III. First-Order Systems of Ordinary Differential Equations
6. Linear Systems: Laplace Transform Methods

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6. Linear Systems: Laplace Transform Methods

6.1. **Introduction.** In Chapter 9 of Part II we studied the Laplace transform can be used to solve initial-value problems for higher-order linear equations with constant coefficients. Here we extend those methods to solve initial-value problems for first-order linear systems with a constant coefficient matrix.

Consider the initial-value problem for a first-order linear system with a constant coefficient matrix $\mathbf{A}$ in the normal form

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}\mathbf{x} + \mathbf{f}(t)\,, \qquad \mathbf{x}(0) = \mathbf{x}^I\,. \tag{6.1}$$

We assume that the forcing $\mathbf{f}(t)$ is defined for every $t \geq 0$, and is sufficiently nice for the following calculations to be valid.

We have already seen that the solution of (6.1) can be expressed in terms of the matrix exponential $e^{t\mathbf{A}}$ as

$$\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{x}^I + \int_0^t e^{(t-r)\mathbf{A}}\,\mathbf{f}(r)\,\mathrm{d}r\,. \tag{6.2}$$

Here we will solve (6.1) using the Laplace transform method. This approach will yield another way to compute $e^{t\mathbf{A}}$. More importantly, it will yield a way to solve nonhomogeneous systems for a wide class of forcings.

Remark. We used r rather than s as the variable of integration in (6.2) because in this chapter we will reserve s for the independent variable of Laplace transforms.

6.2. **Laplace Transform for Systems.** The Laplace transform of either a vector-valued or a matrix-valued function is defined by applying the transform entrywise. Hence, if $\mathbf{x}(t)$ is the vector-valued function and $\mathbf{\Psi}(t)$ is the matrix-valued function given by

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \qquad \mathbf{\Psi}(t) = \begin{pmatrix} \psi_{11}(t) & \psi_{12}(t) & \cdots & \psi_{1n}(t) \\ \psi_{21}(t) & \psi_{22}(t) & \cdots & \psi_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{n1}(t) & \psi_{n2}(t) & \cdots & \psi_{nn}(t) \end{pmatrix}, \tag{6.3a}$$

then their Laplace transforms, $\mathcal{L}[\mathbf{x}](s)$ and $\mathcal{L}[\mathbf{\Psi}](s)$, are given by

$$\begin{gathered} \mathcal{L}[\mathbf{x}](s) = \begin{pmatrix} \mathcal{L}[x_1](s) \\ \mathcal{L}[x_2](s) \\ \vdots \\ \mathcal{L}[x_n](s) \end{pmatrix}, \\ \mathcal{L}[\mathbf{\Psi}](s) = \begin{pmatrix} \mathcal{L}[\psi_{11}](s) & \mathcal{L}[\psi_{12}](s) & \cdots & \mathcal{L}[\psi_{1n}](s) \\ \mathcal{L}[\psi_{21}](s) & \mathcal{L}[\psi_{22}](s) & \cdots & \mathcal{L}[\psi_{2n}](s) \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{L}[\psi_{n1}](s) & \mathcal{L}[\psi_{n2}](s) & \cdots & \mathcal{L}[\psi_{nn}](s) \end{pmatrix}, \end{gathered} \tag{6.3b}$$

wherever the Laplace transform of every entry is defined.

The Laplace transform of vector-valued and matrix-valued functions inherits many properties from the Laplace transform of scalar-valued functions. For example, for every vector-valued functions $\mathbf{x}(t)$ and $\mathbf{y}(t)$, and every constant matrix $\mathbf{A}$ we have the linearity properties

$$\mathcal{L}[\mathbf{x}+\mathbf{y}](s) = \mathcal{L}[\mathbf{x}](s) + \mathcal{L}[\mathbf{y}](s)\,, \qquad \mathcal{L}[\mathbf{A}\mathbf{x}](s) = \mathbf{A}\mathcal{L}[\mathbf{x}](s)\,, \tag{6.4a}$$

and the property

$$\mathcal{L}\left[\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\right](s) = s\,\mathcal{L}[\mathbf{x}](s) - \mathbf{x}(0)\,. \tag{6.4b}$$

Similarly, for every matrix-valued functions $\mathbf{\Phi}(t)$ and $\mathbf{\Psi}(t)$, and every constant matrix $\mathbf{A}$ we have the linearity properties

$$\mathcal{L}[\mathbf{\Phi}+\mathbf{\Psi}](s) = \mathcal{L}[\mathbf{\Phi}](s) + \mathcal{L}[\mathbf{\Psi}](s)\,, \qquad \mathcal{L}[\mathbf{A}\mathbf{\Phi}](s) = \mathbf{A}\mathcal{L}[\mathbf{\Phi}](s)\,. \tag{6.5a}$$

and the property

$$\mathcal{L}\left[\frac{\mathrm{d}\mathbf{\Phi}}{\mathrm{d}t}\right](s) = s\,\mathcal{L}[\mathbf{\Phi}](s) - \mathbf{\Phi}(0)\,. \tag{6.5b}$$

There are many more properties, but these are the ones we need here.

By taking the Laplace transform of the differential system from the initial-value problem (6.1) while using properties (6.5), we obtain

$$s\,\mathcal{L}[\mathbf{x}](s) - \mathbf{x}(0) = \mathbf{A}\mathcal{L}[\mathbf{x}](s) + \mathcal{L}[\mathbf{f}](s)\,,$$

We want to solve this for $\mathbf{X}(s) = \mathcal{L}[\mathbf{x}](s)$. By applying the initial condition from the initial-value problem (6.1) and grouping the terms involving $\mathbf{X}(s)$ on the left-hand side, we obtain

$$s\,\mathbf{X}(s) - \mathbf{A}\mathbf{X}(s) = \mathbf{x}^I + \mathbf{F}(s)\,,$$

where $\mathbf{F}(s) = \mathcal{L}[\mathbf{f}](s)$. This can be recast as

$$(s\,\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{x}^I + \mathbf{F}(s)\,.$$

The matrix $s\,\mathbf{I} - \mathbf{A}$ will be invertible wherever $\det(s\,\mathbf{I} - \mathbf{A}) \neq 0$. At all such points we have

$$\mathbf{X}(s) = (s\,\mathbf{I} - \mathbf{A})^{-1}(\mathbf{x}^I + \mathbf{F}(s))\,. \tag{6.6}$$

Remark. Notice that $\mathbf{X}(s)$ and $\mathbf{F}(s)$ are both vector-valued fuctions even though they are denoted with upper-case letters. This is because if t-dependent function is denoted with a letter in lower-case then it is common in engineering to denote its s-dependent Laplace transform by the same letter in upper case. This convention goes back to a time before vector and matrix notation. It can get confusing when upper-case letters are also being used to denote matrix-valued functions. When a matrix-valued t-dependent function is denoted by an upper-case letter then another upper-case letter must be used for its Laplace transform.

6.3. **Inverse Laplace Transform for Systems.** In order to obtain $\mathbf{x}(t)$ from $\mathbf{X}(s)$, we have to take the inverse Laplace transform of (6.6). The inverse Laplace transform of either a vector-valued or a matrix-valued function is defined by applying the inverse transform entrywise. Hence, if $\mathbf{Y}(s)$ is the vector-valued function and $\mathbf{M}(t)$ is the matrix-valued function given by

$$\mathbf{Y}(s) = \begin{pmatrix} Y_1(s) \\ Y_2(s) \\ \vdots \\ Y_n(s) \end{pmatrix}, \qquad \mathbf{M}(s) = \begin{pmatrix} M_{11}(s) & M_{12}(s) & \cdots & M_{1n}(s) \\ M_{21}(s) & M_{22}(s) & \cdots & M_{2n}(s) \\ \vdots & \vdots & \ddots & \vdots \\ M_{n1}(s) & M_{n2}(s) & \cdots & M_{nn}(s) \end{pmatrix}, \tag{6.7a}$$

then their inverse Laplace transforms, $\mathcal{L}^{-1}[\mathbf{Y}](t)$ and $\mathcal{L}^{-1}[\mathbf{M}](t)$, are given by

$$\begin{aligned} \mathcal{L}^{-1}[\mathbf{Y}](t) &= \begin{pmatrix} \mathcal{L}^{-1}[Y_1](t) \\ \mathcal{L}^{-1}[Y_2](t) \\ \vdots \\ \mathcal{L}^{-1}[Y_n](t) \end{pmatrix}, \\ \mathcal{L}^{-1}[\mathbf{M}](t) &= \begin{pmatrix} \mathcal{L}^{-1}[M_{11}](t) & \mathcal{L}^{-1}[M_{12}](t) & \cdots & \mathcal{L}^{-1}[M_{1n}](t) \\ \mathcal{L}^{-1}[M_{21}](t) & \mathcal{L}^{-1}[M_{22}](t) & \cdots & \mathcal{L}^{-1}[M_{2n}](t) \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{L}^{-1}[M_{n1}](t) & \mathcal{L}^{-1}[M_{n2}](t) & \cdots & \mathcal{L}^{-1}[M_{nn}](t) \end{pmatrix}. \end{aligned} \tag{6.7b}$$

The inverse Laplace transform of vector-valued and matrix-valued functions inherits many properties from the inverse Laplace transform of scalar-valued functions. For example, for every vector-valued functions $\mathbf{Y}(s)$ and $\mathbf{Z}(s)$, and every constant matrix $\mathbf{B}$ we have the linearity properties

$$\mathcal{L}^{-1}[\mathbf{Y} + \mathbf{Z}](t) = \mathcal{L}^{-1}[\mathbf{Y}](t) + \mathcal{L}^{-1}[\mathbf{Z}](t), \qquad \mathcal{L}^{-1}[\mathbf{B}\mathbf{Y}](t) = \mathbf{B}\mathcal{L}^{-1}[\mathbf{Y}](t). \tag{6.8a}$$

Similarly, for every matrix-valued functions $\mathbf{M}(s)$ and $\mathbf{N}(s)$, and every constant vector or matrix $\mathbf{C}$ we have the linearity properties

$$\mathcal{L}^{-1}[\mathbf{M} + \mathbf{N}](t) = \mathcal{L}^{-1}[\mathbf{M}](t) + \mathcal{L}^{-1}[\mathbf{N}](t), \qquad \mathcal{L}^{-1}[\mathbf{M}(s)\mathbf{C}](t) = \mathcal{L}^{-1}[\mathbf{M}](t)\mathbf{C}. \tag{6.8b}$$

There are many more properties, but these are the ones we need here.

By taking the inverse Laplace transform of $\mathbf{X}(s)$ given by (6.6) while using properties (6.8), we obtain

$$\mathbf{x}(t) = \mathcal{L}^{-1}[\mathbf{X}](t) = \mathcal{L}^{-1}\big[(s\,\mathbf{I} - \mathbf{A})^{-1}\big](t)\mathbf{x}^I + \mathcal{L}^{-1}\big[(s\,\mathbf{I} - \mathbf{A})^{-1}\mathbf{F}(s)\big](t).$$

By comparing this formula for $\mathbf{x}(t)$ with formula (6.2), we conclude that

$$e^{t\mathbf{A}} = \mathcal{L}^{-1}\big[(s\,\mathbf{I} - \mathbf{A})^{-1}\big](t), \tag{6.9a}$$

$$\int_0^t e^{(t-r)\mathbf{A}}\,\mathbf{f}(r)\,\mathrm{d}r = \mathcal{L}^{-1}\big[(s\,\mathbf{I} - \mathbf{A})^{-1}\mathbf{F}(s)\big](t). \tag{6.9b}$$

Formula (6.9a) gives a formula for $e^{t\mathbf{A}}$.

6.4. **Computing Matrix Exponentials.** Recall that we defined $e^{t\mathbf{A}}$ to be the solution $\Phi(t)$ of the matrix-valued initial-value problem (4.2):

$$\frac{\mathrm{d}\mathbf{\Phi}}{\mathrm{d}t} = \mathbf{A}\mathbf{\Phi}\,, \qquad \mathbf{\Phi}(0) = \mathbf{I}\,. \tag{6.10}$$

Let us use the Laplace transform method to solve this initial-value problem directly to confirm formula (6.9a).

By taking the Laplace transform of the differential system from the initial-value problem (6.10) while using properties (6.5), we obtain

$$s\,\mathcal{L}[\mathbf{\Phi}](s) - \mathbf{\Phi}(0) = \mathbf{A}\mathcal{L}[\mathbf{\Phi}](s)\,.$$

We want to solve this for $\mathbf{M}(s) = \mathcal{L}[\mathbf{\Phi}](s)$. By applying the initial condition from the initial-value problem (6.10) and grouping the terms involving $\mathbf{M}(s)$ on the left-hand side, we obtain

$$s\,\mathbf{M}(s) - \mathbf{A}\mathbf{M}(s) = \mathbf{I}\,.$$

This can be recast as

$$(s\,\mathbf{I} - \mathbf{A})\,\mathbf{M}(s) = \mathbf{I}\,.$$

The matrix $s\,\mathbf{I} - \mathbf{A}$ will be invertible wherever $\det(s\,\mathbf{I} - \mathbf{A}) \neq 0$. At all such points we have

$$\mathbf{M}(s) = (s\,\mathbf{I} - \mathbf{A})^{-1}\,.$$

Because $e^{t\mathbf{A}} = \mathbf{\Phi}(t)$ and $\mathcal{L}[\mathbf{\Phi}](s) = \mathbf{M}(s)$, we conclude that

$$e^{t\mathbf{A}} = \mathcal{L}^{-1}[\mathbf{M}](t) = \mathcal{L}^{-1}\big[(s\,\mathbf{I} - \mathbf{A})^{-1}\big](t)\,. \tag{6.11}$$

This agrees with formula (6.9a).

Formula (6.11) reduces the task of computing $e^{t\mathbf{A}}$ to two steps.

1. Compute the matrix $(s\,\mathbf{I} - \mathbf{A})^{-1}$.
2. Compute the inverse Laplace transform of the matrix $(s\,\mathbf{I} - \mathbf{A})^{-1}$.

This is the so-called *Laplace transform method* for computing $e^{t\mathbf{A}}$.

The first step of the method requires us to compute the inverse of a matrix. Because you are only expected to know how to do this for 2×2 matrices, here we will restrict our use of this method to that case. Consider the general 2×2 matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Because

$$s\,\mathbf{I} - \mathbf{A} = \begin{pmatrix} s - a_{11} & -a_{12} \\ -a_{21} & s - a_{22} \end{pmatrix},$$

we see that

$$(s\,\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{p(s)} \begin{pmatrix} s - a_{22} & a_{12} \\ a_{21} & s - a_{11} \end{pmatrix}, \tag{6.12}$$

where $p(s) = \det(s\,\mathbf{I} - \mathbf{A})$ is the characteristic polynomial of $\mathbf{A}$. Because $\mathbf{A}$ is a 2×2 matrix, its characteristic polynomial is given by

$$p(s) = s^2 - \mathrm{tr}(\mathbf{A})s + \det(\mathbf{A})\,. \tag{6.13}$$

The second step of the method requires us to compute the inverse Laplace transform of the matrix $(s\,\mathbf{I} - \mathbf{A})^{-1}$ given by (6.12). This means we must compute the inverse Laplace transform of each entry of the matrix $(s\,\mathbf{I} - \mathbf{A})^{-1}$. As we saw in Chapter 9 of Part II, this will often require using partial fraction identities to decompose each entry into elementry forms.

We now illustrate this method on the same 2×2 matrices that appeared in the previous subsection.

Example. Compute $e^{t\mathbf{A}}$ for

$$\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} .$$

Solution. Because $\mathbf{A}$ is 2×2, by (6.13) its characteristic polynomial is

$$p(s) = s^2 - \mathrm{tr}(\mathbf{A})s + \det(\mathbf{A}) = s^2 - 6s + 5 = (s-5)(s-1) .$$

Then by (6.12) we have

$$(s\,\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{(s-5)(s-1)} \begin{pmatrix} s-3 & 2 \\ 2 & s-3 \end{pmatrix} = \begin{pmatrix} \dfrac{s-3}{(s-5)(s-1)} & \dfrac{2}{(s-5)(s-1)} \\ \dfrac{2}{(s-5)(s-1)} & \dfrac{s-3}{(s-5)(s-1)} \end{pmatrix} .$$

By the partial fraction identities

$$\frac{s-3}{(s-5)(s-1)} = \frac{\frac{1}{2}}{s-5} + \frac{\frac{1}{2}}{s-1} , \qquad \frac{2}{(s-5)(s-1)} = \frac{\frac{1}{2}}{s-5} + \frac{-\frac{1}{2}}{s-1} ,$$

we see that

$$(s\,\mathbf{I} - \mathbf{A})^{-1} = \frac{\frac{1}{2}}{s-5} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{\frac{1}{2}}{s+1} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} .$$

Because

$$\mathcal{L}^{-1}\left[\frac{1}{s-5}\right](t) = e^{5t} , \qquad \mathcal{L}^{-1}\left[\frac{1}{s+1}\right](t) = e^{-t} ,$$

formula (6.11) gives

$$e^{t\mathbf{A}} = \mathcal{L}^{-1}\big[(s\,\mathbf{I} - \mathbf{A})^{-1}\big](t) = \frac{e^{5t}}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{e^{-t}}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} e^{5t} + e^{t} & e^{5t} - e^{t} \\ e^{5t} - e^{t} & e^{5t} + e^{t} \end{pmatrix} .$$

Remark. At first glance this may not look like the same answer that we got in the last section — but it is! It is simply expressed in terms of exponentials rather than hyperbolic functions. Either form of the answer is correct.

Example. Compute $e^{t\mathbf{A}}$ for

$$\mathbf{A} = \begin{pmatrix} -3 & -1 \\ 4 & 1 \end{pmatrix} .$$

Solution. Because $\mathbf{A}$ is 2×2, by (6.13) its characteristic polynomial is

$$p(s) = s^2 - \mathrm{tr}(\mathbf{A})s + \det(\mathbf{A}) = s^2 + 2s + 1 = (s+1)^2 .$$

Then by (6.12) we have

$$(s\,\mathbf{I}-\mathbf{A})^{-1} = \frac{1}{(s+1)^2}\begin{pmatrix} s-1 & -1 \\ 4 & s+3 \end{pmatrix} = \begin{pmatrix} \dfrac{s-1}{(s+1)^2} & \dfrac{-1}{(s+1)^2} \\ \dfrac{4}{(s+1)^2} & \dfrac{s+3}{(s+1)^2} \end{pmatrix}.$$

By the partial fraction identities

$$\frac{s-1}{(s+1)^2} = \frac{1}{s+1} + \frac{-2}{(s+1)^2}\,, \qquad \frac{s+3}{(s+1)^2} = \frac{1}{s+1} + \frac{2}{(s+1)^2}\,,$$

we see that

$$(s\,\mathbf{I}-\mathbf{A})^{-1} = \frac{1}{s+1}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{(s+1)^2}\begin{pmatrix} -2 & -1 \\ 4 & 2 \end{pmatrix}.$$

Because

$$\mathcal{L}^{-1}\left[\frac{1}{s+1}\right](t) = e^{-t}\,, \qquad \mathcal{L}^{-1}\left[\frac{1}{(s+1)^2}\right](t) = t\,e^{-t}\,,$$

formula (6.11) gives

$$e^{t\mathbf{A}} = \mathcal{L}^{-1}\big[(s\,\mathbf{I}-\mathbf{A})^{-1}\big](t) = e^{-t}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + te^{-t}\begin{pmatrix} -2 & -1 \\ 4 & 2 \end{pmatrix} = e^{-t}\begin{pmatrix} 1-2t & -t \\ 4t & 1+2t \end{pmatrix}.$$

Example. Compute $e^{t\mathbf{A}}$ for

$$\mathbf{A} = \begin{pmatrix} 6 & -5 \\ 5 & -2 \end{pmatrix}.$$

Solution. Because $\mathbf{A}$ is 2×2, by (6.13) its characteristic polynomial is

$$p(s) = s^2 - \mathrm{tr}(\mathbf{A})s + \det(\mathbf{A}) = s^2 - 4s + 13 = (s-2)^2 + 3^2\,.$$

Then by (6.12) we have

$$(s\,\mathbf{I}-\mathbf{A})^{-1} = \frac{1}{(s-2)^2+3^2}\begin{pmatrix} s+2 & -5 \\ 5 & s-6 \end{pmatrix} = \begin{pmatrix} \dfrac{s+2}{(s-2)^2+3^2} & \dfrac{-5}{(s-2)^2+3^2} \\ \dfrac{5}{(s-2)^2+3^2} & \dfrac{s-6}{(s-2)^2+3^2} \end{pmatrix}.$$

By the partial fraction identities

$$\frac{s+2}{(s-2)^2+3^2} = \frac{s-2}{(s-2)^2+3^2} + \frac{4}{(s-2)^2+3^2}\,,$$
$$\frac{s-6}{(s-2)^2+3^2} = \frac{s-2}{(s-2)^2+3^2} + \frac{-4}{(s-2)^2+3^2}\,,$$

we see that

$$(s\,\mathbf{I}-\mathbf{A})^{-1} = \frac{s-2}{(s-2)^2+3^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{(s-2)^2+3^2}\begin{pmatrix} 4 & -5 \\ 5 & -4 \end{pmatrix}.$$

Because

$$\mathcal{L}^{-1}\left[\frac{s-2}{(s-2)^2+3^2}\right](t) = e^{2t}\cos(3t)\,, \qquad \mathcal{L}^{-1}\left[\frac{3}{(s-2)^2+3^2}\right](t) = e^{2t}\sin(3t)\,,$$

formula (6.11) gives

$$e^{t\mathbf{A}} = \mathcal{L}^{-1}\big[(s\,\mathbf{I} - \mathbf{A})^{-1}\big](t) = e^{2t}\cos(3t)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + e^{2t}\frac{\sin(3t)}{3}\begin{pmatrix} 4 & -5 \\ 5 & -4 \end{pmatrix}$$

$$= e^{2t}\begin{pmatrix} \cos(3t) + \frac{4}{3}\sin(3t) & -\frac{5}{3}\sin(3t) \\ \frac{5}{3}\sin(3t) & \cos(3t) - \frac{4}{3}\sin(3t) \end{pmatrix}.$$

Exercises on Laplace Transform Methods

There are no problems yet.

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