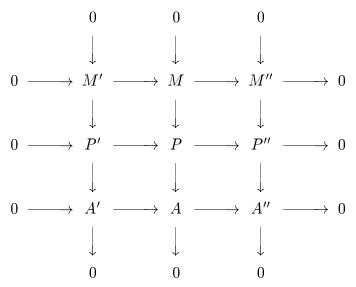
DEPARTMENT OF MATHEMATICS UNIVERSITY OF MARYLAND GRADUATE WRITTEN EXAM August 1995

ALGEBRA (Ph.D. Version)

Instructions to the student

- a. Answer all six questions; each will be assigned a grade from 0 to 10.
- b. Use a different booklet for each question. Write the problem number and your code number (not your name) on the outside of the booklet.
 - c. Keep scratch work on separate pages in the same booklet.
 - 1. Let G be a finite group and assume there is a subgroup $H \neq 1$ such that $H \subseteq K$ for all subgroups $K \neq 1$ of G.
 - (a) Show that G is finite of prime power order.
 - (b) Show that if G is abelian then it is cyclic.
 - (c) Exhibit a non-abelian group G of order 8 with a subgroup H that satisfies the above property.
 - 2. (a) Show that the ideal $I = (2, X^4 + X^2 + 1)$ is not a prime ideal of $\mathbb{Z}[X]$.
 - (b) Find prime ideals $A \neq 0$ and B such that $A \subset I \subset B \subset \mathbb{Z}[X]$.
 - 3. (a) Let L/Q be a non-Galois extension of degree 5, and let K be the smallest Galois extension of Q containing L. Assume K does not contain any subfield F with [F:Q] = 2. Show that Gal(K/Q) ≃ A₅, the group of even permutations of 5 objects.
 (b) Show that if K/Q is a Galois extension with Gal(K/Q) ≃ A₅, then K does not contain any subfield F with [F:Q] = 2 and K does contain a non-Galois subextension L/Q of degree 5.
 - 4. Let A be a real $n \times n$ matrix.
 - (a) Let $y \in \mathbb{R}^n$. Show that $y \perp A^t A x$ (with respect to the standard inner product on \mathbb{R}^n) for all $x \in \mathbb{R}^n$ if and only if Ay = 0.
 - (b) Show that A^t and $A^t A$ have the same range.
 - 5. Let $0 \to A' \to A \to A'' \to 0$, $0 \to M' \to P' \to A' \to 0$, and $0 \to M'' \to P'' \to A'' \to 0$ be exact sequences of modules (over some ring), and assume P'' is projective. Show that these sequences may be put into a commutative

diagram



in which the rows and columns are exact and the middle row splits.

6. Let $\rho: G \to \mathrm{GL}_2(\mathbb{C})$ be a two-dimensional complex representation of the finite group G. Let V be the 4-dimensional vector space of 2×2 complex matrices, and let G act on V by

$$\tilde{\rho}(g)(M) = \rho(g)M\rho(g)^{-1}$$

for $M \in V$.

- (a) Show that if ρ is irreducible then $\tilde{\rho}$ contains the trivial representation exactly once.
- (b) Show that if ρ is the sum of two distinct one-dimensional representations, then $\tilde{\rho}$ contains the trivial representation exactly twice.
- (c) Show that if ρ is the sum of two equal one-dimensional representations, then $\tilde{\rho}$ equals the sum of four copies of the trivial representation.