1. Let $G$ be a finite group and let $H$ be a subgroup.
   (a) Let $N = \{ g \in G | gHg^{-1} = H \}$. Show that the number of distinct subgroups of $G$
   of the form $aHa^{-1}$ with $a \in G$ equals the index $[G : N]$. 
   (b) Assume $H \neq G$. Show that $G \neq \cup_{a \in G} aHa^{-1}$.

2. Let $R \subset \mathbb{C}[X,Y]$ be the ring of polynomials of the form $a + Yg(X,Y)$ with $a \in \mathbb{C}$ and
   $g \in \mathbb{C}[X,Y]$ (so $X \not\in R$ but $3 + XY \in R$).
   (a) Show that $R$ is not a unique factorization domain.
   (b) Show that $R$ is not Noetherian.

3. Let $R$ be a commutative ring with 1 and let $M_1$ and $M_2$ be distinct maximal ideals of $R$. Show that
   \[(R/M_1) \otimes_R (R/M_2) = 0.\]

4. Let $B(x,y)$ be a bilinear form on $\mathbb{R}^n$ (so $B(ax_1 + bx_2, y) = aB(x_1, y) + bB(x_2, y)$ for
   $a, b \in \mathbb{R}$ and $x_1, x_2 \in \mathbb{R}^n$, and similarly for the second variable).
   (a) Show that there exists a unique linear transformation $A : \mathbb{R}^n \to \mathbb{R}^n$ such that
   $B(x, y) = \langle Ax, y \rangle$ for all $x, y \in \mathbb{R}^n$, where $\langle \cdot, \cdot \rangle$ is the usual inner product on $\mathbb{R}^n$.
   (b) Let $\{b_1, \ldots, b_n\}$ be an orthonormal basis of $\mathbb{R}^n$. Show that the sum $\sum_{i=1}^{n} B(b_i, b_i)$
   is independent of the choice of orthonormal basis $\{b_i\}$.

5. Let $K = \mathbb{R}(T)$ be the field of rational functions of one variable over $\mathbb{R}$, and let
   $L = \mathbb{R}(T^{1/4})$.
   (a) Is $L/K$ a Galois extension? If it is, give a proof. If it is not, say why and give a
   Galois extension of $K$ that contains $L$.
   (b) Find, with proofs, all intermediate fields $F$ with $K \subset F \subset L$.

6. (a) Determine the character table of $S_3$, the group of permutations of 3 objects (proofs not needed).
   (b) Let $V$ be the space of homogeneous polynomials of degree 2 in 3 variables with
   coefficients in $\mathbb{C}$. Note that $V$ is a 6-dimensional vector space over $\mathbb{C}$ with basis
   $\{X_iX_j | 1 \leq i \leq j \leq 3\}$. Let $S_3$ act on $V$ by permuting the variables (so the 2-cycle
   $(1,2)$ maps $X_1X_2 + X_1X_3$ to $X_2X_1 + X_2X_3$). Determine the decomposition of this
   representation into irreducible representations of $S_3$. 