Department of Mathematics
University of Maryland
Written Graduate Qualifying Exam
Algebra (M.A. Version)
August, 2001

Instructions

A. Answer all six questions. Each one will be assigned a grade from 0 to 10. In problems with multiple parts, the parts are graded independently of one another. Be sure to go on to subsequent parts even if there is some part you cannot do. You may assume the answer to any part in subsequent parts of the same problem.

B. Unless otherwise stated, you may appeal to a "well known theorem" in your solution to a problem, but if you do, it is your responsibility to make it clear exactly which theorem you are using and why its use is justified.

C. Notation: $\mathbb{Z}$ is the ring of ordinary integers, $\mathbb{Q}$ is the field of rational numbers, $\mathbb{R}$ is the field of real numbers, $\mathbb{C}$ is the field of complex numbers, and $\mathbb{F}_q$ is the finite field with q elements. PID stands for "principal ideal domain." S_n denotes the symmetric group on n letters, and A_n the alternating group.

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1. Let G be a group of order $165 = 11 \cdot 5 \cdot 3$.
   (a) Show that G has a normal subgroup N of order 11.
   (b) Show that with N as in (a), G/N is abelian, and thus that G is solvable.
   (c) Classify all groups of order 165 up to isomorphism.

2. Suppose A is a 3×3 matrix with entries in a field F of characteristic 0, and assume $\operatorname{Tr} A = 6$, $\operatorname{Tr} A^2 = 14$, and $\det A = 6$. (Tr denotes the trace.) Prove that A is similar over F to the diagonal matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

3. You may assume the fact that the ring $R = \mathbb{Z}[\omega]$, where ω is a primitive cube root of unity, is a PID — in fact, a Euclidean ring with respect to the norm N defined by

$$N(a + b\omega) = (a + b\omega)(a + b\omega^2) = a^2 - ab + b^2.$$

Let $p \in \mathbb{Z}$ be an ordinary prime number. Show that:
   (a) An element $y \in R$ is a unit in R if and only if $N(y) = 1$.
   (b) p can be written in the form $a^2 - ab + b^2$, a, $b \in \mathbb{Z}$, if and only if the ideal (p) is not prime in R.
   (c) The ideal (p) is not prime in R if and only if the polynomial $x^2 + x + 1$ is reducible in $\mathbb{F}_p[x]$.

Continue next page.

Deduce that:

(d) (3) and (7) are not prime in R, but that (2) and (5) are prime in R.

4. Let M be a finitely generated module over a PID, R, and suppose that for some distinct nonzero prime ideals P and Q of R, $(P^2Q^2) \cdot M = 0$. Prove that there are unique integers $p_1, p_2, q_1, q_2 \geq 0$ such that M is isomorphic to

$$(R/P)^{p_1} \oplus (R/P^2)^{p_2} \oplus (R/Q)^{q_1} \oplus (R/Q^2)^{q_2}.$$

5. (a) Prove that for every element g of $G = S_n$, g is conjugate in G to g^{-1}.

(b) Show that there is an element of A_4 which is **not** conjugate to its inverse.

6. Show that if M is an $n \times n$ matrix over $\mathbb{C}$ and if $M^2(M+1)^2 = 0$, then M is similar to a direct sum of blocks of the forms

$$(0), \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, (-1), \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}.$$