Solutions
Written Graduate Qualifying Exam
Analysis (Ph.D. and M.A. Versions)

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1. (Ph.D. and M.A.)

(a) By the Baire Category Theorem, one of the one-point sets $\{x_j\}$ has a non-empty interior. Such a set is an isolated point.

(b) Say it is x_1 which is isolated. By the same argument $X - \{x_1\}$ has an isolated point x_2. x_2 is also isolated in X. Continuing, we see that X has infinitely many isolated points.

(c) It is true that $X - J$ has isolated points, i.e., points which are isolated in $X - J$. These need not and won't be isolated in X.

2. (Ph.D. and M.A.) If $|\lambda| < 2$ and $|z| = 1$, then

$$|z^4 + \lambda| < 3 = |3z|.$$

Thus, by Rouché's Theorem, $z^4 - 3z + \lambda = -(3z - (z^4 + \lambda))$ has the same number of zeros (counting multiplicities) as does $3z$ inside the circle $|z| = 1$, and thus $z^4 - 3z + \lambda = 0$ has a unique solution $z(\lambda)$ with $|z(\lambda)| < 1$.

Furthermore, $z(\lambda)$ is holomorphic in λ since it has a complex derivative which can be computed by implicit differentiation:

$$4z^3 \frac{dz}{d\lambda} - 3\frac{dz}{d\lambda} + 1 = 0,$$

or

$$\frac{dz}{d\lambda} = \frac{1}{3 - 4z(\lambda)^3}.$$

Alternatively, if $f(\zeta)$ is holomorphic and has a unique simple zero in $\{|\zeta| < 1\}$ and no zeros on $\{|\zeta| = 1\}$, then

$$\oint_{|\zeta|=1} \frac{\zeta f'(\zeta)}{f(\zeta)} d\zeta$$

gives the value of the zero of f, so applying this we get the integral formula

$$z(\lambda) = \oint_{|\zeta|=1} \frac{\zeta(4\zeta^3 - 3)}{\zeta^4 - 3\zeta + \lambda} d\zeta,$$

from which it follows that $z(\lambda)$ is holomorphic by differentiation under the integral sign.

To find the first few Taylor series coefficients a_n, observe that $z(\lambda) = 0$ when $\lambda = 0$. Thus $a_0 = 0$ and $z(\lambda)^4$ has a Taylor series beginning with the term $(a_1\lambda)^4 = a_1^4\lambda^4$. So substituting in the equation gives

$$a_1^4\lambda^4 + O(\lambda^5) - 3\left(a_1\lambda + a_2\lambda^2 + a_3\lambda^3 + a_4\lambda^4 + O(\lambda^5)\right) + \lambda = 0,$$

so that $-3a_1+1=0$, $a_2=0$, $a_3=0$, and $a_1^4-3a_4=0$. Thus $a_1=\frac{1}{3}$, $a_4=3^{-5}$, and $a_0=a_2=a_3=0$.

3. (Ph.D. and M.A.) Denoting the integrand by $f_n(x)$ we have

$$|f_n(x)| \le \frac{x^{n-2}}{1+x^n}, \qquad x \ge 0.$$

For $0 \le x \le 1$ and $n \ge 2$, $\dfrac{x^{n-2}}{1+x^n} \le \dfrac{1}{1+x^n} \le 1$, while for $x>1$

$$\frac{x^{n-2}}{1+x^n} \le \frac{1}{x^2}.$$

Thus $f_n(x)$ is dominated for each $n \ge 2$ by the integrable function $\min(1, \frac{1}{x^2})$. By the Dominated Convergence Theorem,

$$\lim \int_0^\infty f_n = \int_0^\infty \lim f_n = 0.$$

4. (Ph.D.) Let Γ_R be the contour obtained by traveling from $-R$ to $-\frac{1}{R}$ along the negative real axis, then traveling clockwise along a semicircular arc of radius $\frac{1}{R}$ to the point $+\frac{1}{R}$, then traveling to R along the positive real axis, then traveling counterclockwise along a semicircular arc of radius R back to $-R$. If $\log z$ denotes the branch of the logarithm defined in the complement of the negative imaginary axis and agreeing with the principal branch along the positive real axis, then $\log z$ is holomorphic in a neighborhood of the closed region bounded by Γ_R. So by the Residue Theorem,

$$\begin{aligned}\oint_{\Gamma_R} \frac{(\log z)^2}{z^2+4} dz &= 2\pi i \operatorname{Res}_{z=2i} \frac{(\log z)^2}{z^2+4} \\ &= 2\pi i \frac{(\log 2 + \pi \frac{i}{2})^2}{4i} \\ &= \frac{\pi}{2}\left(\log 2 + \pi \frac{i}{2}\right)^2.\end{aligned}$$

On the other hand, $\oint_{\Gamma_R} \frac{(\log z)^2}{z^2+4} dz$ is the sum of the integrals along the semicircular arcs and the integrals along the line segments. The integrals along the semicircular arcs tend to 0 as $R \to \infty$: in the case of the small arc, because the arc length is $\frac{\pi}{R}$, and thus the integral is bounded by a multiple of $\frac{(\log R)^2}{R} \to 0$, and in the case of the big arc, because the integrand is bounded by a multiple of $\frac{(\log R)^2}{R^2}$, while the arc length is πR, so that the product of the two is bounded by a multiple of $\frac{(\log R)^2}{R} \to 0$. The integrals along the line segments give

$$\begin{aligned}\oint_{\Gamma_R} \frac{(\log z)^2}{z^2+4} dz &\xrightarrow{R\to\infty} \int_0^\infty \frac{(\log x)^2}{x^2+4} dx + \int_{-\infty}^0 \frac{(\pi i + \log(-x))^2}{x^2+4} dx \\ &= \int_0^\infty \frac{(\log x)^2}{x^2+4} dx + \int_0^\infty \frac{(\pi i + \log x)^2}{x^2+4} dx \\ &= \int_0^\infty \frac{2\pi i \log x}{x^2+4} dx + (\text{a real-valued integral}).\end{aligned}$$

Taking imaginary parts, we obtain

$$\int_0^\infty \frac{\log x}{x^2+4}dx = \frac{1}{2\pi}\text{Im} \oint_{\Gamma_R} \frac{(\log z)^2}{z^2+4}dz$$
$$= \frac{1}{2\pi}\text{Im}\,\frac{\pi}{2}\left(\log 2 + \pi\frac{i}{2}\right)^2$$
$$= \frac{1}{4}\pi \log 2.$$

5. (Ph.D.) Define $L : C(X) \to \mathbb{R}$ by

$$L(f) = \lim_{n\to\infty} \int f\, d\mu_n,$$

which limit is assumed to exist. Then L is linear since each $f \mapsto \int f\, d\mu_n$ is linear, and positive (i.e., $L(f) \geq 0$ if $f \geq 0$) since $\int f\, d\mu_n \geq 0$ if $f \geq 0$. Now $\{\int f\, d\mu_n\}$ is bounded for each f, so by the uniform boundedness principle, L is bounded. (Alternatively, positivity implies boundedness since if $M = \sup |f(x)|$, $M - f \geq 0$ and thus $L(M) - L(f) \geq 0$, i.e., $L(f) \leq M \cdot L(1)$, and similarly with $-L(f)$, so that $|L(f)| \leq M \cdot L(1)$.) But by the Riesz representation theorem, every bounded linear functional on $C(X)$ is given by integration against a (finite) signed Baire measure. So $L(f) = \int f\, d\mu$ for some finite signed Baire measure μ. Since L is positive, the negative part of its Jordan decomposition must vanish, and μ is a positive measure.

6. (Ph.D.) The function $\sin z - z^3$ is odd and entire, hence can be written in the form $\sin z - z^3 = zg(z^2)$, where g is entire. Suppose $\sin z - z^3$ has only finitely many zeros. Then $g(\zeta)$ has only finitely many zeros, and so we may write $g(\zeta) = p(\zeta)h(\zeta)$, where p is a polynomial and h is entire without zeros. Now since $\sin z$ is a linear combination of e^{iz} and e^{iz}, $|\sin z| \leq \text{const} \cdot e^{|z|}$. Thus $|\sin z - z^3| \leq Ce^{|z|}$ for some constant $C > 0$, and $|g(\zeta)| \leq C_1 e^{\sqrt{|\zeta|}}$ for some $C_1 > 0$, and $|h(\zeta)| \leq C_2 e^{\sqrt{|\zeta|}}$ for some $C_2 > 0$. Since h is entire of finite order without zeros, it is the exponential of a polynomial, and since the order is $\leq \frac{1}{2}$, this polynomial is a constant. That implies g is a polynomial, and hence $\sin z - z^3$ is a polynomial, a contradiction.

7. (Ph.D.) Let $\sum_{n=-\infty}^{\infty} c_n e^{in\theta}$ be the (formal) Fourier series expansion of $f(e^{i\theta})$, given by

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{-in\theta}d\theta.$$

Since f is C^2, integrating by parts twice in the integral gives the estimate

$$|c_n| \leq \frac{C}{n^2} \qquad (n \neq 0),$$

for some constant $C > 0$ (depending on the sup norms of f, f', and f''). Since

$$\sum_{n \neq 0} \frac{C}{n^2} < \infty,$$

the Fourier series for f converges uniformly to a continuous function (by the Weierstrass M-test). Since on the other hand it converges to f in L^2 (this is true for any function in $L^2(\mathbb{T})$), the series must converge uniformly to f. Let

$$f_+(z) = \sum_{n=0}^{\infty} c_n z^n, \qquad f_-(z) = \sum_{n=1}^{\infty} c_{-n} z^{-n}.$$

Then, again by the Weierstrass M-test, these series converge uniformly to continuous functions on $\overline{D}$ and $\mathbb{C} \setminus D$, respectively. Since these are power series in z and z^{-1}, respectively, each with radius of convergence at least 1, f_+ and f_- are holomorphic in D and $\mathbb{C} \setminus \overline{D}$, respectively. By construction, $f = f_+ + f_-$ on $\mathbb{T}$.

Clearly, we are free to add an arbitrary constant to f_+ and to subtract the same constant from f_-, but this is the extent of the non-uniqueness. Indeed, if g_+ and g_- have the same properties as f_+ and f_- with respect to f, then $g_+ - f_+$ and $g_- - f_-$ have the same properties with respect to the zero-function. But $g_+ - f_+$, being holomorphic in D, has no non-zero Fourier coefficients with negative index, and $g_- - f_-$, being holomorphic in $\mathbb{C} \setminus \overline{D}$, has no non-zero Fourier coefficients with positive index. Thus $g_+ - f_+ = f_- - g_-$ is a constant.

8. (Ph.D.) (a) Let $z_0 \in \Omega$ and consider a closed ball $\overline{B}_r(z_0)$ centered at z_0 and contained in Ω. (Such a ball exists since Ω is open.) By the Mean Value Property,

$$g(z_0) = \frac{1}{\pi r^2} \iint_{B_r(z_0)} g(z)\, dm(z)$$

for any holomorphic function g in Ω, and thus

$$g(z_0) = \left\langle g, \frac{1}{\pi r^2} \chi_{B_r(z_0)} \right\rangle_{L^2(\Omega,\, dm)}$$

for any $g \in B(\Omega)$. Thus

$$\begin{aligned} |f_n(z_0) - f_m(z_0)| &= \left| \left\langle f_n(z_0) - f_m(z_0), \frac{1}{\pi r^2} \chi_{B_r(z_0)} \right\rangle \right| \\ &\leq \|f_n(z_0) - f_m(z_0)\|_{L^2} \left\| \frac{1}{\pi r^2} \chi_{B_r(z_0)} \right\|_{L^2} \qquad \text{(Cauchy-Schwarz)} \\ &= \|f_n(z_0) - f_m(z_0)\|_{L^2} \cdot \frac{1}{\pi r^2} \sqrt{\pi r^2} = \frac{1}{r\sqrt{\pi}} \|f_n(z_0) - f_m(z_0)\|_{L^2}. \end{aligned}$$

Since $\{f_n\}$ is a Cauchy sequence in L^2, this shows that the sequence $\{f_n\}$ is uniformly Cauchy on any compact set $K \subset \Omega$ (take r to be less than the distance from K to the boundary of Ω). Hence $\{f_n\}$ converges uniformly on compacta to an analytic function. Since also $f_n \to f$ in L^2, the limits are the same and (after perhaps changing f on a null set) f is analytic, so $f \in B(\Omega)$. Thus $B(\Omega)$ is closed.

(b) The proof of (a) showed that for $f \in B(\Omega)$, $z_0 \in \Omega$, and $\overline{B}_r(z_0) \subset \Omega$,

$$f(z_0) = \left\langle f,\, \frac{1}{\pi r^2}\chi_{B_r(z_0)} \right\rangle_{L^2(\Omega,\, dm)} .$$

Let P be the orthogonal projection from $L^2(\Omega,\, dm)$ onto the closed subspace $B(\Omega)$, and let

$$k(\_\_,\, z_0) = P\left(\frac{1}{\pi r^2}\chi_{B_r(z_0)}\right) \in B(\Omega).$$

Then

$$\begin{aligned}
\langle f,\, k(__,\, z_0)\rangle &= \left\langle f,\, P\left(\frac{1}{\pi r^2}\chi_{B_r(z_0)}\right)\right\rangle \\
&= \left\langle Pf,\, \left(\frac{1}{\pi r^2}\chi_{B_r(z_0)}\right)\right\rangle \qquad \text{(since } P = P^*\text{)} \\
&= \left\langle f,\, \left(\frac{1}{\pi r^2}\chi_{B_r(z_0)}\right)\right\rangle \qquad \text{(since } f \in \operatorname{range} P\text{)} \\
&= f(z_0).
\end{aligned}$$

4. (M.A.) There are several ways to do this, including contour integration. The fastest method is by change of variables $u = e^x$, $du = u\,dx$:

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{dx}{\cosh x} &= 2\int_0^{\infty} \frac{dx}{\cosh x} \\
&= 2\int_0^{\infty} \frac{2}{e^x + e^{-x}}dx \\
&= 4\int_1^{\infty} \frac{1}{u + u^{-1}}\frac{du}{u} \\
&= 4\int_1^{\infty} \frac{du}{1+u^2} \\
&= 4\arctan u\Big]_1^{\infty} = 4(\frac{\pi}{2} - \frac{\pi}{4}) \\
&= 4\frac{\pi}{4} = \pi.
\end{aligned}$$

5. (M.A.) We claim there are constants $\varepsilon > 0$ and $L > 0$ (independent of n) such that $|f_n(x)| \geq \varepsilon$ for x in a measurable set E_n of measure $\geq L$. Indeed, if not, then for each $\varepsilon > 0$, the measure of $\{x : |f_n(x)| \geq \varepsilon\}$ tends to 0 as $n \to \infty$, which since $\sup_x |f_n(x)| \leq M$ gives

$$\begin{aligned} \|f_n\|_{L^p} &\leq \|\varepsilon \chi_{\{x:|f_n(x)|\leq\varepsilon\}} + M\chi_{\{x:|f_n(x)|\geq\varepsilon\}}\|_{L^p} \\ &\leq \varepsilon + M\|\chi_{\{x:|f_n(x)|\geq\varepsilon\}}\|_{L^p} \to \varepsilon, \end{aligned}$$

a contradiction since we can take ε arbitrarily small.

Suppose $a_n \not\to 0$. Then passing to a subsequence, we may assume $|a_n| > \delta > 0$ for all n. Since [0, 1] has finite measure, there must be a subset E of [0, 1] of positive measure such that for $x \in E$, x lies in infinitely many of the E_n. (If not, for almost all x, x lies in only finitely many of the E_n. Since the set $\mathcal{F}$ of finite subsets of $\mathbb{N}$ is countable, almost all of [0, 1] is partitioned into the countably many sets

$$E_F = \{x : x \in E_n \Leftrightarrow n \in F\}, \qquad F \in \mathcal{F},$$

whose measures have to add up to 1. So there exist $F_1, \ldots, F_j$ whose complement has measure $< L$. This is impossible, since for $n \notin F_1, \ldots, F_j$, E_n is a set of measure at least L which does not meet $F_1, \ldots, F_j$.) Then on E, $\sum_n a_n f_n(x)$ converges a. e. while $|f_n(x)| \geq \varepsilon$ for infinitely many n and $|a_n| > \delta$ for all n, a contradiction.

6. (M.A.) The linear fractional transformation

$$L : z \mapsto \frac{z-1}{z+1}$$

sends $1 \mapsto 0$, $-1 \mapsto \infty$, $0 \mapsto -1$, and $i \mapsto i$. Thus L maps Ω in a one-to-one conformal way onto the domain $\{re^{i\theta} : r > 0, \frac{\pi}{2} < \theta < 2\pi\}$. So

$$g : z \mapsto \left(-i\left(\frac{z-1}{z+1}\right)\right)^{\frac{2}{3}}$$

maps Ω in a one-to-one conformal way onto the upper half-plane. However,

$$g(0) = i^{\frac{2}{3}} = e^{\frac{\pi i}{3}} = \frac{1}{2} + i\frac{\sqrt{3}}{2}.$$

So we need to compose with a conformal automorphism of the upper half-plane sending this point to i. The map

$$z \mapsto \frac{2}{\sqrt{3}}\left(z - \frac{1}{2}\right)$$

is a conformal automorphism of the upper half-plane sending $\frac{1}{2} + i\frac{\sqrt{3}}{2}$ to i. So

$$f : z \mapsto \frac{2}{\sqrt{3}}\left(g(z) - \frac{1}{2}\right) = \frac{2}{\sqrt{3}}\left(-i\left(\frac{z-1}{z+1}\right)\right)^{\frac{2}{3}} - \frac{1}{\sqrt{3}}$$

has all the desired properties.