UMCP Department of Mathematics Qualifying Exam
Partial Differential Equations, August 2011

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(1) Solve all six problems. Each main problem will be assigned a grade from zero to ten.
(2) Begin your answer to each question on a separate sheet.
(3) Write your code number on each page of your answer sheets Do *not* use your name.
(4) Keep any scratch work on separate sheets, which should not be submitted
(5) Carefully explain all your steps. If you invoke a "well-known" theorem, you must make clear which theorem you are using and justify its use.

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1. Consider a bounded domain $U \subset \mathbb{R}^n$ with C^1 boundary ∂U
(i) Prove that the *inequality*

$$\int_U u^2 dx \leq \frac{1}{\lambda_1} \int_U |\nabla u|^2 dx,$$

holds for every $u \in H_0^1(U)$. Here, λ_1 is the smallest eigenvalue of $-\Delta$ with zero boundary conditions, namely, the smallest λ with non-trivial ϕ such that $-\Delta \phi = \lambda \phi$ in U and $\phi_{|\partial U} = 0$.
(ii) Show that there exists a non-trivial solution u for the *equality* $\int_U u^2 dx = \frac{1}{\lambda_1} \int_U |\nabla u|^2 dx$.

2. Let U be a bounded open domain in $\mathbb{R}^n$ with smooth boundary ∂U, and consider the problem

$$\begin{cases} -\Delta u + cu &= 0, \quad x \in U, \\ \frac{\partial u}{\partial \nu} &= g, \quad x \in \partial U. \end{cases} \tag{1}$$

Here, c is a positive constant $c > 0$; ν denotes the normal to ∂U and $g \in L^2(\partial U)$.
(i) Find a weak formulation of problem (1), and prove that a *weak* solution exists.
(ii) Is the *weak* solution of part (i) unique?
(iii) Assume that $c = 0$ in (1). Does a *weak* solution of problem (1) exist for *all* g's? if "yes", justify your answer. If "no", provide a necessary condition for existence

3. Let Ω be a bounded open set of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and let $f : \mathbb{R} \to \mathbb{R}$ be a C^1 function such that $|f'| \leq K$ and $f(0) = 0$. Assume that u is a C^2 solution of

$$\begin{cases} \partial_t u - \Delta u = f(u) & \text{in } \Omega \times (0, \infty), \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0, \infty). \end{cases} \tag{2}$$

(i) Show that if $u(x, 0) \geq 0$ for all $x \in \Omega$ then $u(x,t) \geq 0$ for all $x \in \Omega$ and all $t > 0$.
(ii) Show that if $u(x,0) \leq M$ for all $x \in \Omega$ then $u(x,t) \leq Me^{Kt}$ for all $x \in \Omega$ and all $t > 0$.

4. Let I be a bounded interval in $\mathbb{R}$. Show that there exists a constant C such that

$$||uv||_{H^1(I)} \leq C||u||_{H^1(I)}||v||_{H^1(I)}$$

for all functions u and v in $H^1(I)$.

5. Consider the initial value problem

$$u_t + uu_x + u = 0\,, \qquad -\infty < x < \infty\,,\ t > 0 \tag{3}$$

subject to initial condition $u(x,0) = a \sin x$.

(i) Find the characteristic curves associated with (3) in an explicit form.

(ii) Show that if $a > 1$, then there exists a time $t = t_c > 0$ such that there exists *no* smooth solution of (3) for $t > t_c$. Find this maximal time of smoothness, t_c

6. Let θ be a given constant, $0 < \theta < 1$. Suppose that w solves the nonlinear wave equation

$$w_{tt} - w_{xx} = \theta \frac{w}{1+w^2}, \qquad x \in [0,\pi], \quad t > 0\,,$$

subject to the boundary conditions $w(x=0,t) = w(x=\pi,t) = 0$, and smooth initial data $w(x,0) = w_0(x)$, $w_t(x,0) = w_1(x)$.

(i) Derive an *energy integral*, $E(t)$, in terms of w, w_t and w_x, which is constant in time.
Hint: Multiply by w_t.

(ii) Conclude that there exists a constant C_0 (depending on w_0) such that

$$\max_{0 \le x \le \pi} |w(x,t)| \le C_0 \quad \text{for all } t > 0.$$

Hint: You may use the following three facts. (1) The bound $\ln(1+a^2) \le a^2$; (2) Sobolev inequality; and (3) the following Poincaré inequality: $\int_0^\pi w^2(x)\,dx \le \int_0^\pi w_x^2(x)\,dx$, which holds for all $w(x) \in C^1([0,\pi])$ such that $w(0) = w(\pi) = 0$