk}\,t\right)\,.$$

Therefore the reduced explicit initial-value problem for h is

$$\dot{h} = \sqrt{\frac{k}{g}}\,\tan\!\left(\tan^{-1}\!\left(\sqrt{\frac{k}{g}}v_o\right) - \sqrt{gk}\,t\right)\,, \quad h(0)=0\,.$$

This can be integrated to obtain

$$h = \frac{1}{k}\,\log\!\left(\cos\!\left(\tan^{-1}\!\left(\sqrt{\frac{k}{g}}v_o\right) - \sqrt{gk}\,t\right)\right) - \frac{1}{k}\,\log\!\left(\cos\!\left(\tan^{-1}\!\left(\sqrt{\frac{k}{g}}v_o\right)\right)\right)\,.$$

Second, because the right-hand side of the differential equation in (8.12) is independent of t, we see that it can be solved by the autonomous reduction method. The auxiliary initial-value problem for this method is

$$v\,\frac{\mathrm{d}v}{\mathrm{d}h} = -g - kv^2\,, \qquad v(0) = v_o\,.$$

This problem can be solved by several methods. We will present one here and others later. We begin by noticing that if we set $w = v^2$ then the initial-value problem for w is

$$\tfrac{1}{2}\frac{\mathrm{d}w}{\mathrm{d}h} = -g - k\,w\,, \qquad w(0) = v_o^2\,.$$

This is a nonhomogeneous linear equation with the normal form

$$\frac{\mathrm{d}w}{\mathrm{d}h} + 2k\,w = -2g\,.$$

This equation has the integrating factor e^{2kh} and can be put into the integrating factor form

$$\frac{\mathrm{d}}{\mathrm{d}h}\left(e^{2kh}w\right) = -2ge^{2kh}\,.$$

Upon integrating both sides of this equation and applying the initial condition $w(0) = v_o^2$ we get

$$e^{2kh} w = v_o^2 - \frac{g}{k}\left(e^{2kh} - 1\right) .$$

This has the explicit solution

$$w = v_o^2 e^{-2kh} - \frac{g}{k}\left(1 - e^{-2kh}\right) ,$$

whereby the explicit solution of the auxiliary initial-value problem is

$$v = \sqrt{v_o^2 e^{-2kh} - \frac{g}{k}\left(1 - e^{-2kh}\right)} .$$

Therefore the reduced autonomous initial-value problem is

$$\dot{h} = \sqrt{v_o^2 e^{-2kh} + \frac{g}{k}\left(e^{-2kh} - 1\right)} , \qquad h(0) = 0 .$$

We know that the maximum height of the projectile is reached when $\dot{h} = 0$, which is when

$$e^{2kh} = 1 + \frac{kv_o^2}{g} .$$

Therefore we again obtain a maximum height of

$$h_{\max} = \frac{1}{2k} \log\left(1 + \frac{kv_o^2}{g}\right) .$$

8.4.3. *Turbulent Drag in Variable Density.* Recall that turbulent drag is proportional to the density of air. Because the density of air decreases with altitude, if the projectile goes high enough then we should model this decrease in air density, and the resultant decrease in turbulent drag. The dependence of air density upon altitude h is not simple. Here we adopt the model that

$$\rho_{\text{air}} \propto \frac{1}{1 + \dfrac{h}{\ell}} ,$$

where the density variablity length ℓ is the altitude at which the modelled air density is half of what it is at ground level. This model is reasonable so long as h is a moderate fraction of ℓ. With this approximation then while the projectile is ascending its height is governed by the initial-value problem

$$\ddot{h} = -g - \frac{k}{1 + \dfrac{h}{\ell}} \dot{h}^2 , \qquad h(0) = 0 , \quad \dot{h}(0) = v_o , \tag{8.13}$$

where k is the turbulent drag coefficient at ground level. We know that the projectile should not fly as high for this model as it did for the model without drag. We also know that it should fly higher than it did for the model with turbulent drag in air with a uniform density.

Because the right-hand side of the differential equation in (8.13) depends upon both h and $\dot{h}$, it cannot be integrated explicitly as we did for problem (8.11), which neglected drag. Nor can it be solved by the explicit reduction method as we did for both problems (8.11) and (8.12). However, we can solve this problem by the autonomous reduction method.

Because the right-hand side of the differential equation in (8.12) is independent of t, we see that it can be solved by the autonomous reduction method. The auxiliary initial-value problem for this method is

$$v \frac{\mathrm{d}v}{\mathrm{d}h} = -g - \frac{k}{1 + \dfrac{h}{\ell}} v^2 , \qquad v(0) = v_o .$$

This problem can be solved by several methods. We will present one here and others later. We begin by noticing that if we set $w = v^2$ then the initial-value problem for w is

$$\tfrac{1}{2} \frac{\mathrm{d}w}{\mathrm{d}h} = -g - \frac{k}{1 + \dfrac{h}{\ell}} w , \qquad w(0) = v_o^2 .$$

This is a nonhomogeneous linear equation with the normal form

$$\frac{\mathrm{d}w}{\mathrm{d}h} + \frac{2k}{1 + \dfrac{h}{\ell}} w = -2g .$$

Its integrating factor is

$$\exp\left(\int_0^h \frac{2k}{1 + \dfrac{s}{\ell}} \,\mathrm{d}s \right) = \left(1 + \frac{h}{\ell} \right)^{2k\ell} ,$$

whereby its integrating factor form is

$$\frac{\mathrm{d}}{\mathrm{d}h} \left(\left(1 + \frac{h}{\ell} \right)^{2k\ell} w \right) = -2g \left(1 + \frac{h}{\ell} \right)^{2k\ell} .$$

Upon integrating both sides of this equation we get

$$\left(1 + \frac{h}{\ell} \right)^{2k\ell} w = \ .$$

8.4.4. *Comparison of the Three Models.* We now compare the results for the maximum projectile height found using the three foregoing models. For the model with no drag we found a maximum projectile height of

$$h_{\mathrm{mxn}} = \frac{v_o^2}{2g} . \tag{8.14a}$$

For the model with turbulent drag in air of uniform density we found a maximum projectile height of

$$h_{\mathrm{mxu}}(k) = \frac{1}{2k} \log\left(1 + \frac{k v_o^2}{g} \right) , \tag{8.14b}$$

where $k > 0$ is the drag coefficient. For the model with turbulent drag in air of variable density we found a maximum projectile height of

$$h_{\text{mxv}}(k,\ell) = \ell \left[\left(1 + \frac{kv_o^2}{g} \frac{2k\ell + 1}{2k\ell} \right)^{\frac{1}{2k\ell+1}} - 1 \right], \tag{8.14c}$$

where $k > 0$ is the drag coefficient at $h = 0$ and $\ell > 0$ is the density variablity length.

It can be shown that these indeed satisfy

$$h_{\text{mxu}}(k) < h_{\text{mxv}}(k,\ell) < h_{\text{mxn}}. \tag{8.15}$$

It can also be shown that

$$\lim_{k \to 0} h_{\text{mxu}}(k) = h_{\text{mxn}}, \tag{8.16}$$

which along with (8.15) implies that

$$\lim_{k \to 0} h_{\text{mxv}}(k,\ell) = h_{\text{mxn}} \qquad \text{for every } \ell > 0. \tag{8.17}$$

Finally, it can also be shown for every fixed $k > 0$ that $h_{\text{mxv}}(k,\ell)$ is a decreasing function of ℓ with

$$\lim_{\ell \to 0} h_{\text{mxv}}(k,\ell) = h_{\text{mxn}}, \qquad \lim_{\ell \to \infty} h_{\text{mxv}}(k,\ell) = h_{\text{mxu}}(k). \tag{8.18}$$

We challenge the ambitious student to prove these facts.

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