Numerical Analysis August 2011

Qualifying Exam AMSC/CMSC 666

1. To compute $\sqrt{2}$ we consider the following Eudoxos iterations: starting with $x_0 = y_0 = 1$ we set $x_{n+1} = x_n + y_n$ followed by $y_{n+1} = x_{n+1} + x_n$.

   (a) Show that $y_n/x_n \longrightarrow \sqrt{2}$.

   (b) How many iterations are required for an error $|y_n/x_n - \sqrt{2}| \leq 10^{-6}$?

2. Let $p_N(x) : [-1,1] \mapsto \mathbb{R}$ be an N-degree polynomial given in terms of its Chebyshev expansion $p_N(x) = \sum_{j=0}^{N} a_j T_j(x)$. Set $p^*_{N-1}(x) := \sum_{j=0}^{N-1} a_j T_j(x)$. We claim that p^*_{N-1} is the best $(N-1)$-degree approximation of $p_N(x)$, namely

$$\|p_N(x) - p^*_{N-1}(x)\| \leq \|p_N(x) - q(x)\|, \qquad \text{for all } \ deg\{q\} \leq N-1 \tag{1}$$

   The statement in (1) applies to two different norms $\|\cdot\|$: indicate which two norms, and explain why (no need to provide a detailed proof but clarify your arguments).

3. Assume that $g(x)$ is a smooth function whose first few derivatives are $O(1)$ on $[0,1]$, and we want to approximate $I(g) := \int_0^1 g(x)dx$, in terms of given values, $g_k = g(x_k)$, at the $2N+1$ equally spaced nodes, $x_k := kh$, $k = 0, 1, \ldots, 2N$.

   (a) Write the expressions for the estimates of $\int_0^1 g(x)dx$ by the composite trapezoidal rule, $T(g)$ and the composite Simpson rule, $S(g)$

   (b) Now suppose that instead of the exact values $\{g_k\}$, $k = 0, \ldots, 2N$ we are given the perturbed values,

$$\widetilde{g}_k := g_k + \eta_k, \quad k = 0, \ldots, 2N.$$

   The perturbations $\{\eta_k\}$ are unknown except that: (i) they take the values of $\delta, -\delta$ or 0; (ii) they vanish at the boundaries, $\eta_0 = \eta_{2N} = 0$, and (iii) the other η_k's, $k = 1, \ldots, 2N-1$ have zero average $\sum_{k=1}^{2N-1} \eta_k = 0$.
   Find error bounds for the approximations to $\int_0^1 g(x)dx$ obtained from the composite trapezoidal and Simpson rules using these perturbed data $\{\widetilde{g}_k\}$, $k = 0, \ldots, 2N$.

   Note. *Error expressions for these quadrature rules are given by*

$$I(g) - T(g) = -\frac{g''(\xi)h^2}{12} \quad \text{and} \quad I(g) - S(g) = -\frac{g^{(4)}(\xi)(h/2)^4}{180}$$

   (c) Give a rough estimate of the size of the perturbation δ for which you would prefer to use the composite trapezoidal rule instead of the composite Simpson rule

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4. Let $\lambda > 0$ and u be the solution of the following linear ODE with initial condition $u(0) = u_0$:

$$u' + \lambda u = f(t).$$

Let $\mathcal{T} = \{t_n\}_{n=0}^N$ be a partition of the interval $[0,1]$, with $0 = t_0 < t_1 < \cdots < t_N = 1$. The *continuous Galerkin (cG) method of order* 1 to approximate $u(t)$ reads as follows: find a *continuous* piecewise linear function $U(t)$ over $\mathcal{T}$ satisfying $U(0) = u_0$ and

$$\int_{t_n}^{t_{n+1}} (U'V + \lambda UV)dt = \int_{t_n}^{t_{n+1}} fV dt, \qquad \text{for all } 0 \le n < N,$$

and all piecewise *constant* functions V over $\mathcal{T}$. This method is closely related to the Crank-Nicolson method with step-size $\tau_n = t_{n+1} - t_n$, and this problem explores this connection.

(a) Given $U_n = U(t_n)$, use the cG condition to find an equation satisfied by $U_{n+1} = U(t_{n+1})$. Write the Crank-Nicolson method and compare the two equations.

(b) Derive the Crank-Nicolson method from the cG method using a suitable quadrature.

(c) Prove the energy stability bound for cG

$$\frac{|U_{n+1} - U_n|^2}{2\tau_n} + \frac{\lambda}{2}\left(|U_{n+1}|^2 - |U_n|^2\right) \le \frac{1}{2}\int_{t_n}^{t_{n+1}} |f(t)|^2 dt.$$

5. Given a square matrix A of order n, suppose we have $k+1$ linearly independent vectors $u_1, u_2, \ldots, u_k, u_{k+1}$ that satisfy

$$AU_k = U_k B_k + u_{k+1} b_{k+1}^T$$

where $U_k = [u_1, u_2, \ldots, u_k]$.

(a) Using a QR decomposition, show that there is a matrix Q_k with k orthogonal columns together with a vector q_{k+1} of the form $q_{k+1} = u_{k+1} + w$ with $w \in range(U_k)$ such that

$$\begin{aligned}
&AQ_k = Q_k \tilde{B}_k + q_{k+1}\tilde{b}_{k+1}^T,\\
&Q_k^T q_{k+1} = 0,\\
&range(Q_k) = range(U_k), \quad \text{and}\\
&range([Q_k, q_{k+1}]) = range([U_k, u_{k+1}]).
\end{aligned}$$

(b) Use the result of part (a) to specify a computational strategy for finding k vectors $v \in range(Q_k)$ and scalars μ, which are estimates for eigenvectors and eigenvalues of A, such that $Av - \mu v$ is orthogonal to $range(Q_k)$. You may assume that the matrix $\tilde{B}_k$ has a complete set of linearly independent eigenvectors.

6. Consider the second order elliptic partial differential equation

$$-\text{div}\,(\alpha(x)\text{grad } u) = f \tag{1}$$

posed on a smooth domain $\mathcal{D} \subset \mathbb{R}^2$, subject to Dirichlet boundary conditions $u = 0$ on $\partial\mathcal{D}$.

(a) Define $H_0^1(\mathcal{D})$, and state a weak formulation of the problem (1). This should have the form

$$\text{find } u \in V \text{ such that } a(u,v) = (f,v) \text{ for all } v \in V. \tag{2}$$

where $a(\cdot,\cdot)$ and $(\cdot,\cdot)$ are bilinear forms and $V = H_0^1(\mathcal{D})$.

(b) Identify sufficient conditions on α that guarantee that

$$a(u,u) \geq \gamma \|u\|_V^2, \qquad a(u,v) \leq \Gamma \|u\|_V \|v\|_V$$

for all $u, v \in V$, where $\|\ \|_V$ is a suitable norm defined on V.

(c) Let V_h denote a finite element subspace of V. Define the discrete weak solution of (1) on V_h and show that if u is the solution to (2), then there exists a positive constant $C(\gamma,\Gamma)$ such that

$$\|u - u_h\|_V \leq C \inf_{v_h \in V_h} \|u - v_h\|_V .$$