DEPARTMENT OF MATHEMATICS  
UNIVERSITY OF MARYLAND  
GRADUATE WRITTEN EXAM  
August 1995

ALGEBRA (Ph.D. Version)  

Instructions to the student

a. Answer all six questions; each will be assigned a grade from 0 to 10.

b. Use a different booklet for each question. Write the problem number and your code number (not your name) on the outside of the booklet.

c. Keep scratch work on separate pages in the same booklet.

1. Let $G$ be a finite group and assume there is a subgroup $H \neq 1$ such that $H \subseteq K$ for all subgroups $K \neq 1$ of $G$.
   (a) Show that $G$ is finite of prime power order.
   (b) Show that if $G$ is abelian then it is cyclic.
   (c) Exhibit a non-abelian group $G$ of order 8 with a subgroup $H$ that satisfies the above property.

2. (a) Show that the ideal $I = (2, X^4 + X^2 + 1)$ is not a prime ideal of $\mathbb{Z}[X]$.
   (b) Find prime ideals $A \neq 0$ and $B$ such that $A \subset I \subset B \subset \mathbb{Z}[X]$.

3. (a) Let $L/\mathbb{Q}$ be a non-Galois extension of degree 5, and let $K$ be the smallest Galois extension of $\mathbb{Q}$ containing $L$. Assume $K$ does not contain any subfield $F$ with $[F : \mathbb{Q}] = 2$. Show that $\text{Gal}(K/\mathbb{Q}) \cong A_5$, the group of even permutations of 5 objects.
   (b) Show that if $K/\mathbb{Q}$ is a Galois extension with $\text{Gal}(K/\mathbb{Q}) \cong A_5$, then $K$ does not contain any subfield $F$ with $[F : \mathbb{Q}] = 2$ and $K$ does contain a non-Galois subextension $L/\mathbb{Q}$ of degree 5.

4. Let $A$ be a real $n \times n$ matrix.
   (a) Let $y \in \mathbb{R}^n$. Show that $y \perp A^tAx$ (with respect to the standard inner product on $\mathbb{R}^n$) for all $x \in \mathbb{R}^n$ if and only if $Ay = 0$.
   (b) Show that $A^t$ and $A^tA$ have the same range.

5. Let $0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$,  
   $0 \rightarrow M' \rightarrow P' \rightarrow A' \rightarrow 0$,  
   and $0 \rightarrow M'' \rightarrow P'' \rightarrow A'' \rightarrow 0$ be exact sequences of modules (over some ring), and assume $P''$ is projective. Show that these sequences may be put into a commutative
diagram

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & \rightarrow & M' & \rightarrow & M & \rightarrow & M'' & \rightarrow & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & \rightarrow & P' & \rightarrow & P & \rightarrow & P'' & \rightarrow & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & \rightarrow & A' & \rightarrow & A & \rightarrow & A'' & \rightarrow & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 & 0 & 0
\end{array}
\]

in which the rows and columns are exact and the middle row splits.

6. Let \( \rho : G \rightarrow \text{GL}_2(\mathbb{C}) \) be a two-dimensional complex representation of the finite group \( G \). Let \( V \) be the 4-dimensional vector space of \( 2 \times 2 \) complex matrices, and let \( G \) act on \( V \) by

\[
\tilde{\rho}(g)(M) = \rho(g)M\rho(g)^{-1}
\]

for \( M \in V \).

(a) Show that if \( \rho \) is irreducible then \( \tilde{\rho} \) contains the trivial representation exactly once.

(b) Show that if \( \rho \) is the sum of two distinct one-dimensional representations, then \( \tilde{\rho} \) contains the trivial representation exactly twice.

(c) Show that if \( \rho \) is the sum of two equal one-dimensional representations, then \( \tilde{\rho} \) equals the sum of four copies of the trivial representation.