# Department of Mathematics
# University of Maryland
# Written Graduate Qualifying Exam
# Analysis (M.A. Version)

August, 1995

## Instructions

**1.** Answer all six questions. Each one will be assigned a grade from 0 to 10. In problems with multiple parts, the parts are graded independently of one another. Be sure to go on to subsequent parts even if there is some part you cannot do. You may assume the answer to any part in subsequent parts of the same problem.

**2.** Use a different booklet for each question. Write the problem number and your **code number** (**not** your name) on the outside cover of each booklet.

**3.** Keep scratch work on separate pages in the same booklet.

**4.** Unless otherwise stated, you may appeal to a "well known theorem" in your solution to a problem, but if you do, it is your responsibility to make it clear exactly which theorem you are using and why its use is justified.

1. Let $X = \{x_1, x_2, \dots\}$ be a countably infinite complete metric space.

(a) Show that $X$ has infinitely many isolated points.

(b) Let $J$ be the set of isolated points of $X$. $J$ is open (why?). Thus $X - J$ is a countable complete metric space and, if it is non-empty, has isolated points. But how can that be since to construct $X - J$ we removed all the isolated points from $X$?

2. Show that for any $\lambda \in \mathbb{C}$ with $|\lambda| < 2$, the equation $z^4 - 3z + \lambda = 0$ has a unique solution $z(\lambda) \in \mathbb{C}$ with $|z(\lambda)| < 1$. Prove that $\lambda \mapsto z(\lambda)$ is holomorphic and find the coefficients $a_0, a_1, \dots, a_4$ in the Taylor series development

$$z(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n$$

about $\lambda = 0$.

3. Evaluate

$$\lim_{n \to \infty} \int_0^\infty \frac{x^{n-2}}{1 + x^n} \sin \frac{\pi x}{n} \, dx.$$

Justify each step in your calculation.

4. Evaluate:

$$\int_{-\infty}^{\infty} \frac{dx}{\cosh x}.$$

Justify all intermediate steps.

5. Suppose $1 \le p < \infty$ and $\{f_n\}_{n=1}^{\infty}$ is a sequence of measurable functions on $[0, 1]$ for which there are positive constants $K$, $M$ such that for all $n$,

$$\sup_x |f_n(x)| \le M$$

and

$$\|f_n\|_{L^p} \ge K.$$

Suppose that $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers such that the series

$$\sum_n a_n f_n(x)$$

converges for almost all $x \in [0, 1]$. Prove that $a_n \to 0$ as $n \to \infty$.

6. Find a one-to-one conformal mapping $f$ from

$$\Omega = \{z \in \mathbb{C} : \operatorname{Im} z < 0 \textbf{ or } |z| < 1\}$$

(see below) onto the upper half-plane $H = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$, with $f(0) = i$.