DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAM
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ALGEBRA (Ph.D. Version)

Instructions to the student

a. Answer all six questions; each will be assigned a grade from 0 to 10.

b. Use a different page for each question. Write the problem number and your **code number (not** your name) on the outside of the booklet.

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1. Let G and H be groups and let N be a normal subgroup of $G \times H$. Suppose that $N \cap (G \times 1) = 1$ and $N \cap (1 \times H) = 1$. Show that N is contained in the center of $G \times H$.

2. Let V be a finite-dimensional vector space over a field, and let $T : V \to V$ be a linear transformation. Show that there is a positive integer n such that $\mathrm{Ker}(T^n) \cap \mathrm{Im}(T^n) = 0$.

3. Let L be a subfield of $\mathbb{C}$ such that $L/\mathbb{Q}$ is a finite Galois extension with $\mathrm{Gal}(L/\mathbb{Q})$ cyclic. Let K be a subfield of L and suppose that $K \not\subseteq \mathbb{R}$.
(a) Show that $[K : \mathbb{Q}]$ is even.
(b) Show that $[L : K]$ is odd.

4. Let R be a commutative ring with 1 and let Q be a proper ideal of R. We say that Q is *primary* if it satisfies the following condition: Let $x, y \in R$ with $xy \in Q$. If $x \notin Q$ then $y^n \in Q$ for some positive integer n (depending on y). Define the radical of Q by

$$\mathrm{rad}(Q) = \{x \in R \,|\, x^n \in Q \text{ for some positive integer } n\}.$$

Note that $\mathrm{rad}(Q)$ is also a proper ideal of R (do not prove this). Prove:
(a) If Q is primary then $\mathrm{rad}(Q)$ is a prime ideal.
(b) Suppose R is a principal ideal domain. Find all primary ideals of R (prove that all primary ideals are on your list, and prove that all ideals on your list are primary).

5. Let R be a commutative ring with 1. An R-module F is called *flat* if whenever $f : M \to N$ is an injective homomorphism of R-modules, the induced map $M \otimes_R F \to N \otimes_R F$ is also injective. Suppose R is a principal ideal domain. Show that a finitely generated R-module is flat if and only if it is torsion free.

6. Let p be an odd prime. For x, y, and $z \in \mathbb{F}_p$, the finite field with p elements, let

$$[x, y, z] = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \in GL_3(\mathbb{F}_p).$$

Let G be the finite group

$$G = \{[x, y, z] \mid x,\ y,\ z \in \mathbb{F}_p\}.$$

A straightforward calculation shows that

$$Z = \{[0, 0, z] \mid z \in \mathbb{F}_p\}$$

is the center of G and is the commutator subgroup of G (do not prove these statements). Let $\rho : G \longrightarrow GL(V)$ be an *irreducible* representation of G, where V is a complex vector space of dimension n_V.

(a) Show that there is a homomorphism $\lambda_V : Z \longrightarrow \mathbb{C}^{\times}$ such that $\rho(z)v = \lambda_V(z)v$ for all $z \in Z$ and for all $v \in V$.

(b) Show that $n_V = \dim_{\mathbb{C}} V = 1$ if and only if $\lambda_V(z) = 1$ for all $z \in Z$.

(c) How many distinct 1 dimensional irreducible representations of G are there?

(d) Using the fact that $n_V \mid |G|$, for any irreducible representation V, determine the number of irreducible (non-isomorphic) representations of G and their dimensions.