TOPOLOGY/GEOMETRY QUALIFYING EXAMINATION

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Unless otherwise stated, you may appeal to a "well known theorem" in your solution to a problem, but if you do, it is your responsibility to make it clear exactly which theorem you are using and why its use is justified. In problems with multiple parts, you may assume the answer to any previous part, even if you have not proved it.

As in Bredon, the symbol $\mathbb{Z}_k$ refers to the cyclic group of order k. For real numbers a, b the symbol (a, b) refers to the open interval between a and b.

(1) Let M be a compact, connected orientable smooth 6-dimensional manifold without boundary. Suppose its universal cover $p : M' \to M$ is a 7-fold cover. Suppose also the Euler characteristic of M is 6 and $H_3(M;\mathbb{Z}) = \mathbb{Z}^2 \oplus \mathbb{Z}_2$.
   (a) Compute $H_4(M;\mathbb{Z})$.
   (b) Compute $H^4(M;\mathbb{Z})$.

(2) Let the CW complex Y be obtained from the 2-sphere S^2 by attaching two 3-disks, one via a map of degree 6, and one via a map of degree 9.
   (a) Compute $H_*(Y;\mathbb{Z})$.
   (b) Compute $H_*(Y;\mathbb{Z}_3)$.
   (c) Compute $H^*(Y;\mathbb{Z}_2)$.

(3) Let $W = S^2 \vee S^4$ be the one point union of a 2-sphere and a 4-sphere. Let $f : S^4 \to W$ be inclusion.
   (a) Show that $f^* : H^4(W;\mathbb{Z}) \to H^4(S^4;\mathbb{Z})$ is an isomorphism.
   (b) If $\alpha, \beta \in H^2(W;\mathbb{Z})$ show that $\alpha \cup \beta = 0$.
   (c) Show that W and complex projective space $\mathbb{CP}^2$ are not homotopy equivalent even though they have isomorphic homology and cohomology groups. (You may use standard facts about $\mathbb{CP}^2$.)

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(4) Let $\mathbb{L}$ be the set of **all** lines in the plane (not necessarily passing through the origin). Let $l_0 \in \mathbb{L}$ be a line. Define $U(l_0)$ to be the subset of $\mathbb{L}$ consisting of lines l which intersect l_0 in exactly one point. For $l \in U(l_0)$, let $\psi_{l_0}(l) = (p, \theta)$ where $p \in l_0$ is the point of intersection of l with l_0 and $\theta \in (0, \pi)$ is the angle at which l intersects l_0.
   (a) Show that the collection of all (U_{l_0}, ψ_{l_0}) gives $\mathbb{L}$ the structure of a (topological) manifold.
   (b) Show that this manifold is homeomorphic to the open Möbius band (the compact Möbius band with its boundary removed).

(5) Prove or disprove: The fundamental group of a metric space is commutative.

(6) Let X be a topological space and $X_1 \subset X_2 \subset \cdots \subset X_n \subset \ldots$ a sequence of subsets, each with the subspace topology. Suppose that $X = \cup X_i$, and has the *weak topology* with respect to this union: $\forall O \subset X$, O is open $\Leftrightarrow O \cap X_i$ is open in X_i, $\forall i$.

   Recall that a topological space S is $T_1 \Leftrightarrow \{p\}$ is a closed subset of S, for each $p \in S$. Suppose that each X_i is T_1.
   (a) Let $S \subset X$. Suppose that for each i, the intersection $S \cap X_i$ is finite. Prove that S is closed.
   (b) Suppose that each X_i is T_1. Suppose that K is *sequentially compact* (that is, every infinite sequence has a convergent subsequence). Prove that $K \subset X_n$ for some n.