Geometry/Topology Ph.D Written Examination: August 1995

Instructions: Answer any six questions in separate answer booklets. Make sure the front page of each booklet bears the question number and your examination code number. Under no circumstances will more than six answer booklets be accepted from any student or more than one question graded from any answer booklet. Although your choice of six questions is unrestricted, each question has been labeled for your guidance with the number of the course on which it is based. Recent MATH 730 students may wish to note that question 10 bears two labels.

1. [730] Let

$$X = (0,1) \cup 2 \cup (3,4) \cup 5 \cup \ldots \cup (3n, 3n+1) \cup 3n+2 \cup \ldots, \text{ and}$$
$$Y = (0,1] \cup (3,4) \cup 5 \cup \ldots \cup (3n, 3n+1) \cup 3n+2 \cup \ldots,$$

where both X and Y are topologized as subspaces of $\mathbb{R}$.

(a) Find *continuous* bijections $f: X \to Y$ and $g: Y \to X$.
(b) Show that X and Y are not homeomorphic.

2. [730]

(a) Let $p: \mathbb{R} \to X$ be a covering map. Show that X must be either $\mathbb{R}$ itself or the circle S^1.
(b) Let $p: S^n \to X$ be a covering map. The possible values of the Euler characteristic $\chi(X)$ depends on the value of n. Give examples of all possibilities.

3. [730] Let X be a disk with boundary C_0. Let D_1 and D_2 be disjoint closed disks contained in the interior of X, and let Y be X with the interiors of D_1 and D_2 removed. The boundary of Y consists of three circles: C_0, the outer circle, and C_1 and C_2, the boundaries of D_1 and D_2. Identify points on as Y follows: on C_0 identify antipodal points, on C_1 identify points $120°$ apart, and on C_2 identify points $90°$ apart; points not on the boundary of Y are identified only with themselves. Let W be the resulting quotient space. Calculate the fundamental group of W.

4. [734] Let $f: X \to \mathbb{CP}^n$ be a continuous function from a CW complex X to complex projective space $\mathbb{CP}^n$ (which has real dimension $2n$). Suppose that the map $f_*: H_{2n}(X, \mathbb{Z}) \to H_{2n}(\mathbb{CP}^n, \mathbb{Z})$ is nonzero.

(a) If β is the generator of $H^{2n}(\mathbb{CP}^n, \mathbb{Z})$, show that $f^*(\beta) \neq 0$.
(b) Show that $H^2(X, \mathbb{Z})$ and $H_2(X, \mathbb{Z})$ are both nontrivial groups. (Hint: The cohomology algebra of $\mathbb{CP}^n$ is generated by a single element in $H^2(\mathbb{CP}^n, \mathbb{Z})$.)

5. [734] Let Y be a CW complex so that

$$H_i(Y, \mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } i = 0, 2 \text{ or } 4, \\ \mathbb{Z}/12\mathbb{Z}, & \text{if } i = 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Calculate the cohomology groups $H^*(Y, \mathbb{Z})$ and $H^*(Y, \mathbb{Z}/2\mathbb{Z})$.
(b) Could Y have the homotopy type of a compact orientable manifold without boundary?
(c) Could Y have the homotopy type of a compact nonorientable manifold without boundary?

6. [734] For any positive integer n, let Z_n be the quotient space of the unit disk $\{z \in \mathbb{C} \text{ with } |z| \leq 1\}$ with the identifications $z \sim e^{2\pi i/n} z$ for all z with $|z| = 1$ and also $1 \sim e^{\pi i/n}$. Alternatively, Z_n is the space obtained by gluing the disk to a figure eight via a map which wraps the boundary of the disk n times around the figure eight. Calculate $H_*(Z_n, \mathbb{Z})$ and $H_*(Z_n, \mathbb{Z}/n\mathbb{Z})$ and $\pi_1(Z_n, 1)$.

7. [740] Let (x, y, z) be the standard coordinates on $\mathbb{R}^3$, and let e_1, e_2, and e_3 be the standard coordinate vector fields. Let ρ be the symmetric $(0, 2)$ tensor on $\mathbb{R}^3$ defined by:

$$\rho(e_i, e_j) = \begin{cases} 0 & \text{if } i \neq j, \\ -1 & \text{if } i = j = 3, \\ 1 & \text{otherwise.} \end{cases}$$

Consider the submanifold $\mathbb{H} = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 - z^2 = -1, z > 0\}$. The restriction of ρ to $\mathbb{H}$ is a Riemannian metric on $\mathbb{H}$.

(a) Show that reflection through the x-z plane is an isometry of $\mathbb{H}$.
(b) Write down a parametrization of the unit speed geodesic in $\mathbb{H}$ with initial velocity $(1, 0, 0)$ at the point $(0, 0, 1)$.
(c) Show that there are no closed geodesics in $\mathbb{H}$ through the point $(0, 0, 1)$.

8. [740] Let M be a complete Riemannian manifold, and N_1 and N_2 disjoint compact submanifolds.

(a) Show that there is a length minimizing curve from N_1 to N_2.
(b) Let γ be a length minimizing curve from N_1 to N_2. Show that γ is a geodesic and is orthogonal to both N_1 and N_2.
(c) Suppose that $\dim(N_1) + \dim(N_2) \geq \dim(M)$. Show that there is a nonzero parallel vector field along γ that is tangent to N_1 and N_2 at the two endpoints of γ.

9. [740]

(a) Let M be a three-dimensional Riemannian manifold. Let X, Y, Z be an orthonormal basis of $T_x(M)$. Give an expression for the sectional curvature of the plane spanned by X and Y in terms of the Ricci curvatures of X, Y and Z.
(b) A Riemannian manifold is said to be *Einstein* if $Ric(X, Y) = c\langle X, Y\rangle$, where Ric is the Ricci tensor, c is a constant, and $\langle , \rangle$ is the metric tensor. Show that a three-dimensional Einstein manifold has constant sectional curvature.
(c) Show that a four-dimensional Einstein manifold does not necessarily have constant sectional curvature by considering the manifold $S^2 \times S^2$.

10. [730, 742] Consider the function $f : \mathbb{R}^4 \to \mathbb{R}$ defined by

$$f(x_1, x_2, x_3, x_4) = x_1 x_2 + x_3 x_4.$$

(a) Find the set of regular values of f.
(b) Compute the tangent hyperplane to $f^{-1}(1)$ at the point $(1, 1, 0, 0)$.
(c) Let S^3 be the unit sphere in $\mathbb{R}^4$. Show that $f^{-1}(1)$ is transverse to S^3.

11. **[742]** Let D be the closed unit disk in $\mathbb{R}^2$, and $S^1 = \partial D$. Let f and g be smooth embeddings of S^1 in $\mathbb{R}^3$ such that $f(S^1) \cap g(S^1) = \emptyset$. Define $\lambda : S^1 \times S^1 \to S^2$ by

$$\lambda(x, y) = \frac{f(x) - f(y)}{||f(x) - f(y)||}.$$

(a) Show that λ is a smooth map.
(b) Suppose f extends to a smooth map $\widehat{f} : D \to \mathbb{R}^3$ such that $\widehat{f}(D) \cap g(S^1) = \emptyset$ and $\widehat{f}|S^1 = f$. Show that the (integer) degree of λ is 0.
(c) Prove a partial converse of (b) by showing that if the degree of λ is 0, then λ extends to a smooth map $\widehat{\lambda} : D \times S^1 \to S^2$.

12. **[742]** Let D^k be the closed unit ball in $\mathbb{R}^k$, and $S^k = \partial D^{k+1}$.

(a) Let $h : D^k \to \mathbb{R}^n$, $k \leq n$, be a smooth embedding with $h(0) = 0$. Show that the map $H : D^k \times I \to \mathbb{R}^n$ defined by

$$H(x,t) = \begin{cases} t^{-1}h(tx), & 0 < t \leq 1 \\ Dh(0)x, & t = 0 \end{cases}$$

is an isotopy from h to a linear map.
(b) Let $f, g : D^n \to M^n$ be embeddings into a connected n dimensional manifold M. Suppose M^n is orientable and f and g preserve orientation. Show f is isotopic to g.
(c) Let $\mathrm{Diff}_+(S^n)$ denote the group of smooth, orientation preserving diffeomorphisms of S^n, and G the subgroup consisting of restrictions of orientation preserving diffeomorphisms of D^{n+1}. Let $f \in \mathrm{Diff}_+(S^n)$. Show that if f is isotopic to the identity, then $f \in G$.