Abstract

A Wiener-Wintner Double Recurrence Theorem

Let $(X,\mathcal{F},\mu,T)$ be a standard ergodic dynamical system (i.e. X is a compact metric space, μ a Borel probability measure on X, and T a homeomorphism of X). It is known that all Lebesgue dynamical systems are isomorphic to standard ones. Birkhoff's classical Pointwise Ergodic Theorem states that for μ almost every x, $\frac{1}{N}\sum_{n=0}^{N-1} f(T^n x)$ converges. The classical Wiener-Wintener Ergodic Theorem (one proof can be found in [A]), states that for μ almost every x, $\frac{1}{N}\sum_{n=0}^{N-1} f(T^n x)e^{in\theta}$ converges for every real number θ. The difficulty here is that a single set of full measure works for an uncountable number of θ. Bourgain's almost everywhere Double Recurrence Theorem [B],which answered a difficult conjecture of Furstenberg, states that for every pair of integers a, b, for every $f, g \in L^2(X,\mu)$, and for μ almost every x, $\frac{1}{N}\sum_{n=0}^{N-1} f(T^{an}x)g(T^{bn}x)$ converges. One of the main tools used by Bourgain in his proof, is a uniform version of the Wiener-Winter Ergodic Theorem.

Our purpose is to prove the following. Suppose that T is totally ergodic and that a and b are integers, then $\forall f, g \in L^2(\mu)$ there exist a set $\tilde{X}$ of full measure so that $\forall x \in \tilde{X}$, $\frac{1}{N}\sum_{n=0}^{N-1} f(T^{an}x)g(T^{bn}x)e^{in\theta}$ converges for every real number θ. By taking, f to be the constant one function, we see that this generalizes the Wiener-Wintner Theorem, and taking $\theta = 0$, we see a generalization of Bourgain's Theorem. Each of these theorems are, in turn, generalizations of Birkhoff's Theorem.

The techniques employed here, involve decomposing $L^2(X,\mu)$ into the Conze-Lesigne algebra [CL],[R] (the maximal factor of the type "compact abelian group extension of the Kronecker factor") and its orthogonal complement. A proof has been obtained in the case where either function is in the orthogonal complement of the Conze-Lesigne algebra, and in the case where both functions are in the Conze-Lesigne algebra assuming that the compact abelian group extension is determined by an affine cocycle. A proof of the general result, employing these special cases, is in progress.

References

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