II. Higher-Order Linear Ordinary Differential Equations
4. Homogeneous Equations with Constant Coefficients

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4. Homogeneous Equations with Constant Coefficients

4.1. **Characteristic Polynomials and the Key Identity.** In Chapter 2 we saw how to construct general solutions of homogeneous linear differential equations given a fundamental set of solutions. While there is no general recipe for constructing fundamental sets of solutions, there are recipes for special cases. Here we study the most important such special case — namely, the case where all the coefficients are constants. In that case the n^{th}-order homogeneous linear equation 2.1 becomes

$$\frac{\mathrm{d}^n y}{\mathrm{d}t^n} + a_1 \frac{\mathrm{d}^{n-1} y}{\mathrm{d}t^{n-1}} + \cdots + a_{n-1} \frac{\mathrm{d}y}{\mathrm{d}t} + a_n y = 0\,, \tag{4.1}$$

where $a_1, a_2, \cdots, a_n$ are constants.

Before discussing the general case, let us consider the example

$$y'' - y' - 2y = 0\,.$$

In Chapter 2 we saw that e^{-t} and e^{2t} are solutions of this equation, but we did not see how they were obtained. We can gain insight into how they were obtained by examining how we checked that they are solutions. Those checks went something like

$$\begin{aligned}
(e^{-t})'' - (e^{-t})' - 2e^{-t} &= (-1)^2 e^{-t} - (-1)e^{-t} - 2e^{-t} \\
&= [(-1)^2 - (-1) - 2]e^{-t} = [1 + 1 - 2]e^{-t} = 0\,, \\
(e^{2t})'' - (e^{2t})' - 2e^{2t} &= 2^2 e^{2t} - 2e^{2t} - 2e^{2t} \\
&= [2^2 - 2 - 2]e^{2t} = [4 - 2 - 2]e^{2t} = 0\,.
\end{aligned}$$

Because every derivative of e^{-t} and e^{2t} is proportional to e^{-t} and e^{2t} respectively, we could factor e^{-t} and e^{2t} respectively out of each calculation, leaving simple arithmetic expressions inside the square backets that yielded zero. The calculation is similar for every function of the form e^{zt} where z is a constant. We find that

$$\begin{aligned}
(e^{zt})'' - (e^{zt})' - 2e^{zt} &= z^2 e^{zt} - z e^{zt} - 2e^{zt} \\
&= [z^2 - z - 2]e^{zt} = (z + 1)(z - 2)e^{zt}\,.
\end{aligned} \tag{4.2}$$

We thereby see that e^{zt} is a solution if and only if $z = -1$ or $z = 2$.

The idea is to generalize the calculation in (4.2) to the differential equation (4.1). It is helpful to express equation (4.1) as

$$\mathrm{L}y = 0\,, \tag{4.3a}$$

where L is the n^{th}-order linear differential operator

$$\mathrm{L} = \mathrm{D}^n + a_1 \mathrm{D}^{n-1} + \cdots + a_{n-1}\mathrm{D} + a_n\,, \quad \text{and} \quad \mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}t}\,. \tag{4.3b}$$

We may express L as

$$\mathrm{L} = p(\mathrm{D})\,,$$

where $p(z)$ is the n^{th} degree real polynomial

$$p(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n\,. \tag{4.4}$$

This is called the *characteristic polynomial* of the n^{th}-order differential operator L.

Repeated differentiation of the function e^{zt} yields the identities

$$\mathrm{D}e^{zt} = z\,e^{zt}\,, \qquad \mathrm{D}^2 e^{zt} = z^2\,e^{zt}\,, \qquad \cdots\,,$$

whereby we see that

$$\mathrm{D}^k e^{zt} = z^k\,e^{zt} \quad \text{for every positive integer } k.$$

Hence, we find that

$$\begin{aligned} p(\mathrm{D})e^{zt} &= \big(\mathrm{D}^n + a_1\mathrm{D}^{n-1} + \cdots + a_{n-1}\mathrm{D} + a_n\big)e^{zt} \\ &= \mathrm{D}^n e^{zt} + a_1\mathrm{D}^{n-1}e^{zt} + \cdots + a_{n-1}\mathrm{D}e^{zt} + a_n e^{zt} \\ &= z^n e^{zt} + a_1 z^{n-1}e^{zt} + \cdots + a_{n-1}z\,e^{zt} + a_n e^{zt} \\ &= \big[z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + a_n\big]e^{zt} \\ &= p(z)\,e^{zt}\,. \end{aligned}$$

Therefore every n^{th}-order differential operator L given by (4.3b) has a characteristic polynomial $p(z)$ given by (4.4) such that L and $p(z)$ satisfy the so-called *Key Identity*,

$$\mathrm{L}e^{zt} = p(\mathrm{D})e^{zt} = p(z)\,e^{zt}\,. \tag{4.5}$$

This generalizes the calculation in (4.2). In this chapter we will use the Key Identity to construct a fundamental set of solutions to the homogeneous equation $\mathrm{L}y = 0$. In later chapters we will use the it to obtain solutions of certain nonhomogeneous equations.

4.2. **Real Roots of Characteristic Polynomials.** Let $p(z)$ be the characteristic polynomial of a differential operator L. Notice that if r is a real root of $p(z)$ (i.e. $p(r) = 0$) then the Key Identity (4.5) implies

$$\mathrm{L}e^{rt} = p(r)e^{rt} = 0\,,$$

whereby e^{rt} is a solution of the homogeneous equation $\mathrm{L}y = 0$. This observation allows us to construct a fundamental set of solutions for the homogeneous equation $\mathrm{L}y = 0$.

4.2.1. *Simple Real Roots.* We say that a polynomial $p(z)$ of degree n has n simple real roots r_1, r_2, $\cdots$, r_n if it can be factored as

$$p(z) = (z - r_1)(z - r_2)\cdots(z - r_n)\,, \tag{4.6}$$

where the numbers r_1, r_2, $\cdots$, r_n are real and distinct. (Here "distinct" means that no two of the numbers are equal.)

The above observation implies that if the characteristic polynomial $p(z)$ has n simple real roots r_1, r_2, $\cdots$, r_n then we have n solutions of the homogeneous equation $\mathrm{L}y = 0$:

$$e^{r_1 t}\,, \qquad e^{r_2 t}\,, \qquad \cdots\,, \qquad e^{r_n t}\,.$$

It can be shown that these solutions are linearly independent. For example, when $n = 3$ we see that their Wronskian is

$$\det\begin{pmatrix} e^{r_1 t} & e^{r_2 t} & e^{r_3 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} & r_3 e^{r_3 t} \\ r_1^2 e^{r_1 t} & r_2^2 e^{r_2 t} & r_3^2 e^{r_3 t} \end{pmatrix} = (r_3 - r_2)(r_2 - r_1)(r_3 - r_1)e^{(r_1+r_2+r_3)t} \neq 0\,.$$

The argument for linear independence when $n \geq 4$ goes similarly, but will not be given here because it is more complicated. Given this independence, we conclude that a general solution of $\mathrm{L}y = 0$ is

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t} + \cdots + c_n e^{r_n t} \,.$$

Example. Find a general solution of

$$y'' - y' - 2y = 0 \,.$$

Solution. The characteristic polynomial is

$$p(z) = z^2 - z - 2 = (z+1)(z-2) \,.$$

Its two roots are -1, 2. The solution associated with the root -1 is e^{-t}. The solution associated with the root 2 is e^{2t}. Therefore a general solution is

$$y = c_1 e^{-t} + c_2 e^{2t} \,.$$

Example. Find a general solution of

$$v''' + 2v'' - v' - 2v = 0 \,.$$

Solution. The characteristic polynomial is

$$p(z) = z^3 + 2z^2 - z - 2 = (z-1)(z+1)(z+2) \,.$$

Its three roots are 1, -1, -2. The solution associated with the root 1 is e^{t}. The solution associated with the root -1 is e^{-t}. The solution associated with the root -2 is e^{-2t}. Therefore a general solution is

$$v = c_1 e^{t} + c_2 e^{-t} + c_3 e^{-2t} \,.$$

Remark. Because the characteristic polynomial is easy to read off from the differential equation, the most difficult part of applying this recipe is finding the roots of the characteristic polynomial. Of course, for quadratic polynomials this can be done by completing the square or by using the quadratic formula. In this course characteristic polynomials of degree three or more will generally have some easily found root like 0, ± 1, ± 2, or ± 3. If the coefficients of $p(z)$ are integers, we should first check for roots that are factors of a_n. If we have found a real root r then the characteristic polynomial can be factored as

$$p(z) = (z - r)q(z) \,,$$

where $q(z)$ is an $(n-1)^{\text{th}}$ degree real polynomial

$$q(z) = z^{n-1} + b_1 z^{n-2} + \cdots + b_{n-2} z + b_{n-1} \,.$$

We thereby reduce the problem of finding the remaining roots of $p(z)$ to finding roots of $q(z)$. If a characteristic polynomial has n simple real roots r_1, r_2, $\cdots$, r_n then this procedure is repeated until we have completely factored $p(z)$ into the form (4.6). Of course, if $p(z)$ is given in its factored form, we can just read off the roots!

4.2.2. *Real Roots with any Multiplicity.* Of course, polynomials of degree n do not generally have n simple real roots. There are two ways this can fail to happen. First, a real root might not be simple — that is, it might have multiplicity greater than one. Second, polynomials might have irreducible factors which correspond to complex roots. Here we examine how to treat cases with real roots of multiplicity greater than one.

Recall that r is a double real root of $p(z)$ when $(z-r)^2$ is a factor of $p(z)$ — i.e. when $p(z) = (z-r)^2 q(z)$. Because

$$p'(z) = 2(z-r)q(z) + (z-r)^2 q'(z)\,,$$

we see that $p(r) = p'(r) = 0$. Differentiation of the Key Identity (4.5) with respect to z gives

$$\mathrm{L}\big(t\,e^{zt}\big) = p(z)\,t\,e^{zt} + p'(z)\,e^{zt}\,.$$

Evaluating this at $z = r$ shows that

$$\mathrm{L}\big(t\,e^{rt}\big) = p(r)\,t\,e^{rt} + p'(r)\,e^{rt} = 0\,.$$

Hence, e^{rt} and $t\,e^{rt}$ are solutions of the homogeneous equation $\mathrm{L}y = 0$. Because

$$\mathrm{D}e^{rt} = r\,e^{rt}\,, \qquad \mathrm{D}(t\,e^{rt}) = rt\,e^{rt} + e^{rt}\,,$$

the Wronskian of these solutions is

$$\det\begin{pmatrix} e^{rt} & t\,e^{rt} \\ r\,e^{rt} & rt\,e^{rt} + e^{rt} \end{pmatrix} = e^{rt}(rt\,e^{rt} + e^{rt}) - t\,e^{rt} r\,e^{rt} = e^{2rt} \neq 0\,.$$

Therefore these solutions are linearly independent.

Recall that r is a triple real root of $p(z)$ when $(z-r)^3$ is a factor of $p(z)$ — i.e. when $p(z) = (z-r)^3 q(z)$. Because

$$\begin{aligned} p'(z) &= 3(z-r)^2 q(z) + (z-r)^3 q'(z)\,, \\ p''(z) &= 6(z-r)q(z) + 6(z-r)^2 q'(z) + (z-r)^3 q(z)\,, \end{aligned}$$

we see that $p(r) = p'(r) = p''(r) = 0$. Differentiation of the Key Identity (4.5) twice with respect to z gives

$$\mathrm{L}\big(t^2 e^{zt}\big) = p(z)\,t^2 e^{zt} + 2p'(z)\,t\,e^{zt} + p''(z)\,e^{zt}\,.$$

Evaluating this at $z = r$ shows that

$$\mathrm{L}\big(t^2 e^{rt}\big) = p(r)\,t^2 e^{rt} + 2p'(r)\,t\,e^{rt} + p''(r)\,e^{rt} = 0\,.$$

Hence, e^{rt}, $t\,e^{rt}$, and $t^2 e^{rt}$ are solutions of the homogeneous equation $\mathrm{L}y = 0$. Because

$$\mathrm{D}e^{rt} = r\,e^{rt}\,, \qquad \mathrm{D}(t\,e^{rt}) = rt\,e^{rt} + e^{rt}\,, \qquad \mathrm{D}(t^2 e^{rt}) = rt^2 e^{rt} + 2t\,e^{rt}\,,$$

and

$$\mathrm{D}^2 e^{rt} = r^2 e^{rt}\,, \quad \mathrm{D}^2(t\,e^{rt}) = r^2 t\,e^{rt} + 2r\,e^{rt}\,, \quad \mathrm{D}^2(t^2 e^{rt}) = r^2 t^2 e^{rt} + 4rt\,e^{rt} + 2\,e^{rt}\,,$$

the Wronskian of these solutions is (the details are not shown)

$$\det\begin{pmatrix} e^{rt} & t\,e^{rt} & t^2 e^{rt} \\ r\,e^{rt} & rt\,e^{rt} + e^{rt} & rt^2 e^{rt} + 2t\,e^{rt} \\ r^2 e^{rt} & r^2 t\,e^{rt} + 2r\,e^{rt} & r^2 t^2 e^{rt} + 4rt\,e^{rt} + 2\,e^{rt} \end{pmatrix} = 2e^{3rt} \neq 0\,.$$

Therefore these solutions are linearly independent.

More generally, recall that r is a real root of $p(z)$ of multiplicity m when $(z-r)^m$ is a factor of $p(z)$ — i.e. when $p(z) = (z-r)^m q(z)$. This implies that

$$p(r) = p'(r) = \cdots = p^{(m-1)}(r) = 0\,.$$

Differentiating the Key Identity (4.5) k times with respect to z gives

$$\mathrm{L}\big(t^k e^{zt}\big) = p(z)\, t^k e^{zt} + kp'(z)\, t^{k-1} e^{zt} + \cdots + kp^{(k-1)}(z)\, t\, e^{zt} + p^{(k)}(z)\, e^{zt}\,.$$

Evaluating this at $z = r$ when $k = 1, \cdots, m-1$ shows that

$$\mathrm{L}\big(t^k e^{rt}\big) = p(r)\, t^k e^{rt} + kp'(r)\, t^{k-1} e^{rt} + \cdots + kp^{(k-1)}(r)\, t\, e^{rt} + p^{(k)}(r)\, e^{rt} = 0\,.$$

Therefore

$$e^{rt}\,, \qquad t\, e^{rt}\,, \qquad \cdots \qquad t^{m-1} e^{rt}\,,$$

are m solutions of the homogeneous equation $\mathrm{L}y = 0$. It can be shown that these solutions are linearly independent, but we will not show this here. Moreover, it can be shown that the set of all solutions so constructed from the roots of $p(z)$ are linearly independent.

Example. Find a general solution of

$$u'' + 6u' + 9u = 0\,.$$

Solution. The characteristic polynomial is

$$p(z) = z^2 + 6z + 9 = (z+3)^2\,.$$

Its two roots (counting multiplicity) are -3, -3. The solutions associated with the double root -3 are e^{-3t} and $t\, e^{-3t}$. Therefore a general solution is

$$u = c_1 e^{-3t} + c_2 t\, e^{-3t}\,.$$

Example. Find a general solution of

$$\mathrm{D}^6 w - 5\mathrm{D}^5 w + 6\mathrm{D}^4 w + 4\mathrm{D}^3 w - 8\mathrm{D}^2 w = 0\,, \qquad \text{where} \quad \mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}t}\,.$$

Solution. The characteristic polynomial is

$$p(z) = z^6 - 5z^5 + 6z^4 + 4z^3 - 8z^2 = z^2(z+1)(z-2)^3\,.$$

Its six roots (counting multiplicity) are 0, 0, -1, 2, 2, 2. The solutions associated with the double root 0 are 1 and t. The solution associated with the simple root -1 is e^{-t}. The solutions associated with the triple root 2 are e^{2t}, $t\, e^{2t}$, and $t^2 e^{2t}$. Therefore a general solution is

$$w = c_1 + c_2 t + c_3 e^{-t} + c_4 e^{2t} + c_5 t\, e^{2t} + c_6 t^2 e^{2t}\,.$$

4.3. **Complex Extension of the Key Identity.** Consider the problem of finding a general solution of

$$\mathrm{D}^2 y + 9y = 0\,, \qquad \text{where} \quad \mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}t}\,.$$

The characteristic polynomial is $p(z) = z^2+9$, which clearly has no real roots. However, this can be factored over the complex numbers as

$$p(z) = (z - i3)(z + i3)\,,$$

where $i = \sqrt{-1}$. Its roots are the conjugate pair $i3$ and $-i3$. We claim that e^{i3t} and e^{-i3t} are independent complex-valued solutions of $\mathrm{L}y = 0$. We must first recall what is meant by such complex-valued solutions. We must then see how to generate real-valued solutions from them.

Recall the Euler (pronounced "oiler") identity from calculus. It states that for every real θ we have

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)\,.$$

This identity is the key to making sense of complex exponentials. In particular, for any real number s we have

$$e^{ist} = \cos(st) + i\sin(st)\,.$$

The derivative of this function with respect to t is then

$$\begin{aligned}\mathrm{D}e^{ist} &= \mathrm{D}\big(\cos(st) + i\sin(st)\big) = \mathrm{D}\cos(st) + i\mathrm{D}\sin(st)\\ &= -s\sin(st) + is\cos(st) = is\big(\cos(st) + i\sin(st)\big) = ise^{ist}\,.\end{aligned}$$

More generally, for any real numbers r and s we have

$$e^{(r+is)t} = e^{rt+ist} = e^{rt}e^{ist} = e^{rt}\big(\cos(st) + i\sin(st)\big)\,.$$

By the product rule and our result that $\mathrm{D}e^{ist} = ise^{ist}$, the derivative of this function with respect to t is

$$\begin{aligned}\mathrm{D}e^{(r+is)t} &= \mathrm{D}\big(e^{rt}e^{ist}\big) = \mathrm{D}\big(e^{rt}\big)e^{ist} + e^{rt}\mathrm{D}\big(e^{ist}\big)\\ &= re^{rt}e^{ist} + ise^{rt}e^{ist} = (r + is)e^{(r+is)t}\,.\end{aligned}$$

We thereby see that for any complex number z we have $\mathrm{D}e^{zt} = ze^{zt}$, which implies that $\mathrm{D}^k e^{zt} = z^k e^{zt}$ for every positive integer k. It follows just as it did for the real case that for any polynomial $p(z)$ we have

$$p(\mathrm{D})e^{zt} = p(z)e^{zt}\,. \tag{4.7}$$

In particular, the Key Identity holds for any complex z!

4.4. **Complex Roots of Characteristic Polynomials.** Let $p(z)$ be the characteristic polynomial of the differential operator L given by (4.3). Because $p(z)$ has real coefficients, it has the property that

$$p(\overline{z}) = \overline{p(z)} \quad \text{for every complex } z\,,$$

where the bar denotes complex conjugate — i.e. $\overline{X+iY} = X-iY$ for any real numbers X and Y. Thus, if $p(r+is)=0$ then

$$p(r-is) = p(\overline{r+is}) = \overline{p(r+is)} = 0\,.$$

Therefore roots of $p(z)$ come in conjugate pairs; if $r+is$ is a root with $s \neq 0$ then $r-is$ is another root.

By the Key Identity (4.7), if $p(z)$ has a conjugate pair of roots $r+is$ and $r-is$ then $e^{(r+is)t}$ and $e^{(r-is)t}$ are a pair of complex-valued solutions of equation (4.1) — namely, they satisfy

$$\mathrm{L}e^{(r+is)t} = 0\,, \qquad \mathrm{L}e^{(r-is)t} = 0\,. \tag{4.8}$$

Because $e^{(r+is)t} = e^{rt}\cos(st) + ie^{rt}\sin(st)$, its real and imaginary parts are

$$\mathrm{Re}\big(e^{(r+is)t}\big) = e^{rt}\cos(st)\,, \qquad \mathrm{Im}\big(e^{(r+is)t}\big) = e^{rt}\sin(st)\,.$$

Recall that for any complex Z its real and imaginary parts can be expressed as

$$\mathrm{Re}(Z) = \frac{Z+\overline{Z}}{2}\,, \qquad \mathrm{Im}(Z) = \frac{Z-\overline{Z}}{i2}\,.$$

Therefore we have

$$\begin{aligned} e^{rt}\cos(st) &= \mathrm{Re}\big(e^{(r+is)t}\big) = \frac{e^{(r+is)t}+e^{(r-is)t}}{2}\,,\\ e^{rt}\sin(st) &= \mathrm{Im}\big(e^{(r+is)t}\big) = \frac{e^{(r+is)t}-e^{(r-is)t}}{i2}\,. \end{aligned}$$

It then follows from (4.8) that

$$\begin{aligned} \mathrm{L}\big(e^{rt}\cos(st)\big) &= \mathrm{L}\left(\frac{e^{(r+is)t}+e^{(r-is)t}}{2}\right) = \frac{1}{2}\left(\mathrm{L}e^{(r+is)t}+\mathrm{L}e^{(r-is)t}\right) = 0\,,\\ \mathrm{L}\big(e^{rt}\sin(st)\big) &= \mathrm{L}\left(\frac{e^{(r+is)t}-e^{(r-is)t}}{i2}\right) = \frac{1}{i2}\left(\mathrm{L}e^{(r+is)t}-\mathrm{L}e^{(r-is)t}\right) = 0\,. \end{aligned}$$

In other words, when the characteristic polynomial $p(z)$ of the differential operator L has a conjugate pair of roots $r+is$ and $r-is$ then

$$e^{rt}\cos(st) \qquad \text{and} \qquad e^{rt}\sin(st)$$

are real-valued solutions of equation (4.1). We can easily check that they are linearly independent when $s \neq 0$ by showing that their Wronskian is nonzero.

4.4.1. *Simple Complex Roots.* If $p(z)$ has a conjugate pair of simple roots $r + is$ and $r - is$ with $s > 0$ then it has the pair of complex factors $(z - r - is)$ and $(z - r + is)$. Because

$$(z - r - is)(z - r + is) = (z - r)^2 - (is)^2 = (z - r)^2 + s^2 \,,$$

we see that $p(z)$ has the irreducible real factor $(z - r)^2 + s^2$. Conversely, if $p(z)$ has the irreducible real factor $(z - r)^2 + s^2$ then it has the conjugate pair of roots $r + is$ and $r - is$.

Example. Find a general solution of

$$x'' + 2x' + 5x = 0 \,.$$

Solution. The characteristic polynomial is

$$p(z) = z^2 + 2z + 5 = (z + 1)^2 + 4 = (z + 1)^2 + 2^2 \,,$$

which has the conjugate pair of roots $-1 + i2$ and $-1 - i2$. Therefore a general solution is

$$x = c_1 e^{-t} \cos(2t) + c_2 e^{-t} \sin(2t) \,.$$

Example. Find a general solution of

$$y'' + 9y = 0 \,.$$

Solution. The characteristic polynomial is

$$p(z) = z^2 + 9 = z^2 + 3^2 \,,$$

which has the conjugate pair of roots $i3$ and $-i3$. Therefore a general solution is

$$y = c_1 \cos(3t) + c_2 \sin(3t) \,.$$

Example. Find a general solution of

$$(\mathrm{D} + 5)^3(\mathrm{D}^2 + 4\mathrm{D} + 5)(\mathrm{D}^2 + 4)h = 0 \,, \qquad \text{where} \quad \mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}t} \,.$$

Solution. The characteristic polynomial is

$$p(z) = (z + 5)^3(z^2 + 4z + 5)(z^2 + 4) = (z + 5)^3\big((z + 2)^2 + 1^2\big)\big(z^2 + 2^2\big) \,,$$

which has the real roots -5, -5, -5, the conjugate pair of roots $-2 + i$, $-2 - i$, and the conjugate pair of roots $i2$, $-i2$. Therefore a general solution is

$$h = c_1 e^{-5t} + c_2 t\, e^{-5t} + c_3 t^2 e^{-5t} + c_4 e^{-2t} \cos(t) + c_5 e^{-2t} \sin(t) + c_6 \cos(2t) + c_7 \sin(2t) \,.$$

4.4.2. *Complex Roots with any Multiplicity.* The fundamental theorem of algebra says that any polynomial $p(z)$ of degree n can be written as the product of n linear factors:

$$p(z) = (z - z_1)(z - z_2) \cdots (z - z_n)\,,$$

where z_1, z_2, $\cdots$, z_n are complex numbers that are roots of $p(z)$. Then $p(z_j) = 0$ for each z_j. Conversely, if $p(r + is) = 0$ then $r + is = z_j$ for some z_j. We say $r + is$ is a root of $p(z)$ of multiplicity m if $r + is = z_j$ for m of the z_j. In other words, $r + is$ is a root of $p(z)$ of multiplicity m if $(z - r - is)^m$ is a factor of $p(z)$.

If all the coefficients of a polynomial $p(z)$ are real then it is called a *real polynomial*. Characteristic polynomials of linear differential operators in the form (4.3) are real polynomials. If $p(z)$ is a real polynomial and $r + is$ is a root of $p(z)$ of multiplicity m then its conjugate $r - is$ is also a root of $p(z)$ of multiplicity m. If $s > 0$ then this means that $(z - r - is)^m$ and $(z - r + is)^m$ are distinct complex factors of $p(z)$, which means that $\big((z - r)^2 + s^2\big)^m$ is a real factor of $p(z)$. Conversely, if $\big((z - r)^2 + s^2\big)^m$ is a factor of $p(z)$ for some real r and s and some positive integer m and if $s > 0$ then $r + is$ and $r - is$ are distinct roots of $p(z)$ of multiplicity m.

Example. Find all the roots of $p(z) = (z^3 - 2z^2)(z^2 - 2z + 10)^3(z^2 + 4z + 29)$.

Solution. Because the degree of a factored polynomial is the sum of the degrees of its factors, we see that the degree of $p(z)$ is $3 + 6 + 2 = 11$. Because $p(z)$ has degree 11, it must have 11 roots counting multiplicities. Because

$$p(z) = z^2(z - 2)\big((z - 1)^2 + 3^2\big)^3\big((z + 2)^2 + 5^2\big)\,,$$

the 11 roots are 0, 0, 2, $1 \pm i3$, $1 \pm i3$, $1 \pm i3$, $-2 \pm i5$. Here each $r \pm is$ denotes two distinct roots. The real root 0 has multiplicity 2 while the complex roots $1 + i3$ and $1 - i3$ have multiplicity 3.

Now let $p(z)$ be the characteristic polynomial of a linear n^{th}-order differential operator L in the form (4.3). We know that $p(z)$ has n complex roots counting multiplicities. We already know that if r is a real root of $p(z)$ of multiplicity m then $\mathrm{L}y = 0$ has the m linearly independent real solutions given by

$$e^{rt}\,, \qquad t\,e^{rt}\,, \qquad \cdots \qquad t^{m-1}e^{rt}\,. \tag{4.9}$$

Below we will show that if $r \pm is$ is a conjugate pair of roots of $p(z)$ of multiplicity m then $\mathrm{L}y = 0$ has the $2m$ real solutions

$$\begin{aligned} &e^{rt}\cos(st)\,, \qquad t\,e^{rt}\cos(st)\,, \qquad \cdots \qquad t^{m-1}e^{rt}\cos(st)\,, \\ &e^{rt}\sin(st)\,, \qquad t\,e^{rt}\sin(st)\,, \qquad \cdots \qquad t^{m-1}e^{rt}\sin(st)\,. \end{aligned} \tag{4.10}$$

Therefore the n roots of $p(z)$ generate n solutions of $\mathrm{L}y = 0$ by recipes (4.9) and (4.10). Moreover, it can be shown that these solutions are linearly independent, and thereby are a fundamental set of solutions for the problem.

Remark. The recipe (4.10) of $2m$ real solutions applies to any conjugate pair $r \pm is$ with $s > 0$ of roots of $p(z)$ of multiplicity m. Taking $s > 0$ is customary. Taking $s < 0$ would yield essentially the same real solutions because $\cos(st)$ is even and $\sin(st)$ is odd — i.e. $\cos(-st) = \cos(st)$ and $\sin(-st) = -\sin(st)$.

Example. Find a general solution of

$$\mathrm{D}^4 v + 8\mathrm{D}^2 v + 16v = 0\,, \qquad \text{where} \quad \mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}t}\,.$$

Solution. The characteristic polynomial is

$$p(z) = z^4 + 8z^2 + 16 = \big(z^2 + 4\big)^2 = \big(z^2 + 2^2\big)^2\,.$$

Its 4 roots are $\pm i2$, $\pm i2$. Therefore a general solution is

$$v = c_1 \cos(2t) + c_2 \sin(2t) + c_3 t\,\cos(2t) + c_4 t\,\sin(2t)\,.$$

Example. Find a general solution of

$$\big(\mathrm{D}^3 - 2\mathrm{D}^2\big)\big(\mathrm{D}^2 - 2\mathrm{D} + 10\big)^3\big(\mathrm{D}^2 + 4\mathrm{D} + 29\big)y = 0\,, \qquad \text{where} \quad \mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}t}\,.$$

Solution. The characteristic polynomial is

$$\begin{aligned} p(z) &= (z^3 - 2z^2)(z^2 - 2z + 10)^3(z^2 + 4z + 29) \\ &= z^2(z-2)\big((z-1)^2 + 3^2\big)^3\big((z+2)^2 + 5^2\big)\,. \end{aligned}$$

Its 11 roots are 0, 0, 2, $1 \pm i3$, $1 \pm i3$, $1 \pm i3$, $-2 \pm i5$. Therefore a general solution is

$$\begin{aligned} y = c_1 &+ c_2 t + c_3 e^{2t} + c_4 e^t \cos(3t) + c_5 e^t \sin(3t) + c_6 t\, e^t \cos(3t) + c_7 t\, e^t \sin(3t) \\ &+ c_8 t^2 e^t \cos(3t) + c_9 t^2 e^t \sin(3t) + c_{10} e^{-2t} \cos(5t) + c_{11} e^{-2t} \sin(5t)\,. \end{aligned}$$

Recall that recipe (4.9) was derived by evaluating the Key Identity and its first $m-1$ derivatives at $z = r$, and using the fact that $p(r) = p'(r) = \cdots = p^{(m-1)}(r) = 0$ when r is a real root of $p(z)$ of multiplicity m. Recipe (4.10) is derived in a similar way. If $r + is$ is a complex root of $p(z)$ of multiplicity m then $(z - r - is)^m$ is a factor of $p(z)$. We can differentiate polynomials with respect to the complex variable z exactly as if it were a real variable. Because $(z - r - is)^m$ is a factor of $p(z)$ — i.e. because $p(z) = (z - r - is)^m q(z)$, we can show that

$$p(r + is) = p'(r + is) = \cdots = p^{(m-1)}(r + is) = 0\,.$$

We can also differentiate e^{zt} with respect to the complex variable z exactly as if it were a real variable. Differentiation of the Key Identity (4.7) k times with respect to z gives

$$\mathrm{L}\big(t^k e^{zt}\big) = p(z)\, t^k e^{zt} + kp'(z)\, t^{k-1} e^{zt} + \cdots + kp^{(k-1)}(z)\, t\, e^{zt} + p^{(k)}(z)\, e^{zt}\,.$$

Evaluating this at $z = r + is$ when $k = 1, \cdots, m-1$ shows that

$$\begin{aligned} \mathrm{L}\big(t^k e^{(r+is)t}\big) &= p(r+is)\, t^k e^{(r+is)t} + kp'(r+is)\, t^{k-1} e^{(r+is)t} + \cdots \\ &\qquad \cdots + kp^{(k-1)}(r+is)\, t\, e^{(r+is)t} + p^{(k)}(r+is)\, e^{(r+is)t} = 0\,. \end{aligned}$$

Similarly, we can show that $\mathrm{L}\big(t^k e^{(r-is)t}\big) = 0$. Recipe (4.10) then follows by the taking real and imaginary parts of these complex-valued solutions.

4.5. **Summary of the Recipe.** Consider an n^{th}-order linear differential operator L with real constant coefficients in the normal form

$$\mathrm{L} = \mathrm{D}^n + a_1\mathrm{D}^{n-1} + \cdots + a_{n-1}\mathrm{D} + a_n\,, \quad \text{where} \quad \mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}t}\,. \tag{4.11}$$

We can generate a fundamental set of solutions for the homogeneous equation $\mathrm{L}y = 0$ from the characteristic polynomial of L, which is given by

$$p(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + a_n\,. \tag{4.12}$$

More specifically, we can generate a fundamental set of solutions for the homogeneous equation $\mathrm{L}y = 0$ from the roots of $p(z)$. The following facts should be kept in mind.

- Because $p(z)$ has degree n, the *Fundamental Theorem of Algebra* states that $p(z)$ has n complex roots counting multiplicities.
- Because the coefficients of $p(z)$ are real, these roots must either be real or come in conjugate pairs.
- When the factors of $p(z)$ are known, these roots can be read off as follows.
  - A factor $(z-r)^m$ corresponds to a real root r of multiplicity m.
  - A factor $\big((z-r)^2+s^2\big)^m$ for some $s>0$ corresponds to a conjugate pair of roots $r \pm is$ of multiplicity m.

  A root is called simple when its multiplicity $m = 1$.

After these roots and their multiplicities have been determined, a fundamental set of real solutions for the homogeneous equation $\mathrm{L}y = 0$ is constructed as follows.

- Each real simple root r yields the solution

$$e^{rt}\,. \tag{4.13a}$$

- Each real root r of multiplicity m yields the m solutions

$$e^{rt}\,, \qquad t\,e^{rt}\,, \qquad \cdots \qquad t^{m-1}e^{rt}\,. \tag{4.13b}$$

- Each conjugate pair of simple roots $r \pm is$ yields the two solutions

$$e^{rt}\cos(st)\,, \qquad e^{rt}\sin(st)\,. \tag{4.13c}$$

- Each conjugate pair of roots $r \pm is$ of multiplicity m yields the $2m$ solutions

$$\begin{aligned} &e^{rt}\cos(st)\,, \qquad t\,e^{rt}\cos(st)\,, \qquad \cdots \qquad t^{m-1}e^{rt}\cos(st)\,,\\ &e^{rt}\sin(st)\,, \qquad t\,e^{rt}\sin(st)\,, \qquad \cdots \qquad t^{m-1}e^{rt}\sin(st)\,. \end{aligned} \tag{4.13d}$$

Therefore the n roots of $p(z)$ generate n real solutions of $\mathrm{L}y = 0$ by recipe (4.13). Moreover, these solutions are linearly independent, and thereby are a fundamental set of solutions for the homogeneous equation $\mathrm{L}y = 0$.

Remark. When $n > 2$ the hardest part of applying this recipe is determining the n complex roots of $p(z)$. This task becomes easier when $p(z)$ is given in a factored or partially factored form.

Remark. Recipe (4.13) was derived from the Key Identity (4.7) and its derivatives with respect to z. Knowing this derivation will be useful in Chapter 6 where we will apply the Key Identity to certain nonhomogeneous equations.

Exercises on Homogeneous Equations with Constant Coefficients

Find the general solution to the following differential equations.
(Make sure to find the correct roots to your characteristic polynomial!!!)

(1) $y'' + 3y' + 2y = 0$

Short Answer
Solution

(2) $x'' + 3x' = 0$

Short Answer
Solution

(3) $4y'' - y = 0$

Short Answer
Solution

(4) $y'' + 2y' + 2y = 0$

Short Answer
Solution

(5) $4y'' + 4y' + 5y = 0$

Short Answer
Solution

(6) $\ddot{z} + 9z = 0$

Short Answer
Solution

(7) $y'' - 6y' + 9y = 0$

Short Answer
Solution

(8) $4y'' + 12y' + 9y = 0$

Short Answer
Solution

(9) $5y'' - 4y' + \frac{4}{5}y = 0$

Short Answer
Solution

Find the solution to the following initial value problems. As $t \to \infty$, what happens to the solution of the following problems? (e.g. Does it approach 0? $-\infty$? ∞? a certain value?)

(10) $2y'' + y' - 3y = 0$ $\qquad y(0) = 0, \qquad y'(0) = 1$

Short Answer
Solution

(11) $6\ddot{w} - 5\dot{w} + w = 0$ $\quad w(0) = 4, \quad \dot{w}(0) = 0$

Short Answer
Solution

(12) $y'' + 4y' + 3y = 0$ $\quad y(0) = 2, \quad y'(0) = -1$

Short Answer
Solution

(13) $w'' + 4w' + 5w = 0$ $\quad w(0) = 1, \quad w'(0) = 0$

Short Answer
Solution

(14) $y'' - 2y' + 5y = 0\,,$ $\quad y(\frac{\pi}{2}) = 0\,, \quad y'(\frac{\pi}{2}) = 2\,.$

Short Answer
Solution

(15) $y'' + 4y = 0$ $\quad y(0) = 0, \quad y'(0) = 1$

Short Answer
Solution

(16) $y'' + 4y' + 4y = 0$ $\quad y(-1) = 2, \quad y'(-1) = 1$

Short Answer
Solution

(17) $4z'' + 12z' + 9z = 0$ $\quad z(0) = 1, \quad z'(0) = -4$

Short Answer
Solution

(18) $9y'' - 12y' + 4y = 0$ $\quad y(0) = 2, \quad y'(0) = -1$

Short Answer
Solution

For #19 and #20, which values of α do ALL possible solutions tend to 0 as $t \to \infty$?

(19) $y'' + 2(\alpha - 1)y' + \alpha(\alpha - 2)y = 0$

Short Answer
Solution

(20) $y'' + (5 - \alpha)y' - 3(\alpha - 2)y = 0$

Short Answer
Solution

(21) Find a general solution of $\mathrm{D}^4 z + 18\mathrm{D}^2 z + 81z = 0$, where $\mathrm{D} = \frac{d}{dw}$.

Short Answer
Solution

(22) Given the roots of the characteristic polynomial to be $-1, 0, 2, 2, 2, -3\pm i, -3\pm i$, give a general solution as well as the order of the original differential equation.

Solution

(23) Solve the initial-value problem:

$$w'' - w' - 2w = 0\ , w(0) = \alpha\ , w'(0) = 2\ .$$

Find α so that the solution approaches 0 as z approaches $+\infty$.

Short Answer

Solution

(24) Solve the initial-value problem:

$$4y'' - y = 0 \ , y(0) = 2 \ , y'(0) = \beta.$$

Then find β so that the solution approaches 0 as x approaches $+\infty$.

Short Answer
Solution

(25) The following two problems constitute an exploration of **Euler's formula**.
(a) Use Euler's formula to write e^{3+5i} in the form $a + ib$.
(b) Use Euler's formula to write $e^{2-(\frac{\pi}{2})i}$ in the form $a + ib$.
(c) Use Euler's formula to write 3^{1-i} in the form $a + ib$.

Short Answer
Solution

(26) (a) Show that $x_1(t) = \cos(t)$ and $x_2(t) = \sin(t)$ are a fundamental set of solutions of $x'' + x = 0$; that is, show their are solutions of the differential equation and that their Wronskian is not zero.
(b) Show that $x = e^{it}$ is also a solution of $x'' + x = 0$. Therefore, this implies $e^{it} = c_1 \cos(t) + c_2 \sin(t)$, for some constants c_1 and c_2. Can you explain why?
(c) Set $t = 0$ in the equation above. What do you obtain?
(d) Given that $\frac{d}{dt}(e^{it}) = ie^{it}$, differentiate $e^{it} = c_1 \cos(t) + c_2 \sin(t)$ and set $t = 0$ in this newly obtained equation. What formula do you now obtain? (**Hint** Given the preamble above, you should end up with Euler's Formula. Now you've proved it! :))

Short Answer
Solution

(27) Solve the initial value problem $y'' - 6y' + 9\alpha y = 0$, $y(0) = 0$. Describe and study its behavior for increasing t (i.e. take t approaching $+\infty$) and varrying values of the real parameter α. Is your solution to the second-order linear homogeneous equation above unique? Why? Why not?
Hint : In your exploration of α, your argument should contain three cases.

Short Answer
Solution

(28) Consider the initial-value problem:

$$4y'' + 12y' + 9y = 0 \ , y(0) = 1 \ , y'(0) = -4 \ .$$

(a) Solve the initial-value problem;
(b) Determine where the solutions attains the value 0 ;
(c) Determine the coordinates of the minimum point (if there is one) (x^*, y^*);
(d) Change the second initial condition to $y'(0) = \beta$ and rewrite your solution in terms of β. Next, find the value of β that separates solutions that are always negative from solutions that are always positive (**Hint** : Not a trick question. Such a value of β does indeed exist :)).

Short Answer
Solution

(29) Given the second-order homogeneous ordinary differential equation $w'' + (2 + \alpha)w' + 2\alpha w = 0$, determine what the general solution to the differential equation looks like given varrying values of the real parameter α.

Short Answer
Solution

(30) Consider the initial value problem

$$w'' + 5w' + 6w = 0,\ \ w(0) = 2,\ \ w'(0) = 3.$$

Find the maximum value attained by the solution.

Short Answer
Solution

(31) Write down a second-order, linear homogeneous differential equation with constant coefficients that has solutions:
(a) $X_1(t) = e^t$ and $X_2(t) = e^{-4t}$;
(b) $Z_1(w) = e^{3w}$ and $Z_2(w) = e^{2w}$;
(c) $Y_1(x) = e^{2x}$ and $Y_2(x) = x \cdot e^{2x}$;
(d) $Y_1(t) = e^{\frac{5}{4}t}$ and $Y_2(t) = te^{\frac{5}{4}t}$;
(e) $Z_1(x) = e^x \cos(2x)$ and $Z_2(x) = e^x \sin(2x)$.
(f) $W_1(t) = e^t$ and $W_2(t) = e^{\alpha t}$, where $\alpha \neq 1$, a real parameter.
(g) $W_1(u) = e^{\alpha u} \cos(\beta u)$ and $W_2(u) = e^{\alpha u} \sin(\beta u)$, where α and β are real parameters.

Short Answer
Solution

(32) On page 5 of the Lecture Notes, work out the details in the computation of the Wronskian $det \begin{pmatrix} e^{rt} & re^{rt} & r^2e^{rt} \\ te^{rt} & rte^{rt} + e^{rt} & r^2te^{rt} + 2re^{rt} \\ t^2e^{rt} & rt^2e^{rt} + 2te^{rt} & 2e^{rt} + 4te^{rt} + r^2t^2e^{rt} \end{pmatrix}$, of the solutions $Y_1(t) = e^{rt}, Y_2(t) = te^{rt}$ and $Y_3(t) = t^2e^{rt}$.
What can you conclude from here?

Short Answer
Solution

(33) Suppose that $Y_1(x)$ and $Y_2(x)$ are solutions of the differential equation

$$\ddot{y} + 2\dot{y} + (1 + x^2)y = 0.$$

Suppose you know that $W[Y_1, Y_2](0) = 5$. What is $W[Y_1, Y_2](x)$?

Short Answer
Solution

(34) Suppose that $X_1(t)$ and $X_2(t)$ are solutions of the differential equation

$$x''' + 2cos(t)x' + (e^t + t^2)x = 0.$$

Suppose you know that $W[X_1, X_2](1) = 3578$. What is $W[Y_1, Y_2](25)$?

Short Answer
Solution

(35) Compute the Wronskian of the functions $X_1(u) = \cos(5u)$ and $X_2(u) = \sin(5u)$. (Evaluate the determinant and simplify)

Short Answer
Solution

(36) For each of the following determine if the statement is TRUE or FALSE and justify your response:

(a) The functions $f(w) = w^2 + 3$ and $g(w) = w^2 + 2$ (defined on the real line) are linearly independent.

(b) Let y be the function defined on the real line by the rule $y(u) = 1 - \cos(u)$. This function cannot be the solution of a differential equation of the form

$$\ddot{y} + p(u)\dot{y} + q(u)y = 0$$

where p and q are continuous on the real line.

Remark In answering this question, you might have to cite material (or review :)) material from previous chapters.

Short Answer
Solution

(37) Consider the following statement:

"If $y = f(x)$ and $y = g(x)$ are solutions to the differential equation $y'' + 4y' + 2y = e^x$ then $|f(x) - g(x)|$ goes to zero as t goes to $+\infty$." Is this statement true or false? Briefly justify your answer.

Remark We haven't yet discussed how to solve second-order linear nonhomogeneous differential equations, but we don't need to know how to do that in order to solve this problem.

Short Answer
Solution

(38) Consider the linear differential equation

$$y'' - \frac{1}{z}y' + \frac{3}{z}y = 0 \ , z > 0.$$

Let y_1 and y_2 be the solutions of the differential equation satisfying the initial conditions $y_1(1) = 0, y_1'(1) = 0, y_2(1) = 0, y_2'(1) = 1$. Let W be the Wronskian $W(z) = y_1(z)y_2'(z) - y_1'(z)y_2(z)$.

What is $W(1)$? Without solving the differential equation explicitly, find an exact formula for $W(z)$ (for general $z > 0$).

Short Answer
Solution

(39) Find the Wronskian of w_1, w_2 where

(a) $w_1 = \sin(t)$, $w_2 = \sin(t)\log(t)$; is the Wronskian defined for all t?

(b) $w_1 = x^2$, $w_2 = \log(x)$.

Short Answer
Solution

(40) The Wronskian of two functions $w_1(u)$ and $w_2(u)$ equals $\log(u)$. Can w_1 and w_2 be solutions of a differential equation $w'' + p(u)w' + q(u)w = 0$, where $p(u)$

and $q(u)$ are continuous on the interval $(\frac{1}{2}, \frac{3}{2})$? You should include complete sentences in your justification.

Solution

(41) Find the Wronskian of the pair of functions $\beta\cos^2(\theta), \alpha\,(1+\cos(2\theta))$ and determine the value(s) of α, β for which the Wronskian is nonzero. Can you think of a possible explanation for why you obtain that particular answer?

Solution

(42) Show that if $w = f(u)$ is a solution of the differential equation $w'' + p(u)w' + q(u)w = g(u)$, where $g(u)$ is not always 0, then $w = cf(u)$, where c is any constant other than 1, is not a solution. Why this doesn't contradict the principle of linear superposition?

Solution

(43) For what values of the parameter γ are the characteristic roots to the following differential equation (a) distinct real, (b) real with multiplicity, or (c) complex conjugates?

$$y'' + \gamma y' + 1 = 0.$$

In each case, write down a general solution to the corresponding differential equation.

Short Answer
Solution

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