TOPOLOGY/GEOMETRY QUALIFYING EXAMINATION
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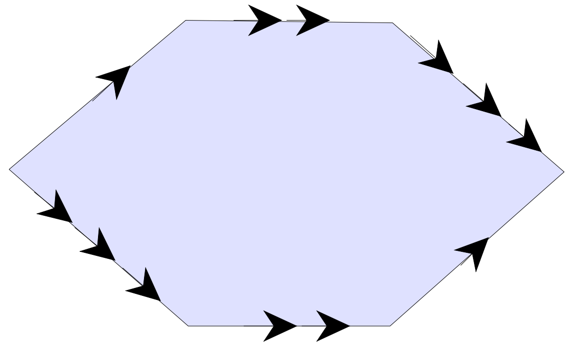
MATHEMATICS DEPARTMENT
UNIVERSITY OF MARYLAND, COLLEGE PARK

Do ANY six problems. Only the first six problems which you do will be graded, so there is no point in turning in more than six problems. This test is designed so that anyone who has covered the syllabi for two of the basic courses (730,734,740,742) should be able to do six problems. The indicated course numbers are included only as a guide and are not intended to restrict your choice of problems (you may safely ignore them if you wish).

Your answers will be graded as mathematical prose. We expect grammatically correct sentences and clear exposition.

1. (**730**) Let (X, d) be a complete metric space. Suppose $A, B \subset X$ are compact subsets.
   (a) Prove that, for every $a \in A$, the minimum $\min_{b\in B} d(a, b)$ exists (i.e., there is a point $b_a \in B$ such that $d(a, b_a) = \min_{b\in B} d(a, b)$). Show that the maximum
   $$h(A, B) = \max_{a\in A} \min_{b\in B} d(a, b)$$
   also exists.
   (b) Show that $h(A, B) = 0$ if and only if $A \subseteq B$.
   (c) Let $\mathcal{K}(X)$ denote the set of all nonempty compact subsets of X. If $A, B \in \mathcal{K}(X)$, define
   $$D(A, B) = \max(h(A, B), h(B, A))$$
   One can show (but we don't ask you to do this) that $(\mathcal{K}(X), D)$ is a complete metric space.
   Let $f : X \longrightarrow \mathcal{K}(X)$ be the map
   $$f(x) = \{x\}$$
   Show that f is an isometric embedding of metric spaces, i.e., $D(f(x), f(y)) = d(x, y)$.
   (d) Let X_n be the subset of $[0, 1]$ consisting of rational numbers whose denominator divides 2^n. Show that X_n is a Cauchy sequence in $\mathcal{K}([0, 1])$, and compute its limit.
2. (**730**)
   (a) Give an example of a surjective continuous map $\mathbb{R} \longrightarrow S^1$ which is a local homeomorphism but not a covering map.
   (b) Let $C = \mathbb{R} \times S^1$ be the cylinder. Find an example of a surjective local homeomorphism $f : X \longrightarrow C$, where X is a surface, for which the induced map
   $$\pi_1(f) : \pi_1(X) \longrightarrow \pi_1(C)$$
   is not injective.
   (c) Show that a contractible topological space is path-connected.

3. (**730**)
   (a) Suppose X is a compact Hausdorff space and $x \in X$. Show that the one-point compactification of the complement $X - \{x\}$ is homeomorphic to X.
   (b) Use the previous part to show the one-point compactification of the Euclidean space $\mathbb{R}^n$ is homeomorphic to the n-sphere
$$S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$$
   (c) Suppose X is homeomorphic to a disjoint union of k circles, and that the cone on X is homeomorphic to a manifold with boundary. Determine k.
4. (**734**) Let (X, x) and (Y, y) be pointed connected topological spaces (i.e., each is a connected space together with a base point). We define the wedge $X \vee Y$ to be the topological space obtained from the disjoint union $X \amalg Y$ by identifying x and y to a single point z. We choose the resulting point z as the base point for $X \vee Y$.
   Assume now that X and Y are manifolds.
   (a) Is $X \vee Y$ a manifold?
   (b) Show that there is an isomorphism of abelian groups
$$\phi \colon \tilde{H}^*(X \vee Y, z) \to \tilde{H}^*(X, x) \oplus \tilde{H}^*(Y, y)$$
   (c) Compute the multiplication law induced on the direct sum by the cup product on $\tilde{H}^*(X \vee Y, z)$, i.e., find a formula for
$$\phi\left(\phi^{-1}(a_1, b_1) \cup \phi^{-1}(a_2, b_2)\right)$$
   for $a_1, a_2 \in \tilde{H}^*(X, x)$ and $b_1, b_2 \in \tilde{H}^*(Y, y)$.
   (d) Deduce that $S^2 \times S^2$ is not homotopy equivalent to $S^2 \vee S^2 \vee S^4$.
5. (**734**) Let Z be a hexagon with opposite sides identified as shown. Calculate $H^*(Z)$.



6. (**734**) Let M be a compact 7 dimensional manifold without boundary. Suppose $a \in H_2(M)$ is nonzero, but $8a = 0$.
   (a) Suppose $a \in H_2(M)$ is nonzero, but $8a = 0$. If M is orientable, show that there is a nonzero $b \in H_4(M)$ so that $8b = 0$.
   (b) If M is connected and not orientable, compute $H^7(M)$. Show that there is a class in $H^6(M; \mathbb{Z}_2)$ that does not lie in the image of the natural map $H^6(M) \to H^6(M; \mathbb{Z}_2)$ induced by the onto homomorphism $\mathbb{Z} \to \mathbb{Z}_2$.

7. (**740**)
   (a) Let M be an n-dimensional Riemannian manifold with metric $\langle\ ,\ \rangle$, and let σ be a differentiable real-valued function. Define a new metric $(\ ,\ )$ on M by
   $$(\ ,\ ) = e^{2\sigma}\langle\ ,\ \rangle.$$
   Denote by ∇ and $\widetilde{\nabla}$ the Riemannian connections of $\langle\ ,\ \rangle$ and $(\ ,\ )$. Show that ∇ and $\widetilde{\nabla}$ are related by the formula
   $$(\widetilde{\nabla}_X Y, Z) = e^{2\sigma}\Big(\langle\nabla_X Y, Z\rangle + X(\sigma)\langle Y, Z\rangle + Y(\sigma)\langle X, Z\rangle - Z(\sigma)\langle X, Y\rangle\Big),$$
   where X, Y and Z are vector fields on M.
   (b) The Riemann curvature tensors R and $\widetilde{R}$ of ∇ and $\widetilde{\nabla}$ are defined by
   $$R_{XY} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y] \qquad \text{and} \qquad \widetilde{R}_{XY} = \widetilde{\nabla}_{[X,Y]} - [\widetilde{\nabla}_X, \widetilde{\nabla}_Y].$$
   In the case that σ is a constant function, show that
   $$\widetilde{R}_{XY} = R_{XY}.$$
   (c) The Ricci curvature tensors ρ and $\widetilde{\rho}$ of R and $\widetilde{R}$ are defined by
   $$\rho(X,Y) = \sum_{i=1}^{n}\langle R_{XE_i}Y, E_i\rangle \qquad \text{and} \qquad \widetilde{\rho}(X,Y) = \sum_{i=1}^{n}\langle \widetilde{R}_{X\widetilde{E}_i}Y, \widetilde{E}_i\rangle,$$
   where $\{E_1, \ldots, E_n\}$ is a local orthonormal frame for $\langle\ ,\ \rangle$ and $\{\widetilde{E}_1, \ldots, \widetilde{E}_n\}$ is a local orthonormal frame for $(\ ,\ )$. In the case that σ is a constant function, show that ρ and $\widetilde{\rho}$ are related by
   $$\widetilde{\rho}(X,Y) = \rho(X,Y).$$

8. (**740**)
   Let M be a Riemannian manifold. Two k-tuples $(p_1, \ldots, p_k)$ and $(q_1, \ldots, q_k)$ of points in M are said to be **isometric** if $d(p_i, p_j) = d(q_i, q_j)$ for any $i, j \in \{1, \ldots, k\}$, where d denotes the distance function of M. The manifold M is said to be k-point homogeneous if for any two isometric k-tuples $(p_1, \ldots, p_k)$ and $(q_1, \ldots, q_k)$ of M there exists an isometry $A : M \to M$ with $A(p_i) = A(q_i)$ for $i = 1, \ldots, k$.
   (a) Show that any 1-point homogeneous manifold is (geodesically) complete.
   (b) Show that any 2-point homogeneous manifold has the property that the covariant derivative of the curvature tensor of M vanishes.

9. (**740**) Show that the following two statements are equivalent for a smooth n-manifold M:
   (a) M admits an atlas where all coordinate changes are (restrictions of) affine maps of $\mathbb{R}^n$
   (b) M admits a torsion free connection with zero curvature tensor.

10. (**742**)
   (a) Let V be a real finite-dimensional vector space, Δ the diagonal of $V \times V$. Suppose $A : V \to V$ is an invertible linear map. Show that $W = \{(v, Av) \mid v \in V\}$ is transverse to Δ if and only if 1 is not an eigenvalue of A.
   (b) Let M^n be a compact smooth manifold without boundary, and $f : M \to M$ a smooth map. Suppose that for each fixed point $x \in M$, the linear map $df_x : T_xM \to T_xM$ has no eigenvalues equal to 1. Show that there are only finitely many fixed points for f.

11. (**742**) Define an $(n-1)$-form on $\mathbb{R}^n - \{0\}$ by

$$\omega = \sum_{i=1}^{n} (-1)^i \frac{x_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n}{r^n}, \quad \text{where } r = \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2},$$

and the "hat" over dx_i indicates that dx_i is omitted.
   (a) Show that ω is closed.
   (b) Let S be a cube in $\mathbb{R}^n$ containing the origin, and B a ball in $\mathbb{R}^n$ containing the origin. Relate the integrals $\int_{\partial S} \omega$ and $\int_{\partial B} \omega$, where ∂S and ∂B have the orientations induced from the standard orientation on $\mathbb{R}^n$.
   (c) Show that ω is not exact.

12. (**742**) Prove or disprove the following statements.
   (a) For any closed, connected, oriented manifold M of dimension n, there is a degree one map $f : M \to S^n$.
   (b) For any closed, connected, oriented manifold M of dimension n, there is a degree one map $f : S^n \to M$.
   (c) If M^n and N^n are closed and connected, and $f : M \to N$ has nonzero degree, then f is surjective.