DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAM
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ALGEBRA (Ph.D. Version)

Instructions to the student

a. Answer all six questions; each will be assigned a grade from 0 to 10.

b. Use a different booklet for each question. Write the problem number and your **code number (not** your name) on the outside of the booklet.

c. Keep scratch work on separate pages in the same booklet.

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1. (a) Let G be a group and let C be its center. Suppose G/C is cyclic. Show that G must be abelian.
   (b) Let G be a group of order $p^2 n$, where p is prime and $p^2 \not| n$. Suppose G has a subgroup of order n that is contained in the center of G. Show that G is abelian.
   (c) Let p and q be distinct primes and suppose G is a non-abelian group of order $p^2 q$. Show that the center of G has order 1 or p.

2. Let $M_n(\mathbb{R})$ be the set of $n \times n$ real matrices, regarded as a vector space of dimension n^2 over $\mathbb{R}$. Let $T : M_n(\mathbb{R}) \to \mathbb{R}$ be a linear functional. Show that there exists a unique matrix $A \in M_n(\mathbb{R})$ such that $T(X) = \text{Trace}(AX)$ for all $X \in M_n(\mathbb{R})$.

3. Let R be a commutative ring with 1 and let S be a non-empty subset of R closed under multiplication, with $0 \notin S$. Let I be maximal among the set of ideals whose intersection with S is empty. Show that I is a prime ideal.

4. (a) Let $f : A \to B$ be a homomorphism of abelian groups, and let $C \subseteq A$ be a subgroup with $[A : C] < \infty$. Let $K = \text{Ker } f$ and $L = K \cap C$. Show that
   $$[A : C] = [f(A) : f(C)][K : L].$$
   (b) Formulate and prove an analogous statement for vector spaces.

5. Let K be a field of characteristic not equal to 2 or 5, and let $a, b \in K$ satisfy $a^2 \neq b^5$, $a \neq 0$, $b \neq 0$. Let $f(X) = X^{10} - 2aX^5 + b^5$.
   (a) Show that f is a separable polynomial.
   (b) Let L be the splitting field of f over K. Show that $[L : K] \leq 40$. (Note that if x is a root of f, then so is b/x.)

6. The following is the character table of a group G ("sizes" refers to the sizes of the conjugacy classes $\{1\}$, A, B, C, D; the characters of G are $\chi_1, \ldots, \chi_5$):
   (a) Compute β.
   (b) Let G^c be the commutator subgroup of G (so G^c =the subgroup generated by $\{aba^{-1}b^{-1} \mid a, b \in G\}$). Show that G/G^c is isomorphic to the Klein 4-group $\mathbb{Z}_2 \times \mathbb{Z}_2$.