ON A CERTAIN SEPARATION CONDITION FOR ATTRACTORS OF FINITE SYSTEMS OF CONTRACTIVE HOMEOMORPHISMS

T. Bedford[1], S. Borodachov[2], J. Geronimo[3]

[1] *University of Strathclyde, Glasgow, Scotland, G1 1QX*
[2] *Towson University, Baltimore, MD, 21252-0001*
[3] *Georgia Institute of Technology, Atlanta, GA, 30332-0160*

Let X be a complete metric space. Denote by $\mathcal{M}(X)$ the set of all contracting homeomorphisms $w : X \to X$ of the whole space X onto itself. Given a system $w_1, \ldots, w_N \in \mathcal{M}(X)$, $N \geq 2$, let $A = A(w_1, \ldots, w_N) \subset X$ be the attractor of this system. Denote $\Sigma = \{1, \ldots, N\}$ and for every vector $\mathbf{i} = \{i_1, \ldots, i_m\} \in \Sigma^m$, let

$$w_{\mathbf{i}}(A) = w_{i_1} \circ \ldots \circ w_{i_m}(A).$$

Our work studies the following separation condition. For the purposes of this paper, we say that a system of mappings $(w_1, \ldots, w_N) \in (\mathcal{M}(X))^N$ satisfies the *strong Markov property* (SMP) if for every $m \in \mathbb{N}$, there is an open set $\mathcal{O}_m \subset X$ such that

1. $\overline{\mathcal{O}_m \cap A} = A$;
2. $w_{\mathbf{i}}(\mathcal{O}_m) \cap w_{\mathbf{j}}(\mathcal{O}_m) = \emptyset$, for every $\mathbf{i} \neq \mathbf{j} \in \Sigma^m$.

This separation condition is weaker than the classical strong open set condition, and not equivalent to the weak separation condition studied in [4] and [7].

We will call a sequence $\{w^k\}_{k\in\mathbb{N}}$ from $\mathcal{M}(X)$ strongly poinwise convergent to a mapping $w \in \mathcal{M}(X)$ and wrtie $w^k \xrightarrow{s.p.} w$, $k \to \infty$, if it converges to w at every point $x \in X$, and the contraction coefficients of the mappings w^k are bounded by a number strictly less than one. If $\{w^k\}_{k\in\mathbb{N}} \subset \mathcal{M}(X)$ is a sequence of similitudes, then the strong pointwise convergence is equivalent to the "usual" pointwise convergence.

We define topology $\mathcal{B}_N$ on the space $(\mathcal{M}(X))^N$ by defining a subset $C \subset (\mathcal{M}(X))^N$ to be closed if $C = \emptyset$ or if for every sequence $\{(w_1^k, \ldots, w_N^k)\}_{k\in\mathbb{N}} \subset C$, such that $\{w_i^k\} \xrightarrow{s.p.} w_i \in \mathcal{M}(X)$, $i = 1, \ldots, N$, we have $(w_1, \ldots, w_N) \in C$.

Theorem 1. *Let X be a complete metric space. The set of systems of mappings $(w_1, \ldots, w_N) \in (\mathcal{M}(X))^N$, which satisfy the SMP is a G-delta set in the topology $\mathcal{B}_N$.*

For every $m \in \mathbb{N}$, denote by $\mathcal{V}_m$ the set of all ordered N-tuples $(w_1, \ldots, w_N) \in (\mathcal{M}(X))^N$ such that for every $\mathbf{i} \in \Sigma^m$, there holds

$$w_{\mathbf{i}}(A) \not\subseteq \bigcup_{\mathbf{j}\in\Sigma^m,\ \mathbf{j}\neq\mathbf{i}} w_{\mathbf{j}}(A).$$

To prove Theorem 1 we show that every set $\mathcal{V}_m$ is open and prove the following statement.

Theorem 2. *Let X be a complete metric space. The system $(w_1, \ldots, w_N)$ of contracting homeomorphisms of X onto itself satisfies the SMP if and only if*

$$(w_1, \ldots, w_N) \in \bigcap_{m=1}^{\infty} \mathcal{V}_m.$$

Theorem 2 is also used to obtain the following sufficient conditions for the SMP.

Proposition 1. *Let $w_1, \ldots, w_N \in \mathcal{M}(X)$ be such that the corresponding attractor A is uncountable and every set $w_i(A) \cap w_j(A)$, $i \neq j$, is at most countable. Then the system $(w_1, \ldots, w_N)$ satisfies the SMP.*

Proposition 2. *Let X be a complete metric space and $w_1, \ldots, w_N \in \mathcal{M}(X)$. Assume that every point in the attractor A of this system has a finite number of addresses. Then the system $(w_1, \ldots, w_N)$ satisfies the SMP.*

Let $W = (w_1, \ldots, w_N)$, where $w_1, \ldots, w_N : \mathbb{R}^d \to \mathbb{R}^d$, $d \in \mathbb{N}$, are similitudes with similarity coefficients $r_1, \ldots, r_N \in (0,1)$ respectively. Denote by $\alpha = \alpha(W)$ the unique positive number such that

$$r_1^\alpha + \ldots + r_N^\alpha = 1.$$

Proposition 3. *Let $W = (w_1, \ldots, w_N)$ be a system of contracting similitudes in $\mathbb{R}^d$, $d \in \mathbb{N}$, and the Hausdorff dimension of $\dim A(W)$ be $\alpha(W)$. Then A satisfies the SMP.*

Given a collection $(a_1, \ldots, a_N) \in ((-1,1) \setminus \{0\})^N$, let $E_N = E_N(a_1, \ldots, a_N)$ be the set of points $(\alpha_1, \ldots, \alpha_N) \in \mathbb{R}^N$ such that the system of mappings

$$u_i(x) = a_i(x - \alpha_i) + \alpha_i, \quad i = 1, \ldots, N,$$

satisfies the SMP. When $|a_1| < 1/2$, ..., $|a_N| < 1/2$, and $\sum_{i=1}^N |a_i| < 1$, the set $E_N(a_1, \ldots, a_N)$ is a subset of $\mathbb{R}^N$ of full measure (and hence, dense in $\mathbb{R}^N$). This follows from the results of Falconer [2, Theorem 5.3] and Solomyak [6, Proposition 3.1], and Proposition 3. We obtain the following density result for $E_N(a_1, \ldots, a_N)$.

Theorem 3. *Let collection $(a_1, \ldots, a_N) \in ((-1,1) \setminus \{0\})^N$ be such that $\sum\limits_{i=1}^{N} |a_i| < 1$. Then the set $E_N(a_1, \ldots, a_N)$ is a dense G-delta subset of $\mathbb{R}^N$.*

Remark 2. The proof of Theorem 3 implies that for every collection $(a_1, \ldots, a_N) \in ((-1,1) \setminus \{0\})^N$ with the property $\sum\limits_{i=1}^{N} |a_i| < 1$, there is a vector $\boldsymbol{\gamma} \in \mathbb{R}^N$ such that the intersection of the complement of $E_N(a_1, \ldots, a_N)$ with every line parallel to $\boldsymbol{\gamma}$ has Hausdorff dimension at most λ, where $\lambda \in (0,1)$ is the unique number such that $\sum_{k=1}^N |a_k|^\lambda = 1$.

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