TOPOLOGY/GEOMETRY QUALIFYING EXAMINATION AUGUST 12, 2002

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Unless otherwise stated, you may appeal to a "well known theorem" in your solution to a problem, but if you do, it is your responsibility to make it clear exactly which theorem you are using and why its use is justified. In problems with multiple parts, you may assume the answer to any previous part, even if you have not proved it.

(1) Let $\mathbb{C}^*$ be the set of nonzero complex numbers. Let $f : \mathbb{C}^* \to \mathbb{C}^*$ be given by $f(z) = z^2$. Show that f is a 2-fold covering map.

(2) Recall the Theorem of "invariance of domain": *If A and B are homeomorphic subsets of $\mathbb{R}^n$ and A is open, then so is B.*
  (a) Use this to show that the sphere S^2 is not homeomorphic to a subset of the plane $\mathbb{R}^2$.
  (b) Show by example that invariance of domain need not hold if $\mathbb{R}^n$ is replaced by a closed interval.

(3) Let X be a topological space. Let $A, B \subset X$ be open subsets such that $X = A \cup B$ and $A \cap B$ is nonempty. Suppose that A and B are each path-connected and simply connected.
  (a) Prove that X is path-connected.
  (b) Assume $A \cap B$ is path-connected. Prove that X is simply connected.
  (c) Find an example where X is not simply connected.

(4) Let $T = S^1 \times S^1$ be the torus, let

$$M = \Big([0,1] \times [0,1]\Big)/\sim$$

be the Möbius band, where the equivalence relation is defined by:

$$(t, 0) \sim (1-t, 1)$$

for $t \in [0,1]$. Let $\mathbb{RP}^2$ be the projective plane.

(a) Let X be the space obtained by attaching the boundary of M to T via a homeomorphism with $S^1 \times \{x_0\}$ where $x_0 \in S^1$ is a point. Compute the homology groups $H_*(X; \mathbb{Z})$.

(b) Compute the homology groups $H_*(X \times \mathbb{RP}^2; \mathbb{Z})$.

(5) Consider a closed oriented 3-dimensional manifold M covered by S^3 where the group G of deck transformations is a group of order 120 which equals its commutator subgroup $[G, G]$ (the normal subgroup generated by $\{ghg^{-1}h^{-1} \mid g, h \in G\}$).

(a) Compute the integral homology groups of M. (Remark: This part is independent of the next two parts.)

(b) If N is any oriented closed 3 manifold and $d \equiv 0 \pmod{120}$ show that there is a map $f : N \to M$ of degree d.

(c) If in addition N is simply connected, show these are the only possible degrees d of maps $f : N \to M$.

(6) Let $f : S^2 \to S^2$ be a map of degree $k > 1$ and $h : S^3 \longrightarrow S^2$ be the Hopf map. Let X be the cell complex $e^0 \cup e^2 \cup e^3 \cup e^4$ where the 2-cell e^2 is attached to the 0-cell e^0 by the constant map and the 4-cell e^4 is attached to the 2-skeleton $e^0 \cup e^2 \approx S^2$ by the Hopf map h, and the 3-cell e^3 is attached to the 2-skeleton by the map f. Let Y be the cell complex $e^0 \cup e^2 \cup e^3 \cup e^4$ where the 2-cell e^2 is attached by the constant map as for X, the 3-cell e^3 is attached to the 2-skeleton $e^0 \cup e^2 \approx S^2$ by the map f of degree k, and the 4-cell e^4 is attached by the constant map $\partial e^4 \longrightarrow e^0$.

(a) Compute the homology and cohomology of X and Y with integer coefficients and show that $H^*(X; \mathbb{Z}) \cong H^*(Y; \mathbb{Z})$ as rings.

(b) Show that $H^*(X; \mathbb{Z}/k)$ and $H^*(Y; \mathbb{Z}/k)$ are isomorphic as **groups** but not isomorphic as **rings**.