DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAM
JANUARY 1991

ALGEBRA (Ph.D. Version)

Instructions to the student

a. Answer all six questions; each will be assigned a grade from 0 to 10.

b. Use a different booklet for each question. Write the problem number and your **code number (not** your name) on the outside of the booklet.

c. Keep scratch work on separate pages in the same booklet.

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1. (a) Let G be a finite group and let K be a normal subgroup. Assume that $i = [G : K]$ and $|K|$ are relatively prime. Show that

$$K = \{x^i | x \in G\}.$$

   (b) Give an example to show that the conclusion of part (a) can be false if K is not assumed to be normal in G.

2. Let A and B be $n \times n$ matrices with complex entries.
   (a) Show that if A is similar to B then A and B have the same minimal polynomial and the same characteristic polynomial.
   (b) Show that if $n = 3$ and A and B have the same minimal polynomials and the same characteristic polynomials, then A and B are similar.
   (c) Give an example of A and B having the same minimal polynomials and the same characteristic polynomials, but with A and B not similar.

3. Recall that a module P is called projective if whenever there is a surjection $A \to B$ and a map $P \to B$ then there is a map $P \to A$ such that the following diagram commutes:

$$\begin{array}{ccccc} & & P & & \\ & & \downarrow & & \\ A & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

   (a) Consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P & \longrightarrow & Q & \longrightarrow & C & \longrightarrow & 0 \\ & & & & & & \downarrow{\scriptstyle id} & & \\ & & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

   where the rows are exact, P and Q are projective, and id denotes the identity map on C. Show that there exist maps $\alpha : P \to A$ and $\beta : Q \to B$ such that the resulting diagram commutes.
   (b) Show by an example that the conclusion of part (a) can be false if P and Q are not assumed to be projective.

4. Let R be a commutative ring with identity and let I and J be ideals of R. Then R/I and R/J are R-modules in the natural way.
   (a) Show that every element of $R/I \otimes_R R/J$ can be written in the form $(1+I)\otimes(r+J)$.
   (b) Show that the map

$$R/I \otimes_R R/J \to R/(I+J)$$
$$\sum (r_i + I) \otimes (s_i + J) \mapsto \sum r_i s_i + (I + J)$$

   is a well-defined isomorphism.

5. Let L/K be a finite extension of fields and let F be a field containing K. Let LF be the compositum of L and F (so LF = the smallest field containing L and F; we assume all the fields L, K, F are contained inside some larger field).
   (a) Assume L/K is Galois. Show that LF/F is Galois.
   (b) Assume L/K is Galois. Show that $[LF : F]$ divides $[L : K]$.
   (c) Give an example to show that the conclusion of part (b) can be false if L/K is not assumed to be Galois.

6. Let R be a commutative Noetherian ring with 1. Let $f = \sum_{n\geq 0} a_n X^n \in R[[X]]$ (=the ring of power series with coefficients in R). Show that f is nilpotent if and only if each a_n is nilpotent.