Written Qualifying Examination
Geometry/Topology
Friday, August 20, 2004

Instructions. Answer each question on a separate numbered answer sheet. In problems with multiple parts, whether the parts are related or not, the parts are graded independently of one another. Be sure to go on to subsequent parts even if there is some part you cannot do.

You are allowed to appeal to "standard theorems" proved in class or in the textbook, but if you do so, it's your responsibility to state clearly exactly what you're using and why it applies.

1. Let U be a connected subset of $\mathbb{R}^n$. Define $d_l\colon U \times U \to \mathbb{R}$ by $d_l(x, y) =$ inf of the lengths of all broken straight line segments joining x and y in U. (If there are no such paths, let $d_l(x, y) = +\infty$.)
   (a) Prove that $d'_l = \min(d_l, 1)$ is a metric.
   (b) Let d be the ordinary Euclidean metric on U. Show that the identity on U, mapping from the topology induced by d'_l to the topology induced by d, is continuous.
   (c) If U is open, show that the map in (b) is a homeomorphism.
   (d) Give a counterexample to part (c) if U is not assumed to be open.

2. Let $h\colon M \to N$ be a submersion from a smooth manifold M onto a smooth manifold N.
   (a) Show that $h^{-1}(x)$ is a smooth manifold without boundary, for each $x \in N$.
   (b) Suppose that h is proper, that is, $h^{-1}(K)$ is compact for any compact set K in N. By (a), $h^{-1}(x)$ is a smooth compact manifold without boundary, for each $x \in N$. In this case, one can show (you do not need to do this) that for each $x \in N$, there is an open neighborhood U such that $h^{-1}(U)$ is diffeomorphic to $U \times P$, where $P = h^{-1}(x)$, in such a way that the restriction of h to $h^{-1}(U)$ can be identified with the projection $U \times P \to U$. In other words, h is *locally* the projection in a product. Give an example where h is proper and is not *globally* the projection in a product, i.e., where M does not split as $N \times P$ for any P.
   (c) If $N = \mathbb{R}$ and h is proper, show (using (b)) that M is homeomorphic to $P \times \mathbb{R}$ for some compact smooth manifold P.
   (d) Give an example of a submersion $h\colon M \to \mathbb{R}$ which is not proper and with M not a product with $\mathbb{R}$.

3. Let M be a connected manifold with $H_1(M, \mathbb{Z}) = 0$. Show that any continuous $f\colon M \to T^2$ is null homotopic, where T^2 is the torus $S^1 \times S^1$.

4. A compact connected 7-manifold M (without boundary) has the following homology groups:

$$\begin{cases} H_1(M, \mathbb{Z}) & \cong \mathbb{Z}/3, \\ H_2(M, \mathbb{Z}) & \cong \mathbb{Z}, \\ H_3(M, \mathbb{Z}) & \cong \mathbb{Z} \oplus \mathbb{Z}/3. \end{cases}$$

(a) Compute all the remaining homology groups of M.
(b) Compute all the cohomology groups of M.
(c) Give a concrete example of a manifold with these homology groups. You can take M to be of the form $N^4 \times L^3$, with N simply connected.

5.
(a) Show that the 2-torus T^2 and $S^1 \vee S^1 \vee S^2$ both have CW decompositions with four cells: one 0-cell, two 1-cells, and a 2-cell. Recall that $\vee$ denotes the "one-point union" of two spaces, obtained from the disjoint union $\coprod$ by identifying basepoints. Then show that T^2 and $S^1 \vee S^1 \vee S^2$ have isomorphic homology groups.
(b) Show that T^2 and $S^1 \vee S^1 \vee S^2$ have different fundamental groups, hence are *not* homotopy equivalent.
(c) Show that the suspensions

$$ST^2, \quad S\left(S^1 \vee S^1 \vee S^2\right) = S^2 \vee S^2 \vee S^3$$

are homotopy equivalent. (The *reduced suspension* of a based space (X, x) is the smash product with S^1, i.e., $SX = (S^1 \times X)/\big(S^1 \times \{x\} \cup \{*\} \times X\big)$.) **Hint**: ST^2 has a CW decomposition with one 0-cell, two 2-cells, and a 3-cell. The attaching map of the 3-cell is the suspension of the attaching map of the 2-cell in T^2. From knowledge of the attaching map of the 2-cell in T^2, show that this attaching map is null-homotopic. You may assume that $\pi_2(S^2 \vee S^2) \cong \mathbb{Z} \oplus \mathbb{Z}$.

6. Let p be the quotient map from $\mathbb{CP}^n$ to $\mathbb{CP}^n/\mathbb{CP}^k$, $k < n$.
(a) Show that p^* is a monomorphism on integral cohomology.
(b) Describe the ring structure on $H^*(\mathbb{CP}^n/\mathbb{CP}^k, \mathbb{Z})$. Give necessary and sufficient conditions on n and k for all cup products to be trivial.
(c) Show that there is no retraction from $\mathbb{CP}^m/\mathbb{CP}^k$ to $\mathbb{CP}^n/\mathbb{CP}^k$, assuming that $n < 2k + 2 \leq m$.
(d) Show that $\mathbb{CP}^n/\mathbb{CP}^{n-1}$ is homeomorphic to S^{2n}.