DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAM
AUGUST 1992

ALGEBRA (Ph.D. Version)

**Instructions to the student**

a. Answer all six questions; each will be assigned a grade from 0 to 10.

b. Use a different booklet for each question. Write the problem number and your **code number (not** your name) on the outside of the booklet.

c. Keep scratch work on separate pages in the same booklet.

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1. Let $C$ be a single conjugacy class of elements in the symmetric group $S_n$.
   (a) Prove that the set $P$ of all products of (not necessarily distinct) elements of $C$ is a subgroup of $S_n$.
   (b) Show that $P$ is either $S_n$, $A_n$, $\{e\}$, or when $n = 4$, the Klein 4-group $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. You may use the fact that $A_n$ is simple for $n \neq 4$ , but you should justify any other assertions you use about subgroups of $S_n$.

2. (a) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that $T(v) \perp v$ for all $v \in \mathbb{R}^3$. Show that $T$ is not invertible.
   (b) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation such that $T(v) \perp v$ for all $v \in \mathbb{R}^n$. Show that $T$ is skew-symmetric ($T^t = -T$). (Hint: Work with $T(v_1 + v_2)$)

3. Let $G$ be the group of automorphisms of the group $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Find the structure of $G$ (that is, identify it with a known group).

4. (parts (a) and (b) are independent)
   (a) Consider an exact sequence of abelian groups
   $$0 \to \mathbb{Q} \to B \to \mathbb{Z}/6\mathbb{Z} \to 0.$$
   Show that $B \simeq \mathbb{Q} \oplus \mathbb{Z}/6\mathbb{Z}$.
   (b) Find rational prime numbers $p$ and $q$ such that $\mathbb{Z}[i]/p\mathbb{Z}[i]$ is an integral domain and $\mathbb{Z}[i]/q\mathbb{Z}[i]$ contains zero divisors .

5. Let $K = \mathbb{Q}(\sqrt{2} - \sqrt{3})$.
   (a) Show that $K/\mathbb{Q}$ is normal.
   (b) Find the quadratic subfields of $K$.
   (c) Compute $\mathrm{Gal}(\mathbb{Q}(\sqrt{2} - \sqrt{3}, \sqrt{5})/\mathbb{Q}))$.

6. (a) Let $G$ be a finite group and let $A$ be a subgroup. Let $r_1, \ldots, r_d$ be a set of coset representatives for $G/A$. Let $\rho : G \to GL(V)$ be a finite dimensional representation of $G$ over the complex numbers and let $W$ be a subspace of $V$ that is stable under the action of $A$ (that is, $\rho(A) \cdot W \subseteq W$). Let
   $$U = \sum_{i=1}^{d} \rho(r_i) \cdot W.$$

Show that $U$ is stable under the action of $G$.

(b) Assume that $(\rho, V)$ is irreducible and that the subgroup $A$ is abelian. Show that $\dim(V) \leq [G : A]$. (Hint: What are the irreducible representations of $A$?)