DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
WRITTEN GRADUATE QUALIFYING EXAM
Analysis (Ph.D. VERSION)
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Instructions

(1) Answer any six of the eight questions. Do not hand in answers for more than six questions. Each will be assigned a grade from 0 to 10. In problems with multiple parts, be sure to go on to subsequent parts even if there is some part you cannot do. You may assume the answer to any part in subsequent parts of the same problem.
(2) Use a different booklet for each question. Write the problem number and your **code number** (not your name) on the outside cover of each booklet.
(3) Keep scratch work on separate pages of the same booklet.
(4) Unless otherwise stated, you may appeal to a "well-known theorem" in your solution to a problem. However, it is your responsibility to make it clear exactly which theorem you are using and to justify its use.

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1. Suppose f is a real-valued function of bounded variation defined on the closed interval $[a, b]$, such that for some positive α and β,

$$m^*\left(\{\, x : D^+f(x) > \alpha\,\}\right) > \beta$$

where D^+f denotes the upper right derivate (= upper right derivative) of f and m^* denotes outer measure.
Prove that the total variation $T_a^b(f) \geq \alpha\beta$.

2. (a) Let $\mathcal{F}$ be a family of complex-valued continuous functions on a compact space X. Define what it means for $\mathcal{F}$ to be an equicontinuous family.
(b) Let Ω be an open subset of $\mathbb{C}$ and $\mathcal{F}$ the family of all holomorphic (analytic) functions f in Ω which satisfy $|f| \leq 1$ in Ω. Prove in detail that $\mathcal{F}$ is equicontinuous on every compact subset of Ω.

3. Suppose that $\{f_n\}$ is a sequence which is bounded in $L^2[0, 1]$, and that the limit $\lim_{n\to\infty} f_n(x) = f(x)$ exists for a.e. x in $[0, 1]$.
Prove that $\lim_{n\to\infty} ||f_n - f||_1 = 0$.

4. Let $f(z) = z + a_2\, z^2 + \cdots$ be holomorphic and 1–1 in $D = \{z \in \mathbb{C} : |z| < 1\}$. Show that if $D \subseteq f(D)$ then $f(z) \equiv z$.

5. Let f be a bounded, uniformly continuous real-valued function on $\mathbb{R}$, suppose that $\{g_n\}$ is a sequence of real-valued Lebesgue integrable functions on $\mathbb{R}$, and suppose that $\int_{\mathbb{R}} |g_n| = 1$ for each positive integer n.
If $h_n(x) = \int_{\mathbb{R}} f(x-y) g_n(y) dy$, prove that there is a bounded continouous function h on $\mathbb{R}$ and a subsequence $\{h_{n_k}\}$ such that $\{h_{n_k}\}$ converges uniformly to h on compact subsets of $\mathbb{R}$.

6. Let Ω be a domain in $\mathbb{C}$ and $F : \Omega \times [0,1] \to \mathbb{C}$ be a measurable function with the following two properties:
(i) for every $t \in [0,\, 1]$, the function $z \longmapsto F(z,\, t)$ is holomorphic in Ω;
(ii) there is a function $g \in L^1\,([0,\, 1])$ such that $|F(z,\, t)| \leq g(t) \quad \forall\, z \in \Omega\, \forall t \in [0,\, 1]$.
Prove that the function
$$z \longmapsto \int_0^1 F(z,\, t)\, dt$$
is a holomorphic function in Ω.

7. Let $(S, \mathcal{M}, \mu)$ and $(T, \mathcal{N}, \nu)$ be complete finite measure spaces. Suppose that the function k defined on $T \times S$ is in $L^2(\nu \times \mu)$, where $\nu \times \mu$ is the product measure. Suppose that $f \in L^2(S, \mu)$. Define $H(t, s) = k(t, s) f(s)$.
(a) Prove that H is in $L^1(\nu \times \mu)$.
(b) Define the operator K on $L^2(T, \nu)$ by
$$(Kf)(t) = \int_S k(t, s) f(s) \mathrm{d}\mu(s).$$
Prove that K is a bounded linear operator mapping $L^2(S, \mu)$ into $L^2(T, \nu)$.

8. Let f be a bounded holomorphic function in the unit disk such that $f(0) \neq 0$. Show that there exists $\rho > 0$ with the property that for any $w \in \mathbb{C}$, $0 < |w| < \rho$, the equation
$$z^3 - w f(z) = 0$$
has exactly three distinct roots in the unit disk.