DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAM
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ALGEBRA (Ph.D. Version)

Instructions to the student

a. Answer all six questions; each will be assigned a grade from 0 to 10.

b. Use a different booklet for each question. Write the problem number and your **code number (not** your name) on the outside of the booklet.

c. Keep scratch work on separate pages in the same booklet.

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1. Let G be a finite group and suppose there is a subgroup H of odd order such that $[G : H]$ is a power of 2.
(a) Show that all elements of G of odd order are contained in H if and only if H is normal in G.
(b) Give an example of a finite group G_1 where the elements of odd order form a subgroup of G_1, and an example of a finite group G_2 where the elements of odd order do not form a subgroup of G_2.

2. Let N be an $n \times n$ matrix with complex entries such that the transpose of N equals the complex conjugate of N.
(a) Let W be a subspace of $\mathbb{C}^n$ such that $N(W) \subseteq W$. Let $W^\perp$ be the orthogonal complement of W under the standard inner product on $\mathbb{C}^n$. Show that $N(W^\perp) \subseteq W^\perp$.
(b) Using part (a), show that N is diagonalizable (you may not use the theorem that says that a matrix that commutes with its conjugate transpose is diagonalizable).

3. Let K be a subfield of the complex numbers containing a primitive cube root of unity ω. Let $f(X) = X^3 + aX^2 + bX + c \in K[X]$ and let α, β, γ be the roots (in $\mathbb{C}$) of $f(X)$. Let $d = (\alpha - \beta)(\alpha - \gamma)(\beta - \gamma)$ and let $e = \alpha + \omega\beta + \omega^2\gamma$. Assume that d and e are both nonzero.
(a) Show that $d^2 \in K$ and $e^3 \in K(d)$.
(b) Show that $K(d, e)$ is the splitting field of $f(X)$.

4. Find all prime ideals of the ring $\mathbb{Z}[X]/(X^2)$.

5. (a) Let R be a commutative ring with 1 and let S be a subset of R that contains 1 and is closed under multiplication. Let M be an R-module. Define $S^{-1}M$ to be the set of equivalence classes of pairs (s, m), where (s_1, m_1) and (s_2, m_2) are equivalent if there exists $s \in S$ with $s(s_2m_1 - s_1m_2) = 0$. Addition is defined by $(s_1, m_1) + (s_2, m_2) = (s_1s_2, s_2m_1 + s_1m_2)$, and the 0-element is $(1, 0)$. The action of $S^{-1}R$ via $(s, r)(s', m) = (ss', rm)$ makes $S^{-1}M$ into an $S^{-1}R$-module. Show that the map

$$\phi : S^{-1}R \otimes_R M \to S^{-1}M$$
$$\sum (s_i, r_i) \otimes m_i \mapsto \sum (s_i, r_im_i)$$

is a well-defined isomorphism of $S^{-1}R$-modules.
(b) Let A be an abelian group. Show that the kernel of the map $A \to \mathbb{Q} \otimes_{\mathbb{Z}} A$, where

$a \mapsto 1 \otimes a$, is exactly the torsion subgroup of A (the torsion subgroup is the set of elements of A of finite order).

6. Let A and B be 2×2 complex matrices such that $A^2 = B^3 = I$ (=the 2×2 identity matrix) and $ABA = B^{-1}$. Assume $\mathrm{Tr}(B) = -1$. Let C be a matrix that commutes with both A and B. Show that C must be a scalar matrix. (Hint: the representation theory of finite groups might be useful)