ALGEBRA (Ph.D. Version)

Instructions to the student

a. Answer all six questions; each will be assigned a grade from 0 to 10.
b. Use a different booklet for each question. Write the problem number and your code number (not your name) on the outside of the booklet.
c. Keep scratch work on separate pages in the same booklet.

1. (a) Let \( G \) be a finite group and let \( K \) be a normal subgroup. Assume that \( i = [G : K] \) and \( |K| \) are relatively prime. Show that

\[
K = \{ x^i | x \in G \}.
\]

(b) Give an example to show that the conclusion of part (a) can be false if \( K \) is not assumed to be normal in \( G \).

2. Let \( A \) and \( B \) be \( n \times n \) matrices with complex entries.

(a) Show that if \( A \) is similar to \( B \) then \( A \) and \( B \) have the same minimal polynomial and the same characteristic polynomial.

(b) Show that if \( n = 3 \) and \( A \) and \( B \) have the same minimal polynomials and the same characteristic polynomials, then \( A \) and \( B \) are similar.

(c) Give an example of \( A \) and \( B \) having the same minimal polynomials and the same characteristic polynomials, but with \( A \) and \( B \) not similar.

3. Recall that a module \( P \) is called projective if whenever there is a surjection \( A \to B \) and a map \( P \to B \) then there is a map \( P \to A \) such that the following diagram commutes:

\[
\begin{array}{c}
P \\
\downarrow \\
A \longrightarrow B \longrightarrow 0
\end{array}
\]

(a) Consider the diagram

\[
\begin{array}{c}
0 \longrightarrow P \longrightarrow Q \longrightarrow C \longrightarrow 0 \\
\downarrow id \\
A \longrightarrow B \longrightarrow C \longrightarrow 0
\end{array}
\]

where the rows are exact, \( P \) and \( Q \) are projective, and \( id \) denotes the identity map on \( C \). Show that there exist maps \( \alpha : P \to A \) and \( \beta : Q \to B \) such that the resulting diagram commutes.

(b) Show by an example that the conclusion of part (a) can be false if \( P \) and \( Q \) are not assumed to be projective.
4. Let $R$ be a commutative ring with identity and let $I$ and $J$ be ideals of $R$. Then $R/I$ and $R/J$ are $R$-modules in the natural way.
   (a) Show that every element of $R/I \otimes_R R/J$ can be written in the form $(1+I) \otimes (r+J)$.
   (b) Show that the map
   \[
   R/I \otimes_R R/J \to R/(I+J)
   \]
   \[
   \sum (r_i + I) \otimes (s_i + J) \mapsto \sum r_i s_i + (I + J)
   \]
   is a well-defined isomorphism.

5. Let $L/K$ be a finite extension of fields and let $F$ be a field containing $K$. Let $LF$ be the compositum of $L$ and $F$ (so $LF =$ the smallest field containing $L$ and $F$; we assume all the fields $L,K,F$ are contained inside some larger field).
   (a) Assume $L/K$ is Galois. Show that $LF/F$ is Galois.
   (c) Give an example to show that the conclusion of part (b) can be false if $L/K$ is not assumed to be Galois.

6. Let $R$ be a commutative Noetherian ring with $1$. Let $f = \sum_{n \geq 0} a_n X^n \in R[[X]]$ (=the ring of power series with coefficients in $R$). Show that $f$ is nilpotent if and only if each $a_n$ is nilpotent.