DEPARTMENT OF MATHEMATICS

UNIVERSITY OF MARYLAND

WRITTEN GRADUATE QUALIFYING EXAM

ANALYSIS (PH.D. VERSION)

January 1998

Instructions

i Answer any six of the eight questions. Do not hand in answers for more than six questions. Your work on each question will be assigned a grade from 0 to 10. Some problems have multiple parts or ask you to do more than one thing. Be sure to go on to subsequent parts even if there is some part you cannot do. If you are asked to prove a result and then apply it to a given situation you may receive partial credit for a correct application even though you do not give a correct proof.

ii Use a different sheet (or different set of sheets) for each question. Write the problem number and your **code number** (not your name) on the top of every sheet.

iii Keep scratch work on separate sheets.

iv Unless otherwise stated, you may appeal to a "well-known theorem"in your solution to a problem. However, it is your responsibility to make it clear exactly which theorem you are using and to justify its use.

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1. (a) Let $\mathcal{F}$ be a family of functions each mapping the metric space (X, ρ) into the metric space (Y, σ).
Define: The family of functions $\mathcal{F}$ is (pointwise) equicontinuous at the point $x \in X$.

   (b) Suppose that the continuous function f maps the open interval $(-1, 1)$ into itself.
Prove: The family $\{f^n\}_{n=1}^{\infty}$ is equicontinuous at each point of $(-1, 1)$. ($f^n(x) = f(x)^n$ for all n and x.)

   (c) Suppose that the continuous function g maps the closed interval $[-1, 1]$ *onto* itself.
Prove: The family $\{g^n\}_{n=1}^{\infty}$ fails to be equicontinuous at one or more points of $[-1, 1]$. ($g^n(x) = g(x)^n$ for all n and x.)

2. Let c_n be the Fibonacci sequence

$$c_n = c_{n-1} + c_{n-2}, \;\; c_0 = c_1 = 1 \; .$$

   (a) Show that

$$f(z) = \sum_{n=0}^{\infty} c_n z^n \; ,$$

   is a rational function with two poles.

   (b) Hence find the infinium of the set of $t \geq 0$ such that for some N depending on t

$$t^n \geq |c_n|, \;\; \text{for all} \;\; n > N \; .$$

3. (a) *Evaluate:* $k = \lim_{n \to \infty} \int_0^{\pi} |\sin(nx)| \, dx$.

   (b) *Prove:* $\lim_{n \to \infty} \int_a^b |\sin(nx)| \, dx = \frac{k}{\pi}(b-a)$ for $-\infty < a < b < \infty$.

   (c) *Prove:* $\lim_{n \to \infty} \int_{-\infty}^{\infty} |\sin(nx)| f(x) \, dx = \frac{k}{\pi} \int_{-\infty}^{\infty} f(x) \, dx$ for $f \in L^1(\mathbf{R})$.

4. If γ is the curve $z = 3\cos(t) + i\, 3^{-1} \sin(t), \; 0 \leq t \leq 2\pi$, compute possible values of

$$\int_{\gamma} \sqrt{z^6 - z^4} \, dz \; ,$$

(where the integrand is a fixed branch of $\sqrt{z^6 - z^4}$ on $\mathbf{C} - [-1, 1]$).

5. Call a measure space $(X, \mathcal{F}, \mu)$ uniformly atomic if there is a $\delta > 0$ such that for every $E \in \mathcal{F}$ either $\mu(E) = 0$ or $\mu(E) > \delta$.

   (a) *Prove:* If a measure space $(X, \mathcal{F}, \mu)$ is not uniformly atomic then there is a sequence of *disjoint* sets $\{E_i\}$ from $\mathcal{F}$ of non-zero measure such that $\mu(E_i) \to 0$ as $i \to \infty$.

   (b) *Prove:* If $L^1(X, \mathcal{F}, \mu) \subseteq L^2(X, \mathcal{F}, \mu)$ then $(X, \mathcal{F}, \mu)$ is uniformly atomic.

   (c) *Prove:* If $L^1(X, \mathcal{F}, \mu) \subseteq L^2(X, \mathcal{F}, \mu)$ then $L^1(X, \mathcal{F}, \mu) \subseteq L^\infty(X, \mathcal{F}, \mu)$.

6. (a) Let $f = u + iv$ be analytic for $|z| \leq R$. Show that for $|z| < R$

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \left\{ \frac{R + ze^{-it}}{R - ze^{-it}} \right\} u(Re^{it})\, dt + ic\,,$$

   where $c = \Im(f(0))$ is a real constant.

   (b) Suppose that f is entire with real part u and there exists positive constants A, C, μ so that

$$\int_0^{2\pi} |u(Re^{it})|\, dt \leq AR^\mu\,,$$

   for all $R > C$. Prove that f is a polynomial of degree $n \leq \mu$.

7. Suppose that $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Suppose that $K : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is a measurable function with $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |K(x,y)|^p\, dx\, dy < \infty$. Suppose that $f : \mathbf{R} \to \mathbf{R}$ is measurable and $\int_{-\infty}^{\infty} |f(x)|^q\, dx < \infty$.

   Prove: The function $g(y) = \int_{-\infty}^{\infty} K(x,y) f(x)\, dx$ is defined for almost all $y \in \mathbf{R}$ and

$$\Big(\int_{-\infty}^{\infty} |g(y)|^p\, dy\Big)^{\frac{1}{p}} \leq \Big(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |K(x,y)|^p\, dx\, dy\Big)^{\frac{1}{p}} \Big(\int_{-\infty}^{\infty} |f(x)|^q\, dx\Big)^{\frac{1}{q}}$$

8. Suppose that $f(z)$ is a conformal mapping of $\{\Im(z) > 0\}$ onto $\{w : 0 < \Re(w) < 1,\ 0 < \Im(w) < 1\}$ so that the boundary points $0,\ 1,\ \infty$ corresponds to $0,\ 1,\ 1+i$ respectively.

   (a) Which point x on the real axis corrresponds to the vertice i?

   (b) Prove that f^{-1} has an analytic continuation to a meromorphic function F.