TOPOLOGY/GEOMETRY QUALIFYING EXAMINATION

AUGUST 5, 2011
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Unless otherwise stated, you may appeal to a "well known theorem" in your solution to a problem, but if you do, it is your responsibility to make it clear which theorem you are using and why its use is justified. In problems with multiple parts, be sure to go on to the rest of the problem even if there is some part you cannot do. In working on any part, you may assume the answer to any previous part, even if you have not proved it.

Problem 1.

Let $\mathbb{RP}^n$ and S^n denote, as usual, the real projective n-space and the n-sphere. Corresponding to the inclusions $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1}$ are inclusions $\mathbb{RP}^n \hookrightarrow \mathbb{RP}^{n+1}$ and $S^n \hookrightarrow S^{n+1}$. Prove or disprove the following statements.

(1) The complement $S^{n+1} \setminus S^n$ is disconnected.
(2) The complement $\mathbb{RP}^{n+1} \setminus \mathbb{RP}^n$ is disconnected.
(3) $\mathbb{RP}^n$ is two-sided in $\mathbb{RP}^{n+1}$. (That is, it has a tubular neighborhood N such that $N \setminus \mathbb{RP}^n$ is disconnected.)

Problem 2.

Let $\mathbb{T}^2 = S^1 \times S^1$ be the torus. Let X be an arc connected, locally arc connected space with finite fundamental group and $f : X \to \mathbb{T}^2$ a continuous map.

(1) Show $f_* : H_2(X) \to H_2(\mathbb{T}^2)$ is the zero homomorphism.
(2) Let $Y = \mathbb{T}^2 \cup_f cX$ the space obtained from the disjoint union of $\mathbb{T}^2$ and the cone on X ($cX = \frac{X \times I}{(x,0) \sim (x',0)}$) by identifying $(x,1) \in cX$ with $f(x) \in \mathbb{T}^2$. Suppose $\pi_1(X)$ is the symmetric group on 3 letters and $H_j(X) = 0$ for $j \geq 2$. Compute the integral homology and cohomology groups of Y.

Problem 3.

Let S^2 and S^3 denote the unit spheres in $\mathbb{R}^3$ and $\mathbb{R}^4$, respectively. Explicitly

$$S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$$

$$S^3 = \{(\zeta, w) \in \mathbb{C}^2 \mid |\zeta|^2 + |w|^2 = 1\} = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}.$$

The *Hopf Fibration* is the map $\pi : S^3 \to S^2$ given by

$$x = 2\operatorname{Re}(\zeta w) = 2(x_1 x_3 - x_2 x_4)$$
$$y = 2\operatorname{Im}(\zeta w) = 2(x_2 x_3 + x_1 x_4)$$
$$z = |\zeta|^2 - |w|^2 = (x_1^2 + x_2^2) - (x_3^2 + x_4^2)$$

(1) Show that the derivative $D\pi(p)$ has maximal rank at every point $p \in S^3$. (Hint: you may use that fact that for any $p, q \in S^3$ there are orthogonal linear maps $A : \mathbb{R}^4 \to \mathbb{R}^4$ and $B : \mathbb{R}^3 \to \mathbb{R}^3$ such that $Ap = q$ and $\pi \circ A = B \circ \pi$.)

(2) Describe the preimage $\pi^{-1}(x, y, z)$ for all $(x, y, z) \in S^2$. (Hint: First do the point $(1, 0, 0)$ and then use the hint above.)

(3) Show that there is no section of π, i.e., there does not exist a continuous map $\sigma : S^2 \to S^3$ with the property that $\pi \circ \sigma$ is the identity (hint: homology).

Problem 4.

Let M be an n-dimensional manifold, with $n > 2$, and let $f : D \to M$ be an embedding of the closed n-disk in M, so f is a homeomorphism onto the image $f(D)$. Prove or disprove: $M \setminus f(D)$ is orientable if and only if M is.

Problem 5.

It is a well known fact (which you can assume) that there is a discrete subgroup G of order 120 in $Sp(1) \cong S^3$, the group of unit quaternions $\{a + bi + cj + dk \mid a^2 + b^2 + c^2 + d^2 = 1\}$ with quaternionic multiplication $ij = k$, etc. G has the presentation

$$G = \langle s, t \mid s^2 = t^3 = (st)^5 \rangle$$

and is *perfect*, meaning that G has no abelian quotients. In fact, the only non-trivial normal subgroup of G is its center, which has order 2.

(1) Show that the homogeneous space $M^3 = Sp(1)/G$ is a compact, oriented manifold without boundary, and that it has the same integral homology groups as S^3.

(2) Suppose you form a 3-dimensional CW-complex X by attaching two 2-cells to M, using loops representing s and t as the attaching maps. Compute the integral homology groups of X.

Problem 6.

Show that every continuous map $f : \mathbb{CP}^2 \to S^2$ induces the zero map in reduced homology. Hint: consider the induced map on cohomology.