TOPOLOGY WRITTEN EXAM - Masters Level

August, 1995

1) Let

$$X = (0,1) \cup \{2\} \cup (3,4) \cup \{5\} \cup \cdots \cup (3n, 3n+1) \cup \{3n+2\} \cup \ldots,$$

$$Y = (0,1] \cup (3,4) \cup \{5\} \cup \cdots \cup (3n, 3n+1) \cup \{3n+2\} \cup \ldots,$$

where both X and Y have the topologies as subspaces of $\mathbb{R}$.

a) Find *continuous* bijections $f : X \to Y$ and $g : Y \to X$.

b) Show that X and Y are not homeomorphic.

2) a) Let $p : \mathbb{R} \to X$ be a covering map. Show that X must be either $\mathbb{R}$ itself or the circle S^1.

b) Let $p : S^n \to X$ be a covering map. What are the possible values of the Euler characteristic $\chi(X)$. This depends on the value of n. Give example of each of these possibilities.

3) Let X be a metric space, and $f : X \to X$ a continuous map. Assume that for every open cover $\mathcal{U}$ of X, there exists a space $U \in \mathcal{U}$ such that $f(U) \cap U \neq \emptyset$. Prove that f has a fixed point. Hint: In many fixed point arguments, it is convenient to assume that there does not exist a fixed point.

4) Let X be a disk with boundary C_0. Let D_1 and D_2 be disjoint closed disks contained in the interior of X, and let Y be X with the interiors of D_1 and D_2 removed. The boundary of Y consists of three circles: C_0, the outer circle, and C_1 and C_2, the boundaries of D_1 and D_2. Identify points on as Y follows: on C_0 identify antipodal points, on C_1 identify points $120°$ apart, and on C_2 identify points $90°$ apart; points not on the boundary of Y are identified only with themselves. Let W be the resulting quotient space. Calculate the fundamental group of W.

5) Let X be the unit interval $[0,1]$ as a set of points, but let the topology be modified so that the only neighborhood of the point 1 is the whole space. Observe that X is not Hausdorff. Let $D = \{(x,x) : x \in X\} \subset X \times X$. Show that D is not closed.

6) Let $\tau : S^1 \times S^1 \to S^1 \times S^1$ be given by $\tau((x,y),(u,v)) = ((-x,-y),(-u,v))$. (Note that it is $+v$ not $-v$ on the right!) Let X be the quotient space of $S^1 \times S^1$ by the identification of any point α with $\tau(\alpha)$.

a) Describe the universal covering space of X.

b) Without computing $\pi_1(X)$, describe how one could use the Van Kampen Theorem to do so, suggesting very specifically how one might write X as the union of two closed subspaces.