DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAM
August 1995

ALGEBRA (Ph.D. Version)

Instructions to the student

a. Answer all six questions; each will be assigned a grade from 0 to 10.

b. Use a different booklet for each question. Write the problem number and your **code number (not** your name) on the outside of the booklet.

c. Keep scratch work on separate pages in the same booklet.

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1. Let G be a finite group and assume there is a subgroup $H \neq 1$ such that $H \subseteq K$ for all subgroups $K \neq 1$ of G.
(a) Show that G is finite of prime power order.
(b) Show that if G is abelian then it is cyclic.
(c) Exhibit a non-abelian group G of order 8 with a subgroup H that satisfies the above property.

2. (a) Show that the ideal $I = (2, X^4 + X^2 + 1)$ is not a prime ideal of $\mathbb{Z}[X]$.
(b) Find prime ideals $A \neq 0$ and B such that $A \subset I \subset B \subset \mathbb{Z}[X]$.

3. (a) Let $L/\mathbb{Q}$ be a non-Galois extension of degree 5, and let K be the smallest Galois extension of $\mathbb{Q}$ containing L. Assume K does not contain any subfield F with $[F : \mathbb{Q}] = 2$. Show that $\mathrm{Gal}(K/\mathbb{Q}) \simeq A_5$, the group of even permutations of 5 objects.
(b) Show that if $K/\mathbb{Q}$ is a Galois extension with $\mathrm{Gal}(K/\mathbb{Q}) \simeq A_5$, then K does not contain any subfield F with $[F : \mathbb{Q}] = 2$ and K does contain a non-Galois subextension $L/\mathbb{Q}$ of degree 5.

4. Let A be a real $n \times n$ matrix.
(a) Let $y \in \mathbb{R}^n$. Show that $y \perp A^t A x$ (with respect to the standard inner product on $\mathbb{R}^n$) for all $x \in \mathbb{R}^n$ if and only if $Ay = 0$.
(b) Show that A^t and $A^t A$ have the same range.

5. Let $0 \to A' \to A \to A'' \to 0, \qquad 0 \to M' \to P' \to A' \to 0,$
and $0 \to M'' \to P'' \to A'' \to 0$ be exact sequences of modules (over some ring), and assume P'' is projective. Show that these sequences may be put into a commutative

diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P' & \longrightarrow & P & \longrightarrow & P'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

in which the rows and columns are exact and the middle row splits.

6. Let $\rho : G \to \mathrm{GL}_2(\mathbb{C})$ be a two-dimensional complex representation of the finite group G. Let V be the 4-dimensional vector space of 2×2 complex matrices, and let G act on V by

$$\tilde{\rho}(g)(M) = \rho(g)M\rho(g)^{-1}$$

for $M \in V$.

(a) Show that if ρ is irreducible then $\tilde{\rho}$ contains the trivial representation exactly once.

(b) Show that if ρ is the sum of two distinct one-dimensional representations, then $\tilde{\rho}$ contains the trivial representation exactly twice.

(c) Show that if ρ is the sum of two equal one-dimensional representations, then $\tilde{\rho}$ equals the sum of four copies of the trivial representation.