TOPOLOGY/GEOMETRY QUALIFYING EXAMINATION

AUGUST 6, 2007
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Unless otherwise stated, you may appeal to a "well known theorem" in your solution to a problem, but if you do, it is your responsibility to make it clear which theorem you are using and why its use is justified. In problems with multiple parts, be sure to go on to the rest of the problem even if there is some part you cannot do. In working on any part, you may assume the answer to any previous part, even if you have not proved it.

Problem 1.

Let U be a connected open set in $\mathbb{R}^n$.

(a) Show that any two points in U can be connected by a piecewise straight line. Define $\mathrm{dist}(p, q)$ to be the infimum of the lengths of all such curves joining points p and q in U.
(b) Show that dist is a metric on U and that dist $\geq D$, where D is the Euclidean distance.
(c) Show that dist defines the same topology as D.
(d) Show that $D =$ dist if and only if the closure of U is convex.

Problem 2.

Represent the two-torus as the quotient Lie group

$$T^2 := \mathbb{R}^2/\mathbb{Z}^2.$$

(a) Prove that the map

$$\mathbb{R}^2 \xrightarrow{\tilde{f}} \mathbb{R}^2$$
$$(x, y) \longmapsto (-x, -y)$$

defines a diffeomorphism $T^2 \xrightarrow{f} T^2$ of period 2.
(b) Notation as in (a), let G be the cyclic group of diffeomorphisms of T^2 generated by f. Prove or disprove: The quotient map $X \xrightarrow{q} X/G$ is a covering space.
(c) Prove or disprove: X/G is a (topological) manifold.
(d) Prove or disprove: X/G is homeomorphic to a 2-sphere.

Problem 3.

In this problem, assume X is a connected finite CW-complex with one 0-cell, two 1-cells, three 2-cells, and some 3-cells.

(a) Suppose X has the homotopy type of a closed 3-dimensional manifold. How many 3-cells are there? (Hint: Euler characteristic)

(b) If X^1 is the 1-skeleton of X, show that $X^1 \cong S^1 \vee S^1$ and that $\pi_1(X^1)$ is a free group on two generators a and b represented by the two 1-cells in X.

(c) If the three 2-cells are attached by (based) maps $g_i : S^1 \to S^1 \vee S^1$ and $[g_i] = w_i(a, b) \in \pi_1(X^1)$, ($w_i(a, b)$ is a word in a and b), give necessary and sufficient conditions on $\{w_i\}$ so that $H_1(X, \mathbb{Z}) = 0$.

Problem 4.

(a) Suppose M^n is any connected, oriented, closed n-manifold. By considering a small disc about a point p in M, show there exists a map $f : M \to S^n$ which induces an isomorphism on H_n.

(b) Suppose M^n and f are as in (a), $n = 3$, and $H_1(M, \mathbb{Z}) = 0$. Show that $f : M^3 \to S^3$ induces an isomorphism on *all* homology groups. Does f have to be a homotopy equivalence?

(c) Show by example if M^n is a connected, oriented, closed n-manifold, there *need not be* a map $g : S^n \to M^n$ which induces an isomorphism on H_n.

Problem 5.

Suppose M is a compact 5-manifold such that $H_0(M) = \mathbb{Z}$, $H_1(M) = \mathbb{Z}/3$ and $H_2(M) = \mathbb{Z}$.

(a) Is M orientable?

(b) What are $H_3(M), H_4(M)$ and $H_5(M)$?

(c) Assume that M can be chosen to be of the form $S^2 \times N$ for some 3-manifold N. What would the homology groups of N be? Find such an N.

Problem 6.

Let V be a closed *complex* submanifold of $\mathbb{CP}^n$ of complex dimension $n - 1$.

(a) Show that the complement of V in $\mathbb{CP}^n$ is connected.

(b) Show by example that the complement of V in $\mathbb{CP}^n$ need not be simply connected.