DEPARTMENT OF MATHEMATICS

UNIVERSITY OF MARYLAND

WRITTEN GRADUATE QUALIFYING EXAM

ANALYSIS

August 2004

Instructions

1. Your work on each question will be assigned a grade from 0 to 10. Some problems have multiple parts or ask you to do more than one thing. Be sure to go on to subsequent parts even if there is some part you cannot do. If you are asked to prove a result and then apply it to a given situation you may receive partial credit for a correct application even though you do not give a correct proof.

2. Use a different sheet (or different set of sheets) for each question. Write the problem number and your **code number** (not your name) on the top of every sheet.

3. Keep scratch work on separate sheets.

4. Unless otherwise stated, you may appeal to a "well-known theorem" in your solution to a problem. However, it is your responsibility to make it clear exactly which theorem you are using and to justify its use.

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1. Let $E \subseteq R$ be a measurable set of positive finite measure. For each $t \geq 0$, define $f(t) = m(E \cap E_t)$ where $E_t = \{t + x : x \in E\}$. Prove that

a) f is continuous on $[0, \infty)$.

b) $\lim_{t \to \infty} f(t) = 0$.

2. Compute the explicit value of

$$\int_0^\infty \frac{1}{x^9 + 1}\, dx.$$

3. Suppose $\{f_n\} \subseteq C^1[0,1]$, and suppose that the sequence $\{f_n\}$ satisfies a, b, and c below:

a) $f_n(0) = 0, \quad$ all n,

b) $|f_n'(x)| \leq \frac{1}{\sqrt{x}}, \quad$ a.e., $\quad 0 < x \leq 1$,

c) There exists a measurable function h on $[0,1]$ such that $f_n'(x) \to h(x)$ for all $x \in [0,1]$.

Prove that $\{f_n\}$ converges uniformly on $[0,1]$ to an absolutely continuous function f.

4. Let $u(x, y)$ be a real valued function harmonic on $\mathbf{C}$. Show that, unless u has constant gradient ∇u, for all nonzero vectors ω there is a point where ∇u is parallel to ω.

5. Suppose $f_0 \in L^1[0,1]$, that $f_0(x) \geq 0$ *a.e.* and that

$$f_{n+1}(x) = \left(\int_0^x f_n\right)^{1/2}, \quad \forall \quad n \geq 0.$$

Assume further that $f_1(x) \leq f_0(x)$ *a.e.*.

a) Prove that for each $x \in [0,1]$ the sequence $\{f_n(x)\}_{n=1}^\infty$ converges monotonically to a non-negative number $f(x)$.

b) Prove that $f(x) = (\int_0^x f)^{1/2}$ for all $x \in [0,1]$.

c) Prove that f is differentiable at all x for which $f(x) > 0$ and calculate $f'(x)$ at these points.

d) In particular, find a simple explicit formula for the function f in the case that $f(x) > 0$ for all $x \in (0,1]$.

6. Let $f(z) = 1 + a_1 z + a_2 z^2 + \ldots$ be analytic on $\mathbf{D} = \{|z| < 1\}$ so that $f(\mathbf{D}) \subset \{z : \Re(z) > 0\}$. Prove that for all $z \in \mathbf{D}$,

$$|f(z)| \leq \frac{1 + |z|}{1 - |z|}.$$