DEPARTMENT OF MATHEMATICS  
UNIVERSITY OF MARYLAND  
GRADUATE WRITTEN EXAM  
August 1996

ALGEBRA (Ph.D. Version)

**Instructions to the student**

a. Answer all six questions; each will be assigned a grade from 0 to 10.
b. Use a different booklet for each question. Write the problem number and your **code number (not your name)** on the outside of the booklet.
c. Keep scratch work on separate pages in the same booklet.

1. Let $G$ be a finite group and suppose there is a subgroup $H$ of odd order such that $[G : H]$ is a power of 2.
   (a) Show that all elements of $G$ of odd order are contained in $H$ if and only if $H$ is normal in $G$.
   (b) Give an example of a finite group $G_1$ where the elements of odd order form a subgroup of $G_1$, and an example of a finite group $G_2$ where the elements of odd order do not form a subgroup of $G_2$.

2. Let $N$ be an $n \times n$ matrix with complex entries such that the transpose of $N$ equals the complex conjugate of $N$.
   (a) Let $W$ be a subspace of $\mathbb{C}^n$ such that $N(W) \subseteq W$. Let $W^\perp$ be the orthogonal complement of $W$ under the standard inner product on $\mathbb{C}^n$. Show that $N(W^\perp) \subseteq W^\perp$.
   (b) Using part (a), show that $N$ is diagonalizable (you may not use the theorem that says that a matrix that commutes with its conjugate transpose is diagonalizable).

3. Let $K$ be a subfield of the complex numbers containing a primitive cube root of unity $\omega$. Let $f(X) = X^3 + aX^2 + bX + c \in K[X]$ and let $\alpha, \beta, \gamma$ be the roots (in $\mathbb{C}$) of $f(X)$. Let $d = (\alpha - \beta)(\alpha - \gamma)(\beta - \gamma)$ and let $e = \alpha + \omega\beta + \omega^2\gamma$. Assume that $d$ and $e$ are both nonzero.
   (a) Show that $d^2 \in K$ and $e^3 \in K(d)$.
   (b) Show that $K(d, e)$ is the splitting field of $f(X)$.

4. Find all prime ideals of the ring $\mathbb{Z}[X]/(X^2)$.

5. (a) Let $R$ be a commutative ring with 1 and let $S$ be a subset of $R$ that contains 1 and is closed under multiplication. Let $M$ be an $R$-module. Define $S^{-1}M$ to be the set of equivalence classes of pairs $(s, m)$, where $(s_1, m_1)$ and $(s_2, m_2)$ are equivalent if there exists $s \in S$ with $s(s_2m_1 - s_1m_2) = 0$. Addition is defined by $(s_1, m_1) + (s_2, m_2) = (s_1s_2, s_2m_1 + s_1m_2)$, and the 0-element is $(1, 0)$. The action of $S^{-1}R$ via $(s, r)(s', m) = (ss', rm)$ makes $S^{-1}M$ into an $S^{-1}R$-module. Show that the map 
$$
\phi : S^{-1}R \otimes_R M \to S^{-1}M \quad \sum (s_i, r_i) \otimes m_i \mapsto \sum (s_i, r_i m_i)
$$

is a well-defined isomorphism of $S^{-1}R$-modules.
   (b) Let $A$ be an abelian group. Show that the kernel of the map $A \to \mathbb{Q} \otimes_{\mathbb{Z}} A$, where
$a \mapsto 1 \otimes a$, is exactly the torsion subgroup of $A$ (the torsion subgroup is the set of elements of $A$ of finite order).

6. Let $A$ and $B$ be $2 \times 2$ complex matrices such that $A^2 = B^3 = I$ (= the $2 \times 2$ identity matrix) and $ABA = B^{-1}$. Assume $\text{Tr}(B) = -1$. Let $C$ be a matrix that commutes with both $A$ and $B$. Show that $C$ must be a scalar matrix. (Hint: the representation theory of finite groups might be useful)