DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAM
August 2011
ALGEBRA (Ph.D. Version)

Instructions to the student

a. Answer all six questions; each will be assigned a grade from 0 to 10.

b. Use a different booklet for each question. Write the problem number and your **code number** (**not** your name) on the outside of the booklet.

c. Keep scratch work on separate pages in the same booklet.

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1. (a) Let G be a group of order 27. Suppose G acts on a set X with 50 elements. Show that there are at least two elements of X that are fixed by G.
(b) Given an exact sequence of groups
$$1 \to A \to B \to C \to 1$$
with $|A| = 50$ and $|C| = 27$, show that there exists a subgroup $D \subset B$ that maps isomorphically to C.
(c) Let A, B, C be as in part (b). Assume that A is abelian. Show that the center of B is nontrivial.

2. (a) Let S be a finite subset of a field with $0 \notin S$. Suppose that there is an integer $m \geq 2$ such that the mth power map maps S into S. Show that every element of S is a root of unity.
(b) Let $n \geq 1$ and let M be an $n \times n$ invertible complex matrix. Suppose that M^m is similar to (that is, conjugate to) M for some $m \geq 2$. Show that the eigenvalues of M are roots of unity.

3. Let R be a unique factorization domain with field of fractions K. Let $\pi \in R$ be irreducible, and let
$$R_\pi = \{a/b \in K \mid a, b \in R, \pi \nmid b\}.$$
Let $0 \neq x \in R$ and let π^r be the highest power of π dividing x. Show that
$$xR_\pi \cap R = \pi^r R.$$

4. Let R be a commutative ring with 1 and let M be an R-module. Recall that M is injective if whenever $\phi : N_1 \to M$ is an R-module homomorphism, and $N_1 \subseteq N_2$ is an inclusion of R-modules, then ϕ can be extended to $\phi : N_2 \to M$. It is well known that every R-module is isomorphic to a submodule of an injective R-module.
(a) Suppose M_i, $i \in I$, for some index set I, is a collection of R-modules and $R \to \oplus_{i \in I} M_i$ is a homomorphism of R-modules. Show that there is a finite subset F of I such that the image of R is contained in $\oplus_{i \in F} M_i$.
(b) Suppose that $J_1 \subseteq J_2 \subseteq J_3 \cdots$ is an ascending chain of ideals in R and let $J = \cup_{i \geq 1} J_i$. For each i, let J/J_i be isomorphic to a submodule of an injective module M_i. Assume that $\oplus_{i=1}^{\infty} M_i$ is an injective R-module. Show that there is an N such that the image of J under the natural map
$$J \to \oplus_{i=1}^{\infty} J/J_i \to \oplus_{i=1}^{\infty} M_i$$

is contained in $\oplus_{i\leq N} M_i$. (*Hint:* injectivity is needed here.)
(c) Show that if R has the property that every countable direct sum of injective R-modules is injective then R is Noetherian.

5. Let ζ be a primitive 11-th root of unity and let $\alpha = \zeta + \zeta^3 + \zeta^4 + \zeta^5 + \zeta^9$. It is well known that $\mathrm{Gal}(\mathbf{Q}(\zeta)/\mathbf{Q}) \simeq (\mathbf{Z}/11\mathbf{Z})^\times$, where $k \mod 11$ corresponds to the map $\zeta \mapsto \zeta^k$.
(a) Let S be a subset of $\{1, \zeta, \zeta^2, \ldots, \zeta^{10}\}$. Show that if S has 10 elements then S is linearly independent.
(b) Show that $\alpha \notin \mathbf{Q}$.
(c) Show that $[\mathbf{Q}(\alpha) : \mathbf{Q}] = 2$.

6. (a) Let G be a finite group such that G/H is abelian whenever H is a nontrivial normal subgroup of G. Let

$$\rho : G \longrightarrow GL_n(\mathbf{C})$$

be an irreducible representation of G with $n \geq 2$. Show that ρ is injective.
(b) Give an example of a finite group G and an irreducible representation

$$\rho : G \longrightarrow GL_n(\mathbf{C})$$

with $n \geq 2$ such that ρ is not injective.