DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAM
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LOGIC (PhD/MA version)

1. (a) Let L be the language whose only non-logical symbol is the binary relation symbol E. Let $\mathbf{K}$ be the class of all L-structures $\mathfrak{A}$ such that $E^{\mathfrak{A}}$ is an equivalence relation with at least one finite equivalence class. Prove that there is no L-theory T such that $\mathbf{K} = Mod(T)$.

   Hint: Assuming that $\mathbf{K} \subseteq Mod(T)$ show that T has a model which is not in $\mathbf{K}$.

   (b) Let L be a language containing at least the binary relation symbol E. Let T be a theory of L such that E defines an equivalence relation in each model of T. Assume that T has some model which has arbitrarily large finite E-classes but no infinite E-class. Prove that T has a model with infinitely many infinite E-classes.

2. (a) Let T be a complete theory in a countable language L Assume there is a type $\Phi(x)$ consistent with T such that any two countable models of T realizing Φ are isomorphic. Prove that T has a countable ω-saturated model.

   (b) Let $\mathfrak{A}_0$, $\mathfrak{B}$, and $\mathfrak{A}_1$ be L-structures. Assume that $\mathfrak{A}_0 \prec \mathfrak{A}_1$ and $\mathfrak{A}_0 \subseteq \mathfrak{B} \subseteq \mathfrak{A}_1$. Prove that if $\mathfrak{A}_0 \models \sigma$ then $\mathfrak{B} \models \sigma$ for every L-sentence σ of the form
$$\exists x_1 \ldots \exists x_n \forall y_1 \ldots \forall y_m \alpha(x_1, \ldots x_n, y_1 \ldots y_m),$$
   where α is an open L-formula.

3. (a) Let T be a complete theory in a countable language L, and assume that T has a prime model $\mathfrak{A}$. Assume that no (countable) proper elementary extension of $\mathfrak{A}$ is also prime. Prove that no uncountable model of T is atomic.

   (b) Let T be a complete theory in a language L, and let $\Phi(x)$ be an L-type. Assume that Φ is realized by *at most* two elements in every model of

T and by *exactly* two elements in some model of T. Prove that Φ is realized by *exactly* two elements in every model of T.

4. (a) Prove that there is some $H : \omega \to \omega$ such that for every recursive $f : \omega \to \omega$ there is some $n_f \in \omega$ such that $f(k) < H(k)$ for all $k > n_f$.

   (b) Let $A \subseteq \omega$ be an infinite r.e. set. Prove that there are *disjoint* infinite r.e. sets A_0 and A_1 with $A = A_0 \cup A_1$.

5. (a) Let L be the language whose only non-logical symbol is the binary relation symbol E. Prove that there is an undecidable theory T of L such that $E^{\mathfrak{A}}$ is an equivalence relation in every $\mathfrak{A} \models T$.

   (b) Prove that there is some $e \in \omega$ such that
   $$\{e\}(e+1) = \{e+2\}(e).$$

   Note: You may not use the special properties of any particular enumeration of the partial recursive functions.

6. (a) Let $f, g : \omega \to \omega$ be total recursive functions. Let $I_f = \{e : \{e\} = f\}$ and let $I_g = \{e : \{e\} = g\}$. Prove that $I_f \equiv_m I_g$.

   (b) Let $f : \omega \to \omega$ be a total recursive function and let $I_f = \{e : \{e\} = f\}$. Prove that $I_f \leq_T \mathbf{0}''$.