DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAM
August 1993

ALGEBRA (Ph.D. Version)

Instructions to the student

a. Answer all six questions; each will be assigned a grade from 0 to 10.

b. Use a different booklet for each question. Write the problem number and your **code number (not** your name) on the outside of the booklet.

c. Keep scratch work on separate pages in the same booklet.

---

1. Let G be a finite group and let H be a subgroup.
(a) Let $N = \{g \in G | gHg^{-1} = H\}$. Show that the number of distinct subgroups of G of the form aHa^{-1} with $a \in G$ equals the index $[G : N]$.
(b) Assume $H \neq G$. Show that $G \neq \cup_{a\in G}\, aHa^{-1}$.

2. Let $R \subset \mathbb{C}[X, Y]$ be the ring of polynomials of the form $a + Yg(X, Y)$ with $a \in \mathbb{C}$ and $g \in \mathbb{C}[X, Y]$ (so $X \notin R$ but $3 + XY \in R$).
(a) Show that R is not a unique factorization domain.
(b) Show that R is not Noetherian.

3. Let R be a commutative ring with 1 and let M_1 and M_2 be distinct maximal ideals of R. Show that
$$(R/M_1) \otimes_R (R/M_2) = 0.$$

4. Let $B(x, y)$ be a bilinear form on $\mathbb{R}^n$ (so $B(ax_1 + bx_2, y) = aB(x_1, y) + bB(x_2, y)$ for $a, b \in \mathbb{R}$ and $x_1, x_2 \in \mathbb{R}^n$, and similarly for the second variable).
(a) Show that there exists a unique linear transformation $A : \mathbb{R}^n \to \mathbb{R}^n$ such that $B(x, y) =< Ax, y >$ for all $x, y \in \mathbb{R}^n$, where $<,>$ is the usual inner product on $\mathbb{R}^n$.
(b) Let $\{b_1, \ldots, b_n\}$ be an orthonormal basis of $\mathbb{R}^n$. Show that the sum $\sum_{i=1}^n B(b_i, b_i)$ is independent of the choice of orthonormal basis $\{b_i\}$.

5. Let $K = \mathbb{R}(T)$ be the field of rational functions of one variable over $\mathbb{R}$, and let $L = \mathbb{R}(T^{1/4})$.
(a) Is L/K a Galois extension? If it is, give a proof. If it is not, say why and give a Galois extension of K that contains L.
(b) Find, with proofs, all intermediate fields F with $K \subset F \subset L$.

6. (a) Determine the character table of S_3, the group of permutations of 3 objects (proofs not needed).
(b) Let V be the space of homogeneous polynomials of degree 2 in 3 variables with coefficients in $\mathbb{C}$. Note that V is a 6-dimensional vector space over $\mathbb{C}$ with basis $\{X_iX_j | 1 \leq i \leq j \leq 3\}$. Let S_3 act on V by permuting the variables (so the 2-cycle (1,2) maps $X_1X_2 + X_1X_3$ to $X_2X_1 + X_2X_3$). Determine the decomposition of this representation into irreducible representations of S_3.