

ANALYSIS QUALIFYING
EXAMINATION

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Unless otherwise stated, you may appeal to a “well known theorem” in your solution to a problem, but if you do, it is your responsibility to make it clear which theorem you are using and why its use is justified. You may use any given hint without proving it. In problems with multiple parts, be sure to go on to the rest of the problem even if there is some part you cannot do. In working on any part, you may assume the answer to any previous part, even if you have not proved it.

1. For a function $f : [0, 1] \rightarrow \mathbb{R}$, prove that f is Lipschitz if and only if it is absolutely continuous and there is an $M \geq 0$ for which

$$|f'(x)| \leq M \quad \text{for almost all } x \in [0, 1].$$

Here “almost all” refers to Lebesgue measure.

2. For $|z| < 1$, evaluate the integral

$$\int_{|\zeta|=1} \frac{1}{\zeta(\zeta - z)} d\zeta.$$

Use the function $f(\zeta) = 1/\zeta$, to test if for $|z_0| = 1$, does

$$\lim_{z \rightarrow z_0, |z| < 1} \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{f(\zeta)}{\zeta - z} d\zeta$$

necessarily equal $f(z_0)$?

3. Let m denote the Lebesgue measure on $[0, 1]$. Let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence of Lebesgue measurable functions defined on $[0, 1]$, such that for each $n \geq 1$, $0 \leq f_n \leq M$ for a positive constant M . Moreover, assume that $\|f_n\|_1 = \int_0^1 f_n(x) dm(x) = 1$ for all $n \geq 1$. Consider a sequence $\{a_n\}$ such that $a_n \geq 0$ for all $n \geq 1$, and assume that $\sum_{n=1}^{\infty} a_n = \infty$.

- State Egoroff's theorem.
- Prove that there exists a subset A of positive measure in $[0, 1]$ such that $\sum_{n=1}^{\infty} a_n f_n(x) = \infty$ for each $x \in A$.

4. Let $D \subset \mathbb{C}$ be the unit disc and Q the strip $\{z \in \mathbb{C} : |\operatorname{Im} z| < \pi/2\}$.
- Find a conformal mapping $f : D \rightarrow Q$ with $f(0) = 0$, $f'(0) = 2$.
 - Show that if $g : D \rightarrow Q$ is analytic with $g(0) = 0$, then $|g'(0)| \leq 2$.
5. Fix positive real numbers p, q, a , satisfying $1 \leq p < q < \infty$ (there is no condition on a except $a > 0$), and give the interval $[0, a]$ the usual Lebesgue measure (of total mass a). Let $\|\cdot\|_p$ and $\|\cdot\|_q$ denote the norms for $L^p([0, a])$ and for $L^q([0, a])$, respectively.
- Show that there is a constant $C_a > 0$, such that if $f \in L^q([0, a])$, then $f \in L^p([0, a])$, and $\|f\|_p \leq C_a \|f\|_q$. Find the minimal value of C_a , and show that it cannot be improved.
 - Show that $C_a \rightarrow \infty$ as $a \rightarrow \infty$.
 - Regardless of the value of $a > 0$, show that there is NO constant $C > 0$ such that if $f \in L^p([0, a]) \cap L^q([0, a])$,
- $$f \in L^p([0, a]) \cap L^q([0, a]),$$
- then $\|f\|_q \leq C \|f\|_p$.

6. Let f be a 2π -periodic, entire function satisfying the inequality

$$|f(z)| \leq 1 + |\operatorname{Im} z|.$$

Show that f is constant.