

Department of Mathematics
University of Maryland
Written Graduate Qualifying Exam
Algebra (Ph.D. Version)
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Instructions

A. Answer all six questions. Each one will be assigned a grade from 0 to 10. In problems with multiple parts, the parts are graded independently of one another. Be sure to go on to subsequent parts even if there is some part you cannot do. You may assume the answer to any part in subsequent parts of the same problem.

B. Unless otherwise stated, you may appeal to a “well known theorem” in your solution to a problem, but if you do, it is your responsibility to make it clear exactly which theorem you are using and why its use is justified.

C. Notation: \mathbb{Z} is the ring of ordinary integers, \mathbb{Q} is the field of rational numbers, \mathbb{R} is the field of real numbers, \mathbb{C} is the field of complex numbers, and \mathbb{F}_q is the finite field with q elements. PID stands for “principal ideal domain.” S_n denotes the symmetric group on n letters, and A_n the alternating group. M_n denotes the $n \times n$ matrices, and \bar{z} is the complex conjugate of $z \in \mathbb{C}$.

1. Let G be a group of order $165 = 11 \cdot 5 \cdot 3$.

- (a) Show that G has a normal subgroup N of order 11.
- (b) Show that with N as in (a), G/N is abelian, and thus that G is solvable.
- (c) Classify all groups of order 165 up to isomorphism.

2. Suppose A is a 3×3 matrix with entries in a field F of characteristic 0, and assume $\text{Tr } A = 6$, $\text{Tr } A^2 = 14$, and $\det A = 6$. (Tr denotes the trace.) Prove that A is similar over F to the diagonal matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

3. You may assume the fact that the ring $R = \mathbb{Z}[\omega]$, where ω is a primitive cube root of unity, is a PID — in fact, a Euclidean ring with respect to the norm N defined by

$$N(a + b\omega) = (a + b\omega)(a + b\omega^2) = a^2 - ab + b^2.$$

Let $p \in \mathbb{Z}$ be an ordinary prime number. Show that:

- (a) An element $y \in R$ is a unit in R if and only if $N(y) = 1$.
- (b) p can be written in the form $a^2 - ab + b^2$, $a, b \in \mathbb{Z}$, if and only if the ideal (p) is not prime in R .
- (c) The ideal (p) is not prime in R if and only if the polynomial $x^2 + x + 1$ is reducible in $\mathbb{F}_p[x]$.

Continue next page.

Deduce that:

(d) (3) and (7) are not prime in R , but that (2) and (5) are prime in R .

4. Let R be a commutative ring. An R -module M is called *flat* if, for all short exact sequences

$$(1) \quad 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

of R -modules, the sequence

$$(2) \quad 0 \rightarrow M \otimes_R A \rightarrow M \otimes_R B \rightarrow M \otimes_R C \rightarrow 0$$

is exact. An R -module M is called *faithfully flat* if M is flat, and if in addition, exactness of sequence (2) implies (1) is exact.

(a) Show that a free R -module is faithfully flat.

(b) Take $R = \mathbb{Z}$. Show that the R -module $\mathbb{Z}/(2)$ is not flat, and that the R -module \mathbb{Q} is flat but not faithfully flat.

5. Let $f(x) = x^5 - 6x + 2$.

(a) Show that f is irreducible in $\mathbb{Q}[x]$, and that in \mathbb{C} , it has exactly three real roots. (For the last assertion you need freshman calculus.)

(b) Deduce that if L is the splitting field of f over \mathbb{Q} , $G = \text{Gal}(L/\mathbb{Q})$, when identified with a subgroup of S_5 , contains a 5-cycle and a 2-cycle. (Remark: This then implies that $G = S_5$, but you don't need to prove this.)

6. Let G be a finite group and let $g \in G$.

(a) Let $\pi : G \rightarrow M_n(\mathbb{C})$ be a representation of G and let χ_π be its character. Show that $\chi_\pi(g^{-1}) = \overline{\chi_\pi(g)}$.

(b) Prove that g is conjugate in G to g^{-1} if and only if the following condition is satisfied: for every irreducible complex representation π of G , the character χ_π of π is real-valued on g .

(c) Show that the condition of (b) is satisfied for all elements of S_n , and thus that all characters of S_n are real-valued.