III. First-Order Systems of Ordinary Differential Equations
1. Introduction to First-Order Systems

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Contents

Return to Main Webpage

1. Introduction to First-Order Systems

1.1. **Normal Forms and Solutions.** For the remainder of the course we will study first-order systems of n ordinary differential equations for functions $x_j(t)$, $j = 1, 2, \cdots, n$ that can be put into the normal form

$$\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}t} &= f_1(t, x_1, x_2, \cdots, x_n)\,, \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= f_2(t, x_1, x_2, \cdots, x_n)\,, \\ &\vdots \\ \frac{\mathrm{d}x_n}{\mathrm{d}t} &= f_n(t, x_1, x_2, \cdots, x_n)\,. \end{aligned} \tag{1.1}$$

We say that n is the *dimension* of this system.

System (1.1) can be expressed more compactly in *vector notation* as

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(t, \mathbf{x})\,, \tag{1.2}$$

where $\mathbf{x}$ and $\mathbf{f}(t, \mathbf{x})$ are given by the n-dimensional *column vectors*

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \qquad \mathbf{f}(t, \mathbf{x}) = \begin{pmatrix} f_1(t, x_1, x_2, \cdots, x_n) \\ f_2(t, x_1, x_2, \cdots, x_n) \\ \vdots \\ f_n(t, x_1, x_2, \cdots, x_n) \end{pmatrix}.$$

We thereby express the system of n equations (1.1) as the single vector equation (1.2). We say x_1, x_2, $\cdots$, x_n are the *entries* of the vector $\mathbf{x}$. Similarly, we say that the functions $f_1(t, x_1, x_2, \cdots, x_n)$, $f_2(t, x_1, x_2, \cdots, x_n)$, $\cdots$, $f_n(t, x_1, x_2, \cdots, x_n)$ are the entries of the vector-valued function $\mathbf{f}(t, \mathbf{x})$.

Remark. We will use boldface, lowercase letters like $\mathbf{x}$ and $\mathbf{f}$ to denote column vectors. Other common notations include an underline like $\underline{x}$ and $\underline{f}$, or an arrow like $\vec{x}$ and $\vec{f}$. Many advanced books do not use any special notation for vectors, but expect the reader to recall what each letter represents from when it was introduced.

Here we recall from multi-variable calculus what it means for a vector-valued function $\mathbf{u}(t)$ to be either continuous or differentiable at a point.

- We say $\mathbf{u}(t)$ is *continuous at* time t if *every entry* of $\mathbf{u}(t)$ is continuous at t.
- We say $\mathbf{u}(t)$ is *differentiable at* time t if *every entry* of $\mathbf{u}(t)$ is differentiable at t.

Given these definitions, we define what it means for a vector-valued function $\mathbf{u}(t)$ to be either continuous, differentiable, or continuously differentiable over a time interval.

- We say $\mathbf{u}(t)$ is *continuous over* a time interval (t_L, t_R) if it is continuous at every t in (t_L, t_R).
- We say $\mathbf{u}(t)$ is *differentiable over* a time interval (t_L, t_R) if it is differentiable at every t in (t_L, t_R).
- We say $\mathbf{u}(t)$ is *continuously differentiable over* a time interval (t_L, t_R) if it is differentiable over (t_L, t_R) and its derivative is continuous over (t_L, t_R).

We are now ready to define what we mean by a solution of system (1.2).

Definition. We say that $\mathbf{x}(t)$ is a *solution* of system (1.2) over a time interval $(t_{\text{L}}, t_{\text{R}})$ when

1. $\mathbf{x}(t)$ is differentiable at every t in $(t_{\text{L}}, t_{\text{R}})$;
2. $\mathbf{f}(t, \mathbf{x}(t))$ is defined for every t in $(t_{\text{L}}, t_{\text{R}})$;
3. equation (1.2) holds at every t in $(t_{\text{L}}, t_{\text{R}})$.

Remark. This definition is similar to definitions of solutions to single differential equations that we gave earlier. The first point states that the left-hand side of the equation makes sense. The second point states that the right-hand side of the equation makes sense. The third point states that the two sides are equal.

1.2. **Initial-Value Problems.** We will consider initial-value problems of the form

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(t, \mathbf{x}), \qquad \mathbf{x}(t_I) = \mathbf{x}^I. \tag{1.3}$$

Here t_I is the *initial time*, $\mathbf{x}^I$ is the *initial value* or *initial data*, and $\mathbf{x}(t_I) = \mathbf{x}^I$ is the *initial condition*. Below we will give conditions on $\mathbf{f}(t, \mathbf{x})$ that insure this problem has a unique solution that exists over some time interval that contains t_I.

We begin with a definition.

Definition 1.1. *Let S be a set in $\mathbb{R} \times \mathbb{R}^n$. A point $(t_o, \mathbf{x}_o)$ is said to be in the* interior *of S if there exists a box $(t_L, t_R) \times (x_1^L, x_1^R) \times \cdots \times (x_n^L, x_n^R)$ that contains the point $(t_o, \mathbf{x}_o)$ and also lies within the set S.*

Our basic existence and uniqueness theorem is the following.

Theorem 1.1. *Let $\mathbf{f}(t, \mathbf{x})$ be a vector-valued function defined over a set S in $\mathbb{R} \times \mathbb{R}^n$ such that*

- *$\mathbf{f}$ is continuous over S,*
- *$\mathbf{f}$ is differentiable with respect to each x_i over S,*
- *each $\partial_{x_i}\mathbf{f}$ is continuous over S.*

Then for every inital time t_I and every initial value $\mathbf{x}^I$ such that $(t_I, \mathbf{x}^I)$ is in the interior of S there exists a unique solution $\mathbf{x}(t)$ to initial-value problem (1.3) that is defined over some time interval (a, b) such that

- *t_I is in (a, b),*
- *$\{(t, \mathbf{x}(t)) : t \in (a, b)\}$ lies within the interior of S.*

Moreover, $\mathbf{x}(t)$ extends to the largest such time interval and $\mathbf{x}'(t)$ is continuous over that time interval.

Remark. This is not the most general theorem we could state, but it applies to the first-order systems you will face in this course. It asserts that the initial-value problem (1.3) has a unique solution $\mathbf{x}(t)$ that will exist until $(t, \mathbf{x}(t))$ leaves the interior of S.

1.3. **Recasting Higher-Order Problems as First-Order Systems.** Many higher-order differential equation problems can be recast in terms of a first-order system in the normal form (1.2). For example, every n^{th}-order ordinary differential equation in the normal form

$$y^{(n)} = g\big(t, y, y', \cdots, y^{(n-1)}\big)\ ,$$

can be expressed as an n-dimensional first-order system in the form (1.2) with

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(t,\mathbf{x}) = \begin{pmatrix} x_2 \\ \vdots \\ x_n \\ g(t, x_1, x_2, \cdots, x_n) \end{pmatrix}, \qquad \text{where} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y \\ y' \\ \vdots \\ y^{(n-1)} \end{pmatrix}.$$

Notice that the first-order system is expressed solely in terms of the entries of $\mathbf{x}$. The "dictionary" that relates $\mathbf{x}$ to y, y', $\cdots$, $y^{(n-1)}$ is given as a separate equation.

Example. Recast as a first-order system

$$y''' + yy' + e^t y^2 = \cos(3t)\,.$$

Solution. Because this single equation is third order, the first-order system will have dimension three. It will be

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ \cos(3t) - x_1 x_2 - e^t x_1^2 \end{pmatrix}, \qquad \text{where} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y \\ y' \\ y'' \end{pmatrix}.$$

More generally, every d-dimensional m^{th}-order ordinary differential system in the normal form

$$\mathbf{y}^{(m)} = \mathbf{g}\big(t, \mathbf{y}, \mathbf{y}', \cdots, \mathbf{y}^{(m-1)}\big)\ ,$$

can be expressed as an md-dimensional first-order system in the form (1.2) with

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(t,\mathbf{x}) = \begin{pmatrix} \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_m \\ \mathbf{g}(t, \mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_m) \end{pmatrix}, \qquad \text{where} \quad \mathbf{x} = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_m \end{pmatrix} = \begin{pmatrix} \mathbf{y} \\ \mathbf{y}' \\ \vdots \\ \mathbf{y}^{(m-1)} \end{pmatrix}.$$

Here each $\mathbf{x}_k$ is a d-dimensional vector while $\mathbf{x}$ is the md-dimensional vector constructed by stacking the vectors $\mathbf{x}_1$ through $\mathbf{x}_m$ on top of each other.

Example. Recast as a first-order system

$$q_1'' + f_1(q_1, q_2) = 0\,, \qquad q_2'' + f_2(q_1, q_2) = 0\,.$$

Solution. Because this two dimensional system is second order, the first-order system will have dimension four. It will be

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_3 \\ x_4 \\ -f_1(x_1, x_2) \\ -f_2(x_1, x_2) \end{pmatrix}, \qquad \text{where} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \\ q_1' \\ q_2' \end{pmatrix}.$$

When faced with a higher-order initial-value problem, we use the dictionary to obtain the initial values for the first-order system from those for the higher-order problem.

Example. Recast as an initial-value problem for a first-order system

$$y'''' - e^y = 0\,, \qquad y(0) = 2\,, \quad y'(0) = -1\,, \quad y''(0) = 5\,, \quad y'''(0) = -4\,.$$

Solution. The first-order initial-value problem is

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ e^{x_1} \end{pmatrix}, \qquad \begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ x_4(0) \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 5 \\ -4 \end{pmatrix}, \qquad \text{where} \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} y \\ y' \\ y'' \\ y''' \end{pmatrix},$$

Remark. We can also find single higher-order equations that are satisfied by the entries of a first-order system. We will not discuss how this is done because it is not as useful.

1.4. **Numerical Methods.** One advantage of expressing an initial-value problem in the form of a first-order system is that we can then apply all the numerical methods that we studied earlier in the setting of single first-order equations. In fact, the most common way in which numerical methods are applied to construct a numerical approximation of the solution to an initial-value problem for a higher-order equation is to recast it as an initial-value problem for a first-order system and then apply such numerical methods.

Suppose we wish to construct a numerical approximation over the time interval $[t_I, t_F]$ to the solution $\mathbf{x}(t)$ of the initial-value problem

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(t, \mathbf{x})\,, \qquad \mathbf{x}(t_I) = \mathbf{x}^I\,. \tag{1.4}$$

A numerical method selects times $\{t_n\}_{n=0}^N$ such that

$$t_I = t_0 < t_1 < t_2 < \cdots < t_{N-1} < t_N = t_F\,,$$

and computes vectors $\{\mathbf{x}_n\}_{n=0}^N$ such that $\mathbf{x}_0 = \mathbf{x}^I$ and $\mathbf{x}_n$ approximates $\mathbf{x}(t_n)$ for $n = 1, 2, \cdots, N$. If we do this by using N uniform time steps (as we did earlier) then we set

$$h = \frac{t_F - t_I}{N}\,, \qquad \text{and} \qquad t_n = t_I + nh \quad \text{for } n = 0, 1, \cdots, N\,,$$

where h is called the *time step.* Below we show that the vectors $\{\mathbf{x}_n\}_{n=0}^N$ can be computed easily by the four explicit methods that we studied earlier for scalar-valued equations: the explicit Euler, Runge-trapezoidal, Runge-midpoint, and Runge-Kutta methods. The only modification that we need to make in these explicit methods is to replace the scalar-valued dependent variables and functions with vector-valued ones. The justifications of these methods also carry over upon making the same modification.

Remark. Implicit methods such as the implicit Euler method are often extremely useful for computing approximate solutions for first-order systems, but are more complicated to implement because they require the numerical solution of algebraic systems, which is beyond the scope of this course.

Explicit Euler Method. The vector-valued version of this method is as follows.

Set $\mathbf{x}_0 = \mathbf{x}^I$ and then for $n = 0, \cdots, N-1$ cycle through

$$\mathbf{f}_n = \mathbf{f}(t_n, \mathbf{x}_n)\,, \qquad \mathbf{x}_{n+1} = \mathbf{x}_n + h\mathbf{f}_n\,,$$

where $t_n = t_I + nh$.

Remark. Like its scalar-valued version, this method is first-order.

Runge-Trapezoidal Method. The vector-valued version of this method is as follows.

Set $\mathbf{x}_0 = \mathbf{x}^I$ and then for $n = 0, \cdots, N-1$ cycle through

$$\begin{aligned} \mathbf{f}_n &= \mathbf{f}(t_n, \mathbf{x}_n)\,, & \tilde{\mathbf{x}}_{n+1} &= \mathbf{x}_n + h\mathbf{f}_n\,, \\ \tilde{\mathbf{f}}_{n+1} &= \mathbf{f}(t_{n+1}, \tilde{\mathbf{x}}_{n+1})\,, & \mathbf{x}_{n+1} &= \mathbf{x}_n + \tfrac{1}{2}h[\mathbf{f}_n + \tilde{\mathbf{f}}_{n+1}]\,, \end{aligned}$$

where $t_n = t_I + nh$.

Remark. Like its scalar-valued version, this method is second-order. It requires twice as many function evaluations per time step as the explicit Euler method. Because it is second order, this method often outperforms the explicit Euler method because the same error often can be realized with a time step that is more than twice as large.

Runge-Midpoint Method. The vector-valued version of this method is as follows.

Set $\mathbf{x}_0 = \mathbf{x}^I$ and then for $n = 0, \cdots, N-1$ cycle through

$$\begin{aligned} \mathbf{f}_n &= \mathbf{f}(t_n, \mathbf{x}_n)\,, & \mathbf{x}_{n+\frac{1}{2}} &= \mathbf{x}_n + \tfrac{1}{2}h\mathbf{f}_n\,, \\ \mathbf{f}_{n+\frac{1}{2}} &= \mathbf{f}(t_{n+\frac{1}{2}}, \mathbf{x}_{n+\frac{1}{2}})\,, & \mathbf{x}_{n+1} &= \mathbf{x}_n + h\mathbf{f}_{n+\frac{1}{2}}\,, \end{aligned}$$

where $t_n = t_I + nh$ and $t_{n+\frac{1}{2}} = t_I + (n + \frac{1}{2})h$.

Remark. Like its scalar-valued version, this method is second-order. It has the same number of function evaluations per time step as the Runge-trapeziodal method. Because they are the same order, their performances are comparable.

Runge-Kutta Method. The vector-valued version of this method is as follows.

Set $\mathbf{x}_0 = \mathbf{x}^I$ and then for $n = 0, \cdots, N-1$ cycle through

$$\begin{aligned} \mathbf{f}_n &= \mathbf{f}(t_n, \mathbf{x}_n)\,, & \tilde{\mathbf{x}}_{n+\frac{1}{2}} &= \mathbf{x}_n + \tfrac{1}{2}h\mathbf{f}_n\,, \\ \tilde{\mathbf{f}}_{n+\frac{1}{2}} &= \mathbf{f}(t_{n+\frac{1}{2}}, \tilde{\mathbf{x}}_{n+\frac{1}{2}})\,, & \mathbf{x}_{n+\frac{1}{2}} &= \mathbf{x}_n + \tfrac{1}{2}h\tilde{\mathbf{f}}_{n+\frac{1}{2}}\,, \\ \mathbf{f}_{n+\frac{1}{2}} &= \mathbf{f}(t_{n+\frac{1}{2}}, \mathbf{x}_{n+\frac{1}{2}})\,, & \tilde{\mathbf{x}}_{n+1} &= \mathbf{x}_n + h\mathbf{f}_{n+\frac{1}{2}}\,, \\ \tilde{\mathbf{f}}_{n+1} &= \mathbf{f}(t_{n+1}, \tilde{\mathbf{x}}_{n+1})\,, & \mathbf{x}_{n+1} &= \mathbf{x}_n + \tfrac{1}{6}h\big[\mathbf{f}_n + 2\tilde{\mathbf{f}}_{n+\frac{1}{2}} + 2\mathbf{f}_{n+\frac{1}{2}} + \tilde{\mathbf{f}}_{n+1}\big]\,, \end{aligned}$$

where $t_n = t_I + nh$ and $t_{n+\frac{1}{2}} = t_I + (n + \frac{1}{2})h$.

Remark. Like its scalar-valued version, this method is fourth-order. It requires twice as many function evaluations per time step as either second-order method and four times more than the explicit Euler method. However, because it is fourth order, the same error often can be realized with a time step that is more than twice as large as that for either second-order method. In such cases, this method outperforms all the foregoing methods. Variants of this method are among the most widely used numerical methods for approximating the solution of initial-value problems. The variant used by the MATLAB command “ode45” is the Dormand-Prince method.

Remark. In addition to the four methods given above, here are vector-valued versions of the classical third-order methods due to *Heun* and *Kutta*.

Heun Method. Set $\mathbf{x}_0 = \mathbf{x}^I$ and then for $n = 0, \cdots, N-1$ cycle through

$$\begin{aligned}
\mathbf{f}_n &= \mathbf{f}(t_n, \mathbf{x}_n)\,, & \mathbf{x}_{n+\frac{1}{3}} &= \mathbf{x}_n + \tfrac{1}{3}h\mathbf{f}_n\,,\\
\mathbf{f}_{n+\frac{1}{3}} &= \mathbf{f}\big(t_{n+\frac{1}{3}}, \mathbf{x}_{n+\frac{1}{3}}\big)\,, & \mathbf{x}_{n+\frac{2}{3}} &= \mathbf{x}_n + \tfrac{2}{3}h\mathbf{f}_{n+\frac{1}{3}}\,,\\
\mathbf{f}_{n+\frac{2}{3}} &= \mathbf{f}\big(t_{n+\frac{2}{3}}, \mathbf{x}_{n+\frac{2}{3}}\big)\,, & \mathbf{x}_{n+1} &= \mathbf{x}_n + \tfrac{1}{4}h\big[\mathbf{f}_n + 3\mathbf{f}_{n+\frac{2}{3}}\big]\,,
\end{aligned}$$

where $t_n = t_I + nh$, $t_{n+\frac{1}{3}} = t_I + (n+\frac{1}{3})h$, and $t_{n+\frac{2}{3}} = t_I + (n+\frac{2}{3})h$.

Kutta Method. Set $\mathbf{x}_0 = \mathbf{x}^I$ and then for $n = 0, \cdots, N-1$ cycle through

$$\begin{aligned}
\mathbf{f}_n &= \mathbf{f}(t_n, \mathbf{x}_n)\,, & \mathbf{x}_{n+\frac{1}{2}} &= \mathbf{x}_n + \tfrac{1}{2}h\mathbf{f}_n\,,\\
\mathbf{f}_{n+\frac{1}{2}} &= \mathbf{f}\big(t_{n+\frac{1}{2}}, \mathbf{x}_{n+\frac{1}{2}}\big)\,, & \tilde{\mathbf{x}}_{n+1} &= \mathbf{x}_n + h\big[-\mathbf{f}_n + 2\mathbf{f}_{n+\frac{1}{2}}\big]\,,\\
\tilde{\mathbf{f}}_{n+1} &= \mathbf{f}\big(t_{n+1}, \tilde{\mathbf{x}}_{n+1}\big)\,, & \mathbf{x}_{n+1} &= \mathbf{x}_n + \tfrac{1}{6}h\big[\mathbf{f}_n + 4\mathbf{f}_{n+\frac{1}{2}} + \tilde{\mathbf{f}}_{n+1}\big]\,,
\end{aligned}$$

where $t_n = t_I + nh$ and $t_{n+\frac{1}{2}} = t_I + (n+\frac{1}{2})h$.

1.5. **Application: Tank Problems.** First-order systems arise in many applications. In this section we show they arise from problems of describing interconnected tanks. These represent a broad class of problems that describe the transport of some quantity into and out of tanks or other volumes. The quantity might be a fluid like water, oil, or air, or it might be a substance like a solute or pollutant that is carried along by a fluid. The tanks might be well-defined volumes like ponds, lakes, or rooms in a building. These problems lie at the heart of many numerical simulations of fluids.

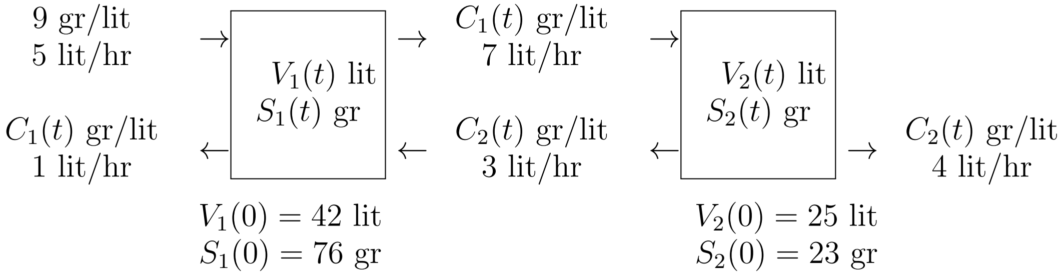
In such problems we construct an initial-value problem satisfied by the amounts Q_i of some quantity in the tank i. The associated system of ordinary differential equations will consist of equations in the form

$$\frac{\mathrm{d}Q_i}{\mathrm{d}t} = \text{RATE IN}_i - \text{RATE OUT}_i\,,$$

where RATE IN_i is the rate the quantity enters tank i while RATE OUT_i is the rate the quantity exits the tank i. There will be one such equation for each tank. For some problems RATE IN_i and RATE OUT_i will be given explicitly in the problem. At other times they will be given in terms of other variables in the problem. The way in which this is done is similar to the way we did it for the tank problems with just one tank that we studied earlier.

Example. Consider two interconnected tanks filled with brine (salt water). The first tank contains 42 liters and the second contains 25 liters. Brine with a salt concentration of 9 grams per liter flows into the first tank at 5 liters per hour. Well-stirred brine flows from the first tank into the second at 7 liters per hour, from the second into the first at 3 liters per hour, from the first into a drain at 1 liter per hour, and from the second into a drain at 4 liters per hour. At $t = 0$ there are 76 grams of salt in the first tank and 23 grams in the second. Give an initial-value problem that governs the amount of salt in each tank as a function of time.

Solution. Let $V_1(t)$ and $V_2(t)$ be the volumes (lit) of brine in the first and second tank at time t minutes. Let $S_1(t)$ and $S_2(t)$ be the mass (gr) of salt in the first and second tank at time t minutes. Because mixtures are assumed to be well-stirred, the salt concentration of the brine in the tanks at time t are $C_1(t) = S_1(t)/V_1(t)$ and $C_2(t) = S_2(t)/V_2(t)$ respectively. In particular, these will be the concentrations of the brine that flows out of the respective tank. We have the following picture.



We are asked to write down an initial-value problem that governs $S_1(t)$ and $S_2(t)$.

The rates work out so there will always be $V_1(t) = 42$ liters of brine in the first tank and $V_2(t) = 25$ liters in the second. Then $S_1(t)$ and $S_2(t)$ are governed by the initial-value problem

$$\begin{aligned} \frac{\mathrm{d}S_1}{\mathrm{d}t} &= 9\cdot 5 + \frac{S_2}{25}3 - \frac{S_1}{42}7 - \frac{S_1}{42}1\,, && S_1(0) = 76\,,\\ \frac{\mathrm{d}S_2}{\mathrm{d}t} &= \frac{S_1}{42}7 - \frac{S_2}{25}3 - \frac{S_2}{25}4\,, && S_2(0) = 23\,. \end{aligned}$$

This can be simplified to

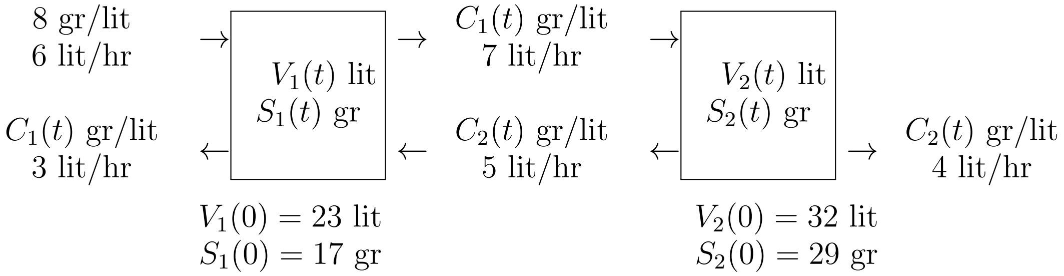
$$\begin{aligned} \frac{\mathrm{d}S_1}{\mathrm{d}t} &= 45 + \tfrac{3}{25}S_2 - \tfrac{4}{21}S_1\,, && S_1(0) = 76\,,\\ \frac{\mathrm{d}S_2}{\mathrm{d}t} &= \tfrac{1}{6}S_1 - \tfrac{7}{25}S_2\,, && S_2(0) = 23\,. \end{aligned}$$

The next example shows that the amount of brine in each tank may vary with time.

Example. Two interconnected tanks are filled with brine (salt water). At $t = 0$ the first tank contains 23 liters and the second contains 32 liters. Brine with a salt concentration of 8 grams per liter flows into the first tank at 6 liters per hour. Well-stirred brine flows from the first tank into the second at 7 liters per hour, from the second into the first at 5 liters per hour, from the first into a drain at 3 liter per hour, and from the second into a drain at 4 liters per hour. At $t = 0$ there are 17 grams of salt in the first tank and 29 grams in the second. Give an initial-value problem that governs the amount of salt in each tank as a function of time.

Solution. Let $V_1(t)$ and $V_2(t)$ be the volumes (lit) of brine in the first and second tank at time t hours. Let $S_1(t)$ and $S_2(t)$ be the mass (gr) of salt in the first and second tank at time t hours. Because the mixtures are assumed to be well-stirred, the salt concentration of the brine in the tanks at time t are $C_1(t) = S_1(t)/V_1(t)$ and

$C_2(t) = S_2(t)/V_2(t)$ respectively. In particular, these are the concentrations of the brine that flows out of these tanks. We have the following picture.



We are asked to write down an initial-value problem that governs $S_1(t)$ and $S_2(t)$.

The rates work out so there will be $V_1(t) = 23 + t$ liters of brine in the first tank and $V_2(t) = 32 - 2t$ liters in the second. Then $S_1(t)$ and $S_2(t)$ are governed by the initial-value problem

$$\begin{aligned}
\frac{\mathrm{d}S_1}{\mathrm{d}t} &= 8 \cdot 6 + \frac{S_2}{32 - 2t}\,5 - \frac{S_1}{23 + t}\,7 - \frac{S_1}{23 + t}\,3\,, \qquad & S_1(0) = 17\,,\\
\frac{\mathrm{d}S_2}{\mathrm{d}t} &= \frac{S_1}{23 + t}\,7 - \frac{S_2}{32 - 2t}\,5 - \frac{S_2}{32 - 2t}\,4\,, \qquad & S_2(0) = 29\,.
\end{aligned}$$

Your answer could be left in the above form. However, it can be simplified to

$$\begin{aligned}
\frac{\mathrm{d}S_1}{\mathrm{d}t} &= 48 + \frac{5}{32 - 2t}\,S_2 - \frac{10}{23 + t}\,S_1\,, \qquad & S_1(0) = 17\,,\\
\frac{\mathrm{d}S_2}{\mathrm{d}t} &= \frac{7}{23 + t}\,S_1 - \frac{9}{32 - 2t}\,S_2\,, \qquad & S_2(0) = 29\,.
\end{aligned}$$

Remark. The systems of ordinary differential equations derived in the last two examples are linear, which is the type of system that we will study in the next chapter.

1.6. **Overview of First-Order Systems.** We will begin with the study of linear first-order systems in the form

$$\begin{aligned}
\frac{\mathrm{d}y_1}{\mathrm{d}t} &= a_{11}(t)y_1 + a_{12}(t)y_2 + \cdots + a_{1n}(t)y_n + f_1(t)\,,\\
\frac{\mathrm{d}y_2}{\mathrm{d}t} &= a_{21}(t)y_1 + a_{22}(t)y_2 + \cdots + a_{2n}(t)y_n + f_2(t)\,,\\
&\ \vdots\\
\frac{\mathrm{d}y_n}{\mathrm{d}t} &= a_{n1}(t)y_1 + a_{n2}(t)y_2 + \cdots + a_{nn}(t)y_n + f_n(t)\,.
\end{aligned} \tag{1.5}$$

This system is called *homogeneous* if $f_1(t) = f_2(t) = \cdots = f_n(t) = 0$, and is called *nonhomogeneous* or *inhomogeneous* otherwise.

Many of the methods that we will apply to system (1.5) have analogue methods that we have already applied to single higher-order linear ordinary differential equations. For example, the interval of definition for solutions of system (1.5) can be read off by looking at it.

Such systems are best studied using vector and matrix notation from linear algebra. After introducing this notation we will study **analytic methods** that allows us to construct a general solution to (1.5) from a sufficiently nice set of n solutions to the associated homogeneous system. Such a set is called a *fundamental set of solutions*. When system (1.5) is homogeneous the construction is simply linear superposition. When system (1.5) is nonhomogeneous the construction requires finding up to n^2 primitives. Its analogue for nonhomogeneous higher-order linear equations is the Green function method.

We then give several analytical methods for constructing fundamental sets of solutions to the homogeneous system associated with (1.5) when is has *constant coefficients*. One method will construct a matrix-valued exponential function. Another method will build upon the concepts of eigenvalues and eigenvectors from linear algebra.

We then study **graphical methods** that apply when system (1.5) is homogeneous, has constant coefficients, and has just two equations ($n = 2$). These methods build upon the aforementioned analytical methods.

The last class of problems that we study are nonlinear first-order systems in the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x,y)\,, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = g(x,y)\,. \tag{1.6}$$

We begin with some analytic methods that utilize many of the methods for single first-order equations that we studied early in the course. These methods can yield either explicit special solutions or analytic relations satified by general solutions. They seldom yield explicit general solutions. Rather, we will use them to gain a **graphical** understanding of how general solutions behave.

Next, we show how analytic methods for homogeneous linear first-order systems with constant coeffients can yield information about how solutions of (1.6) behave near stationary solutions. We also study other graphical methods that help draw a more global picture of how solutions of (1.6) behave.

Finally, we apply some of the aforementioned methods to study mathematical models of population dynamics. The new ingredient here is learning the biological system and the associated system of differential equations.

EXERCISES ON FIRST-ORDER SYSTEMS

(1) When can a general nth order differential equation be cast as a system of n first-order equations? Give an example of a second order equation that cannot be written this way.

Solution

For problems 2- 8, recast the given higher-order differential equation into a system of an appropriate number of first-order equations.

(2) $y'' + 3y' + 7 = 0$

Solution

(3) $y''' + yy'' = 3y^2 \sin(y')$

Solution

(4) $y^{(5)} = \tan(2t) + 3$

Solution

(5) $ty''' + (y')^2 = ye^t + te^y$

Solution

(6) $y''' + y^2y' + 3y = t^2e^{-5t}$

Solution

(7) $x'' + 3x' - ax = 0$

Solution

(8) $v^{(4)} + \cos(v'')v - v'v + (v''')^3 = 0$

Solution

For problems 9- 11, recast the initial value problems into initial value problems for a system of first order equations.

(9) $u^{(4)} = 0$, $u(0) = 4$, $u'(0) = 0$, $u''(0) = -5$, $u'''(0) = 0$

Solution

(10) $y'' + 3ty = \cos(4t)$, $y(1) = 2$, $y'(1) = -2$

Solution

(11) $y^{(5)} + \frac{2}{y} = t$, $y(1) = 0$, $y'(1) = -1$, $y''(1) = 3$, $y'''(1) = -7$, $y''''(1) = 2$

Solution

For problems 12- 14, recast the systems of higher-order differential equations as first order systems of differential equations.

(12)

$$\begin{aligned} u''' &= u + v \\ v'' &= 2tuv' - v + t\sin t \end{aligned}$$

Solution

(13)

$$\begin{aligned} x'' &= 4(y - z) + ty' - \sin(2t) \\ y' &= y + zz' \\ z'' &= 2y + 16x'y - 10yx \end{aligned}$$

Solution

(14)

$$x''' = ty' - 3y + x^2e^{3t}$$
$$y'' = 6xy - 2.$$

With initial conditions

$$x(-2) = 0, \quad x'(-2) = 3, \quad x''(-2) = -1,$$
$$y(-2) = 8, \quad y'(-2) = 4.$$

Solution

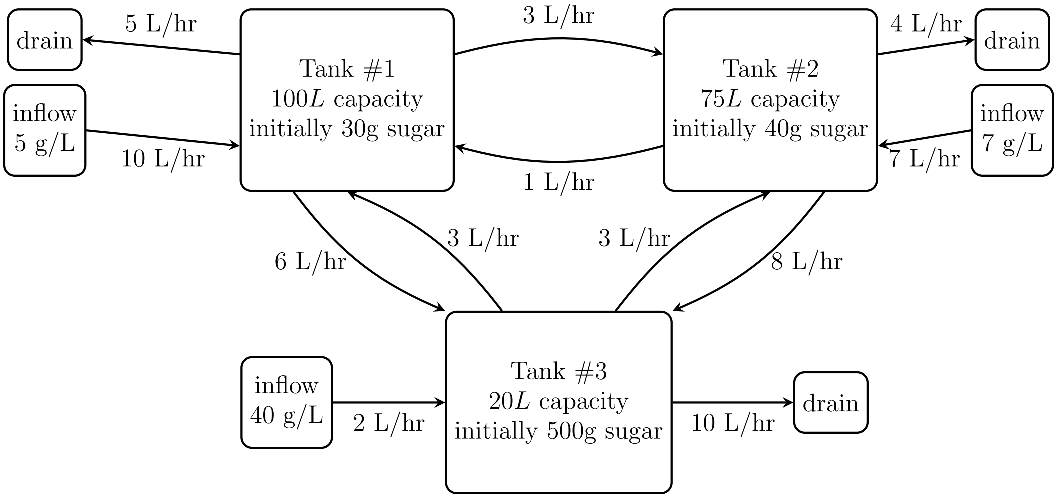
(15) Consider two interconnected tanks filled with brine (salt water). The first tank contains 60 liters and the second tank contains 90 liters. Brine with a concentration of 9 grams of salt per liter flows into the first tank at a rate of 6 liters per hour, from the second tank into the first tank at a rate of 3 liters per hour, from the first tank into a drain at a rate of 4 liters per hour, and from the second tank into a drain at a rate of 2 liters per hour. At $t = 0$ there are 90 grams of salt in the first tank and 30 grams of salt in the second tank. Assume that the solution flows between the tanks in such a way that the volumes of the tanks remain constant. Give (but don't solve) an initial value problem that governs the amount of salt in each tank as a function of time.

Short Answer
Solution

(16) A factory has a pair of interconnected tanks as part of a manufacturing process. The tanks contain a fluid (more of a sludge, but let's be generous) that has important chemicals in it for whatever it is they're manufacturing, but we're interested in the quantity of salt in the tanks because too much salt in the sludge can damage the equipment. The volume of the first tank is 50 liters and the volume of the second tank is 30 liters; both tanks are full of fluid when the machine is initially turned on. The initial amount of salt in the first tank is 120 grams, and initially there are 64 grams of salt in the second tank. After the machine is turned on, the well-mixed fluid flows from the first tank to the second one at a rate of 10 liters per hour, and in a separate pipe from the second tank back to the first one at a rate of 3 liters per hour. The second tank has a drain that leads to the rest of the manufacturing process, and fluid flows down the drain at a rate of 7 liters per hour. Fresh sludge is put back into the first tank at a rate of 7 liters per hour, and in that fresh sludge there is a salt concentration of 4 grams per liter. Write down an initial value problem that governs the amount of salt in each tank as a function of time.

Short Answer
Solution

(17) The following is a schematic for a system of three interconnected tanks with sugar water flowing between them in rates as listed. Write down a initial-value problem that describes how much sugar is in each tank as a function of time.



Short Answer
Solution

(18) Use Euler's method with two steps to estimate where the solution to the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 + ty \\ x + y^2 \end{pmatrix},$$

with initial conditions $x(0) = 1$, $y(0) = 2$ will be at time $t = 1$.

Short Answer
Solution

(19) Use Runge's Trapezoidal method with on step to estimate the value of the following system of differential equations at $t = .5$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3x - z \\ tx + y - 3z \\ t^2(x - z) + 1 \end{pmatrix},$$

with initial conditions $x(0) = 1$, $y(0) = 1$, $z(0) = 0$.

Solution

(20) Use MATLAB and `ode45` to determine where the solution to the initial value problem

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = -2x + 3y \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = 3x - 2y\,, \end{cases} \qquad \begin{pmatrix} x(0) = -1 \\ y(0) = 0 \end{pmatrix}$$

is at time $t = 5$. Repeat this for the initial conditions $x(0) = 2$, $y(0) = 2$.

Solution

(21) This problem uses MATLAB's command `ode45` to investigate the system of differential equations

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = y \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = -y - \sin(x) \end{cases}$$

with initial conditions that are nearby.

(a) Take as initial conditions at time $t = 0$ the point $(x, y) = (0.5, 3.75)$. Where does the solution wind up after 20 seconds?

(b) Repeat the process but use as initial conditions $(x, y) = (0.5, 3.8)$. (You should notice that the two solutions have headed in different directions despite starting close to each other.)

(c) Graph the results of the previous two parts against each other to see where and how they diverge.

Solution

(22) Consider a system of two differential equations $\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(t, \mathbf{x})$, with

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad \mathbf{f}(t, \mathbf{x}) = \begin{pmatrix} f_1(t, x_1, x_2) \\ f_2(t, x_1, x_2) \end{pmatrix}.$$

Suppose further that the function $\mathbf{f}$ is continuous, differentiable with respect to each x_i everywhere, and all its partial derivatives are continuous on $\mathbb{R}^3$. Suppose also that

$$\mathbf{a} = (a_1(t, x_1, x_2), a_2(t, x_1, x_2)) \text{ and } \mathbf{b} = (b_1(t, x_1, x_2), b_2(t, x_1, x_2)),$$

are two different solutions to the system, corresponding to the initial conditions $\mathbf{a}(0) = (0, 0)$ and $\mathbf{b}(0) = (0, 1)$. The functions $\mathbf{a}$ and $\mathbf{b}$ can be viewed as parametrizing curves in the plane. For instance representing the trajectories of particles in two dimensional space. Which of the following statements are true?

(a) The particles could potentially collide, i.e., there can exist a time t_c with $\mathbf{a}(t_c) = \mathbf{b}(t_c)$. (In this case both particles are in the same place at the same time.)

(b) The paths of the particles could potentially cross, i.e., there can exist times t_a and t_b with $\mathbf{a}(t_a) = \mathbf{b}(t_b)$, but t_a is not necessarily equal to t_b. (In this case both particles pass through the same place, but at different times.)

Solution

NAVIGATION TO OTHER CHAPTERS

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